*Vietnam Journal of Mathematics* **26**:4 (1998) 301–309

Vietnam Journal
of
MATHEMATICS


# On a Nonlinear Boundary Value Problem with a Mixed Nonhomogeneous Condition

**Nguyen Hoi Nghia**
*Department of Mathematics and Computer Science*
*College of Natural Science, University of Ho Chi Minh City*
*227 Nguyen Van Cu, Distr. 5, Ho Chi Minh City, Vietnam*

**Nguyen Thanh Long**
*Department of Mathematics and Computer Science*
*College of General Studies, University of Ho Chi Minh City*
*268 Ly Thuong Kiet, Distr. 10, Ho Chi Minh City, Vietnam*

Received March 5, 1997
Revised May 5, 1998

**Abstract.** We study the following nonlinear boundary value problem

$$\begin{cases} \dfrac{-1}{x^\gamma} \cdot \dfrac{d}{dx}(x^\gamma . u'(x)) + f(x, u(x)) = F(x), \ \ 0 < x < 1, \\ \big| \lim\limits_{x \to 0_+} x^{\gamma/2} u'(x) \big| < \infty, \ \ u'(1) + h.u(1) = g. \end{cases} \tag{$*$}$$

In Sec. 1, we prove by the Galerkin method the existence and uniqueness of the weak solution of $(*)$ in appropriate Sobolev spaces with weight. In Sec. 2, we study asymptotic behavior of the solution depending on $h$ as $h \to 0_+$.

## 1. Introduction

We consider the following nonlinear boundary value problem:

$$\frac{-1}{x^\gamma} \cdot \frac{d}{dx}(x^\gamma . u'(x)) + f(x, u(x)) = F(x), \ \ 0 < x < 1, \tag{1.1}$$

$$\big| \lim_{x \to 0_+} x^{\gamma/2} u'(x) \big| < \infty, \ \ u'(1) + h.u(1) = g \tag{1.2}$$

where $\gamma > 0$, $h > 0$, $g$ are given constants. $f$, $F$ are given functions.

In [1], Tucsnak has considered the equation

$$-\frac{d}{dx}M(x,u'(x)) + g(x)\,\sin u(x) = 0\,,\ 0 < x < 1\,. \qquad (1.3)$$

Equation (1.3) has its motivation in the mathematical sense of the buckling of a nonlinear elastic bar immersed in a fluid. We note that Eq. (1.3), with $u'.M(x,u') \geq c_1|u'|^p$, $p > 1$, $C_1 > 0$ independent of $x$, had been considered by the authors in [2]. We consider here Problem (1.1) with $M(x,u') = x^\gamma.u'$, where $C_1 = C_1(x) = x^\gamma \geq 0$.

In [3, 4], the authors have studied the following nonlinear Bessel differential equation

$$\frac{-1}{x}\cdot\frac{d}{dx}(x.u'(x)) + u^2 - u = 0, \quad x > 0. \qquad (1.4)$$

In this paper, we use the Galerkin and compactness method in appropriate Sobolev spaces with weight to prove the existence of a unique weak solution. This result slightly generalizes [1–4]. We also study the asymptotic behavior of the solution $u_h$ depending on $h$ as $h \to 0_+$. We also obtain that the function $h \mapsto |u_h(1)|$ is nonincreasing on $(0, +\infty)$.

## 2. Theorem of Existence and Uniqueness

Put $\Omega = (0,1)$, we omit the definitions of usual function spaces $c^m(\overline{\Omega})$, $L^p(\Omega)$, and $H^m(\Omega)$. We denote by $H$ the class of all measurable functions $u$, defined on $\Omega$, for which

$$\int_0^1 x^\gamma |u(x)|^2\,dx < +\infty. \qquad (2.1)$$

We identify in $H$ functions that are equal almost everywhere (a.e. for short) on $\Omega$. The elements of $H$ are thus actually equivalence classes of measurable functions satisfying (2.1) , two functions are equivalent if they are equal a.e in $\Omega$. Then $H$ is also a Hilbert space with respect to the scalar product

$$\langle u, v\rangle = \int_0^1 x^\gamma.u(x)\,v(x)\,dx. \qquad (2.2)$$

We denote

$$V = \{v \in H \,:\, v' \in H\} \qquad (2.3)$$

the real Hilbert space with the scalar product

$$\langle u, v\rangle + \langle u', v'\rangle \qquad (2.4)$$

with derivatives in the sense of distributions [6].

The norms in $H$ and $V$ induced by the corresponding scalar products are denoted by $\|\cdot\|$ and $\|\cdot\|_V$, respectively. $V$ is continuously and densely embedded in $H$. Identifying $H$ with $H'$ (the dual of $H$), we have $V \hookrightarrow H \hookrightarrow V'$; on the other hand, the notation $\langle .,.\rangle$ is used for the pairing between $V$ and $V'$.

*Remark 1.* In defining the function space $V$ with weight $x^\gamma$, we can also define $V$ as the completion of the space

$$S = \left\{u \in C^1([0,1]) : \|u\|_V^2 = \int_0^1 x^\gamma (|u(x)|^2 + |u'(x)|^2)dx < \infty\right\}$$

with respect to the norm $\|.\|_V$ (see [6]).

We then have the following lemmas.

**Lemma 1.** *There exist two constants $K_1 > 0$ and $K_2 > 0$ (depending only on $\gamma$) such that*

$$\|u'\|^2 + u^2(1) \geq K_1 \|u\|_V^2, \quad \forall u \in C^1([0,1]); \tag{2.5}$$

$$x^{\gamma/2}\,|u(x)| \leq K_2\,\|u\|_V, \quad \forall u \in C^1([0,1]), \quad \forall x \in [0,1]. \tag{2.6}$$

**Lemma 2.** *The embedding $V \hookrightarrow H$ is compact.*

The proof of Lemmas 1 and 2 can be found in [5].

*Remark 2.* We also note that

$$\lim_{x\to 0_+} x^{\gamma/2}\,u(x) = 0, \quad \forall u \in V. \tag{2.7}$$

(see [6, Lemma 5.40, p. 128]).

On the other hand, by $H^1(\varepsilon,1) \hookrightarrow C^0([\varepsilon,1]),\ 0 < \varepsilon < 1$, and

$$\varepsilon^{\gamma/2}\,\|u\|_{H^1(\varepsilon,1)} \leq \|u\|_V, \ \forall u \in V, \ 0 < \varepsilon < 1, \tag{2.8}$$

it follows that

$$u|_{[\varepsilon,1]} \in C^0([\varepsilon,1]), \ \forall \varepsilon, \ 0 < \varepsilon < 1. \tag{2.9}$$

From (2.7) and (2.9), we deduce that

$$x^{\gamma/2}\,u \in C^0([0,1]), \ \forall u \in V. \tag{2.10}$$

We shall make the following assumptions.

($H_1$) $f : (0,1) \times R \longrightarrow R$ satisfies the Caratheodory condition, i.e., $f(.,u)$ is measurable on $(0,1)$ for every $u \in R$, and $f(x,.)$ is continuous on $R$ for a.e. $x \in (0,1)$.

($H_2$) There exist positive constants $C_1,\ C_1',\ C_2$ and $p > 1$ such that
(i) $u.f(x,u) \geq C_1\,|u|^p - C_1'$;
(ii) $|f(x,u)| \leq C_2(1 + |u|^{p-1})$.

The weak solution of Problems (1.1) and (1.2) is formed from the following variational problem.

Find $u \in V$ such that

$$\langle u', v'\rangle + h.u(1)\,v(1) + \langle f(x,u), v\rangle = g.v(1) + \langle F, v\rangle, \quad \forall v \in V. \tag{2.11}$$

*Remark 3.* By (2.10), the terms $u(1)$ and $v(1)$ appearing in (2.11) are defined for every $u, v \in V$. We obtain (2.11) by formally multiplying both sides of (1.1) by $v$ and then integrating by part having conditions (1.2) and (2.7) in mind.

Then we have the following result.

**Theorem 1.** *Let $h > 0$, $g \in R$, $F \in V'$ and $(H_1)$, $(H_2)$ hold. Then there exists a solution $u$ of the variational problem (2.11) such that*

$$u \in V \quad \text{and} \quad x^{\gamma/p}\, u \in L^p(\Omega). \tag{2.12}$$

*Furthermore, if $f(x, u)$ is nondecreasing with respect to $u$, i.e.,*

$$(\mathrm{H}_3) \quad (f(x,u) - f(x,v)).(u - v) \geq 0 \;\; \forall u, v \in R, \;\; \text{a.e. } x \in (0,1),$$

*then the solution is unique.*

*Proof.* Denote by $\{W_j\}$ the infinite orthonormal base in the separable Hilbert space $V$. We find $u_m$ of the form

$$u_m(x) = \sum_{j=1}^{m} c_{mj}\, W_j(x), \tag{2.13}$$

where $c_{mj}$ satisfy the following nonlinear equation system:

$$\begin{aligned} &\langle u_m', W_j' \rangle + h u_m(1)\, W_j(1) + \langle f(x, u_m), W_j \rangle \\ &= g\, W_j(1) + \langle F, W_j \rangle, \quad 1 \leq j \leq m. \end{aligned} \tag{2.14}$$

By Brouwer's lemma (see [7, Lemma 4.3, p. 53]), it follows from the hypotheses $(\mathrm{H}_1)$ and $(\mathrm{H}_2)$ that the system (2.13) and (2.14) has a solution $u_m$.

Multiplying the $j$th equation of system (2.14) by $c_{mj}$, then adding these equations for $j = 1, 2, ..., m$, we have

$$\|u_m'\|^2 + h\, u_m^2(1) + \langle f(x, u_m), u_m \rangle = g\, u_m(1) + \langle F, u_m \rangle. \tag{2.15}$$

By using the inequalities (2.5) and (2.6) and by the hypothesis $(\mathrm{H}_2)$(i), we obtain

$$C_0\, \|u_m\|_V^2 + C_1 \int_0^1 x^\gamma\, |u_m(x)|^p\, dx \leq (|g| K_2 + \|F\|_{V'})\, \|u_m\|_V + \frac{C_1'}{\gamma + 1}, \tag{2.16}$$

where $C_0 = K_1. \min\{1, h\}$.

Hence, we deduce from (2.16) that

$$\|u_m\|_V \leq C, \tag{2.17}$$

$$\|x^{\gamma/p}\, u_m\|_{L^p(\Omega)} \leq C. \tag{2.18}$$

$C$ is a constant independent of $m$.

By means of (2.17) and (2.18) and Lemma 1, the sequence $\{u_m\}$ has a subsequence still denoted by $u_m$ such that

$$u_m \to u \quad \text{in } V \text{ weakly}, \tag{2.19}$$

$$u_m \to u \quad \text{in } H \text{ strongly and a.e. in } \Omega, \tag{2.20}$$

$$x^{\gamma/p} u_m \to x^{\gamma/p} u \quad \text{in } L^p(\Omega) \text{ weakly.} \tag{2.21}$$

On the other hand, by (2.20) and the hypothesis ($H_1$), we have

$$f(x, u_m) \to f(x, u) \quad \text{a.e. in } \Omega. \tag{2.22}$$

We also deduce from the hypothesis ($H_2$)(ii) and from (2.18) that

$$\|x^{\gamma/p'} f(x, u_m)\|_{L^{p'}(\Omega)}^{p'} \leq C_2^{p'} 2^{p'-1}(1 + \|x^{\gamma/p} u_m\|_{L^p(\Omega)}^p) \leq C, \tag{2.23}$$

where $p' = p/(p-1)$. $C$ is a constant independent of $m$.

We shall need the following lemma, the proof of which can be found in [7].

**Lemma 3.** *Let $Q$ be an open bounded set of $R^N$ and*

$$G_m, G \in L^q(Q), 1 < q < \infty \ \ \textit{such that}\ \ G_m \to G \ \textit{a.e. in}\ Q$$

*and $\|G_m\|_{L^q(Q)} \leq C$, with $C$ being a constant independent of $m$. Then $G_m \to G$ weakly in $L^q(Q)$.*

Applying Lemma 3 with $N = 1$, $q = p'$, $Q = \Omega$, $G_m = x^{\gamma/p'} f(x, u_m)$, $G = x^{\gamma/p'} f(x, u)$, we deduce from (2.22) and (2.23) that

$$x^{\gamma/p'} f(x, u_m) \to x^{\gamma/p'} f(x, u) \quad \text{weakly in } L^{p'}(\Omega). \tag{2.24}$$

Passing to the limit in Eq. (2.14), we find without difficulty from (2.19) and (2.24) that $u$ satisfies the equation

$$\langle u', W_j' \rangle + h\, u(1) W_j(1) + \langle f(x, u), W_j \rangle = g\, W_j(1) + \langle F, W_j \rangle. \tag{2.25}$$

Equation (2.25) holds for every $j \in N$, i.e., (2.11) holds.

*Proof of Uniqueness.* Let $u_1$, $u_2$ be two solutions of the problem (2.11) and let $u = u_1 - u_2$. Then $u$ satisfies

$$\langle u', v' \rangle + h\, u(1) + \langle f(x, u_1) - f(x, u_2), v \rangle = 0. \tag{2.26}$$

Taking $v = u$ in (2.26) and using (2.5) and ($H_3$), we have

$$C_0 \|u\|_V^2 \leq \|u'\|^2 + h\, u^2(1) + \langle f(x, u_1) - f(x, u_2), u \rangle = 0.$$

Then this inequality implies $u = 0$, i.e., $u_1 = u_2$.

This completes the proof of Theorem 1. ■

*Remark 4.* In [3], we have proved that the nonlinear Bessel differential equation (1.4) associated with the boundary condition $u(0) = 1$, $u(+\infty) = 0$ has at least one solution. There, the nonlinear term $u^2 - u$ is non-monotonic. One of the solutions above is established from the boundary value problem (1.4) in the interval $a < x < b$ associated with the boundary condition $u(a) = 1$, $u(b) = 0$, wherein, $x_i < a < b < x_{i+1}$ and $x_i$, $x_{i+1}$ are two consecutive zeros of the first order Bessel function $J_0(x)$. Formation of a counterexample for the function $f(x, u)$ not satisfying the assumption ($H_3$) so that the solution of (2.11) is not unique is an open problem.

## 3. Asymptotic Behavior of the Solution as $h \to 0_+$

In this section, let ($H_1$)–($H_3$) hold. The variational problem (2.11) according to Theorem 1 admits a unique solution $u = u_h$, $h > 0$. We shall study asymptotic behavior of solution $u_h$ as $h \to 0_+$.

We make the following additional assumption on the function $f$.

($H_4$) There exist constants $p \geq 2$, $C_3 > 0$ such that

$$(f(x, u) - f(x, v)).(u - v) \geq C_3\, |u - v|^p, \quad \forall u, v \in R, \quad \text{a.e. } x \in (0, 1).$$

We have the following result.

**Theorem 2.** *Let ($H_1$)–($H_3$) hold and $F \in V'$, $g \in R$. Then Problem (2.11) with $h = 0$ has a unique solution $u_0$ such that*

$$u_0 \in V \quad \text{and} \quad x^{\gamma/p}\, u_0 \in L^p(\Omega).$$

*Furthermore,*

$$\|u_h - u_0\|_V + \|x^{\gamma/p}\, u_h - x^{\gamma/p}\, u_0\|_{L^p(\Omega)} \leq C.h^{1/p-1},$$

*with $h > 0$ small enough, where $C$ is a constant depending on $\gamma$, $p$, $C_1$, $C_1'$, $C_2$, $C_3$, $g$, $\|F\|_{V'}$ only.*

*Proof.* First, we prove that the solution $u_h$ of (2.11) is bounded by a constant independent of $h > 0$.

Taking $v = u_h$ in (2.11) and using ($H_2$)(i) and (2.6), we obtain

$$\|u_h'\|^2 + C_1\, \|x^{\gamma/p}\, u_h\|^p_{L^p(\Omega)} \leq \mathcal{C}_1\, \|u_h\|_V + \frac{C_1'}{\gamma + 1}, \tag{3.1}$$

where $\mathcal{C}_1 = |g|\, K_2 + \|F\|_{V'}$.

On the other hand, using Hölder's inequality, we obtain

$$\|u_h\|^2 \leq \frac{1}{(1+\gamma)^{(p-2)/p}}\, \|x^{\gamma/p}\, u_h\|^2_{L^p(\Omega)} \leq \|x^{\gamma/p}\, u_h\|^2_{L^p(\Omega)}. \tag{3.2}$$

It follows from (3.1) and (3.2) that

$$\|u_h'\|^2 \leq \beta_1\, \|u_h\|^2_V + \frac{1}{4\beta_1}\, \mathcal{C}_1^2 + \frac{C_1'}{\gamma + 1}, \ \forall \beta_1 > 0. \tag{3.3}$$

$$\|u_h\|^2 \le \left(\frac{\mathcal{C}_1}{C_1}\,\|u_h\|_V + \frac{C_1'}{C_1(\gamma+1)}\right)^{2/p}$$
$$\le \frac{1}{p}(\beta_2\,\|u_h\|_V^{2/p})^p + \frac{1}{p'}(C_4/\beta_2)^{p'} + \left(\frac{C_1'}{C_1(\gamma+1)}\right)^{2/p}, \quad \forall \beta_2 > 0 \tag{3.4}$$

where $C_4 = (\mathcal{C}_1/C_1)^{2/p}$.

Choosing $\beta_1 + \frac{\beta_2^p}{p} < \frac{1}{2}$, we have from (3.3) and (3.4) that

$$\|u_h\|_V^2 \le \frac{1}{2\beta_1}\,\mathcal{C}_1^2 + \frac{2}{p'}(C_4/\beta_2)^{p'} + \widetilde{C}_2(C_1, C_1')$$
$$\le C_5 = \widetilde{C}_1(p, C_1).\max\left\{\mathcal{C}_1^2, \mathcal{C}_1^{2/p-1}\right\} + \widetilde{C}_2(C_1, C_1'), \tag{3.5}$$

where

$$\begin{cases} \widetilde{C}_1(p, C_1) = \frac{1}{2\beta_1} + \frac{2}{p'\,\beta_2^{p'}\,C_1^{2/p-1}}, \\ \widetilde{C}_2(C_1, C_1') = \frac{2C_1'}{\gamma+1} + 2\left(\frac{C_1'}{C_1(\gamma+1)}\right)^{2/p}. \end{cases} \tag{3.6}$$

$C_5$ is a constant independent of $h > 0$.

Now, let $u_h$ (resp. $u_{h'}$) be the solution of Problem (2.11) with the parameter $h$ (resp. $h'$). Let $v = u_h - u_{h'}$, $\widetilde{h} = h - h'$. Then $v$ satisfies

$$\langle v', w'\rangle + h\,v(1)\,w(1) + \langle f(x, u_h) - f(x, u_{h'}), w\rangle = -\widetilde{h}\,u_{h'}(1)\,w(1), \ \forall w \in V. \tag{3.7}$$

Proceeding as in the proof of the first part, we deduce from (2.6), (3.5) and ($H_4$) that

$$\|v'\|^2 + C_3\,\|x^{\gamma/p}\,v\|_{L^p(\Omega)}^p \le |\widetilde{h}|\,K_2^2\,\sqrt{C_5}\,\|v\|_V. \tag{3.8}$$

Applying (3.1), (3.5), and (3.6) with $C_1 = C_3$, $C_1' = 0$, $\mathcal{C}_1 = |\widetilde{h}|\,K_2^2\,\sqrt{C_5}$, we deduce from (3.8) that

$$\|v\|_V^2 \le \widetilde{C}_1(p, C_3).\max\left\{\left(|\widetilde{h}|\,K_2^2\,\sqrt{C_5}\right)^2, \left(|\widetilde{h}|\,K_2^2\,\sqrt{C_5}\right)^{2/p-1}\right\}. \tag{3.9}$$

We note that, if $p \ge 2$, then

$$\left(|\widetilde{h}|\,K_2^2\,\sqrt{C_5}\right)^2 \le \left(|\widetilde{h}|\,K_2^2\,\sqrt{C_5}\right)^{2/p-1}$$

as $|\widetilde{h}|$ is small enough.

Hence,

$$\|u_h - u_{h'}\|_V = \|v\|_V \le C_6\,|h - h'|^{1/p-1}, \tag{3.10}$$

where

$$C_6 = C_6(\gamma, p, C_3, C_5, g, \|F\|_{V'}). \tag{3.11}$$

It follows from (3.8) and (3.10) that

$$\|x^{\gamma/p} u_h - x^{\gamma/p} u_{h'}\|_{L^p(\Omega)} = \|x^{\gamma/p} v\|_{L^p(\Omega)} \leq C_7 \, |h - h'|^{1/p-1}, \tag{3.12}$$

where

$$C_7 = (K_2^2 \sqrt{C_5} \, C_6 \, / \, C_3)^{1/p}. \tag{3.13}$$

Thus, we obtain from (3.10) and (3.12) that

$$\|u_h - u_{h'}\|_V + \|x^{\gamma/p} u_h - x^{\gamma/p} u_{h'}\|_{L^p(\Omega)} \leq C \, |h - h'|^{1/p-1}. \tag{3.14}$$

Let us consider the space

$$W = \{v \in V \; : \; x^{\gamma/p} v \in L^p(\Omega)\}.$$

$W$ is a Banach space with the norm

$$\|v\|_W = \|v\|_V + \|x^{\gamma/p} v\|_{L^p(\Omega)}.$$

Let $h_m$ be a sequence such that $h_m > 0$, $h_m \to 0$ as $m \to \infty$. It follows from (3.14) that $\{u_{h_m}\}$ is a Cauchy sequence in $W$. Hence, there exists $u_0 \in W$ such that

$$u_{h_m} \to u_0 \quad \text{strongly in } W. \tag{3.15}$$

By passing to the limit as in the proof of Theorem 1, we deduce that $u_0$ satisfies the following variational equation:

$$\langle u_0', w' \rangle + \langle f(x, u_0), w \rangle = g \, w(1) + \langle F, w \rangle, \; \forall w \in V.$$

The uniqueness is proved in a standard manner as in the proof of Theorem 1. Then, letting $h' \to 0_+$ in (3.14), we have

$$\|u_h - u_0\|_V + \|x^{\gamma/p} u_h - x^{\gamma/p} u_0\|_{L^p(\Omega)} \leq C \, h^{1/p-1}.$$

Therefore, Theorem 2 is proved completely. ■

**Theorem 3.** *Under the assumptions of Theorem 2, we have that*
(i) *The function* $h \mapsto |u_h(1)|$ *is nonincreasing on* $(0, +\infty)$;
(ii) $|u_0(1)| = \sup_{h>0} |u_h(1)|$.

*Proof.* Let $0 < h < h'$, $\widetilde{h} = h - h' < 0$. Then $v = u_h - u_{h'}$ satisfies (3.7). Taking $w = v$ in (3.7), we obtain

$$-\widetilde{h} \, u_{h'}(1) \, (u_h(1) - u_{h'}(1)) \geq 0.$$

Hence,

$$|u_{h'}(1)|^2 \leq u_{h'}(1) \, u_h(1).$$

Therefore,

$$|u_{h'}(1)| \leq |u_h(1)|, \tag{3.16}$$

and (i) is proved.

Letting $h \to 0_+$ in (3.16), we obtain (ii). Theorem 3 is completely proved. ■

*Acknowledgement.* The authors wish to thank the referees for their constructive and useful remarks.

## References

1. M. Tucsnak, Buckling of nonlinearly elastic rods immersed in a fluid, *Bull. Math. Soc. Sci. Math. R. S. Roumanie* **33(81)**(2) (1989) 173–181.
2. N.T. Long and T.V. Lang, The problem of buckling of a nonlinearly elastic bar immersed in a fluid, *Vietnam J. Math.* **24**(4) (1996) 131–142.
3. N.T. Long, E.L. Ortiz, and A.Ph.Ng. Dinh, On the existence of a solution of a boundary value problem for a nonlinear Bessel equation on an unbounded interval, *Proc. Royal Irish Acad.* **95A**(2) (1995) 237–247.
4. N.T. Long, E.L. Ortiz, and A.Ph.Ng. Dinh, A nonlinear Bessel differential equation associated with Cauchy condition, *Computers Math. Appl.* **31** (1996) 131–139.
5. N.T. Long and A.Ph.Ng. Dinh, Periodic solutions for a nonlinear parabolic equation associated with the penetration of a magnetic field into a substance, *Computers Math. Appl.* **30**(1) (1995) 63–78.
6. R.A. Adam, *Sobolev Spaces*, Academic Press, New York, 1975.
7. J.L. Lions, *Quelques Méthodes de Résolution des Problèmes aux Limites Non-Linéaires*, Dunod, Gauthier–Villars, Paris, 1969.