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(DFN)-valued Meromorphic Functions on (DFN)-spaces and the Properties $(\overline{\overline{\Omega}})$, $(\underline{\mathrm{DN}})$

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Abstract. The following question has been studied: Under what conditions is every (DFN)-valued meromorphic function on the dual space E^* of a nuclear Frechet space E of uniform type? Some sufficient conditions are given in terms of the invariants $(\overline{\overline{\Omega}})$, $(\underline{\mathrm{DN}})$.

Let E, F be locally convex spaces and D an open subset of E. A holomorphic function $f : D_0 \longrightarrow F$ on a dense open subset D_0 of D with values in F is called meromorphic on D if, for every $z \in D$, there exists a neighborhood U of z and holomorphic functions $h : U \longrightarrow F, \sigma : U \longrightarrow \mathbf{C}$ with $\sigma \neq 0$ such that

$$f\Big|_{D_0 \cap U} = \frac{h}{\sigma}\Big|_{D_0 \cap U}.$$

By $\mathcal{M}(D, F)$, we denote the space of F-valued meromorphic functions on D. Write $\mathcal{M}(D)$ for $F = \mathbf{C}$. A function $f \in \mathcal{M}(E, F)$ is said to be of *uniform type* if f can be meromorphically factorized through a Banach space. This also means that there exists a continuous semi-norm ρ on E and a meromorphic function g from E_ρ, the canonical Banach space associated to ρ, into F such that $f = g\omega_\rho$, where $\omega_\rho : E \longrightarrow E_\rho$ is the canonical map.

Put $\mathcal{M}_u(E, F) = \{f \in \mathcal{M}(E, F) \mid f \text{ is of uniform type}\}$.

The uniformity of holomorphic functions between locally convex spaces is defined similarly as for meromorphic functions. In 1982, Colombeau and Mujica [1] have proved that every Frechet-valued holomorphic function on dual spaces of Frechet–Montel spaces (DFM-spaces for short) is of uniform type. Later, Meise and Vogt [9] have obtained an important result of this type for scalar holomorphic functions on nuclear Frechet spaces in an inter-relation with linear topological invariants. Let us note that a counterexample for this problem was given by Narchbin [10]. Recently, Ha [3] has extended the above results of Meise and Vogt for Frechet-valued holomorphic functions on Frechet–Schwartz spaces having absolutely Schauder basis.

Recently, the uniformity of meromorphic functions was considered by Hai [4] for the dual Frechet–Schwartz case with an absolute basis in an inter-relation with linear topological invariants. The aim of this paper is to consider this problem in the dual nuclear Frechet case. To formulate the main result, we recall the definitions of such invariants, which were introduced and investigated by Vogt [9, 12, 13].

Let E be a Frechet space with an increasing fundamental system of semi-norms $\{\|\cdot\|_k\}$. For each subset B of E, we define a general semi-norm $\|\cdot\|_B^*$ on E^*, the dual space of E, by $\|u\|_B^* = \sup\{|\langle x, u\rangle| : x \in B\}$.

We write

$$\|\cdot\|_q^* = \|\cdot\|_{U_q}^* \quad \text{where} \quad U_q = \{x \in E \mid \|x\|_q \leq 1\}.$$

Note that E has the property

$$(\overline{\overline{\Omega}}) \quad \text{if } \forall p\ \exists q\ \forall k, d > 0\ \exists C > 0 \ : \ \|\cdot\|_q^{*1+d} \leq C\|\cdot\|_k^* \|\cdot\|_p^{*d}.$$

$$(\underline{DN}) \quad \text{if } \exists p\ \forall q\ \exists k, d, C > 0 \ : \ \|\cdot\|_q^{1+d} \leq C\|\cdot\|_k \|\cdot\|_p^d.$$

Theorem. *Let E and F be nuclear Frechet spaces. Then*

$$\mathcal{M}_u(E^*, F^*) = \mathcal{M}(E^*, F^*) \ \ if \ \ F \in (\overline{\overline{\Omega}}) \ and \ \mathcal{H}(E^*) \in (\underline{DN}).$$

The proof of the theorem is given in Sec. 2. Some propositions, which are necessary for this proof, are presented in Sec. 1. Finally, in Sec. 3, we give some examples about spaces having ($\underline{DN}$)-property.

1. Some Propositions

Proposition 1.1. *Let E be a nuclear Frechet space and F a Frechet space. Then every holomorphic function on E with values in F can be factorized holomorphically through a Banach space if*

$$E \in (\overline{\overline{\Omega}}) \text{ and } F \in (\underline{DN}).$$

Proof. Given $f : E \longrightarrow F$, a holomorphic function. $\{\|\cdot\|_k\}$is a fundamental system of semi-norms of E.

Choose $p \geq 1$ such that ($\underline{DN}$) holds. Let $\alpha \geq 1$ such that

$$\sup\{\|f(z)\|_p : z \in U_\alpha\} < \infty.$$

Consider the space $E/\ker\|\cdot\|_\alpha$ equipped with the quotient topology. By the Taylor expansion of f at $O \in E$, it follows that f can be considered as a holomorphic function from $E/\ker\|\cdot\|_\alpha$ into F. Thus, without loss of generality, we can assume $\|\cdot\|_\alpha$ is a norm on E. Since $E \in (\overline{\overline{\Omega}})$, by [12], there exists a bounded balanced convex set B in E and $\beta \geq 1$ such that $E(B)$ is a Hilbert space, $E(B)$ is dense in E and

$$\forall d > 0\ \exists C_d > 0 : \|\cdot\|_\beta^{*1+d} \leq C_d \|\cdot\|_B^* \|\cdot\|_\alpha^{*d}. \qquad (*)$$

Let $\omega_\alpha : E \longrightarrow E_\alpha$, $\omega_\beta : E \longrightarrow E_\beta$, and $\omega_{\beta\alpha} : E_\beta \longrightarrow E_\alpha$ be the canonical maps. Moreover, from the nuclearity of E, we may assume E_α is a Hilbert space for every α.

Put $A = \omega_\alpha|_{E(B)}$, then A is injective and of type s, because E is nuclear and $\|\cdot\|_\alpha$ is a norm on E.

By the spectral mapping theorem [11], there exist a complete orthonormal system $\{y_j\}_{j\in\mathbf{N}}$ in $E(B)$, an orthonormal system $\{z_j\}_{j\in\mathbf{N}}$ in E_α and a decreasing sequence $\lambda = (\lambda_j)_{j\in\mathbf{N}} \in s$ (with $\lambda_j > 0 \; \forall j \geq 1$) such that

$$Ax = \sum_{j=1}^{\infty} \lambda_j \langle x, y_j\rangle_{E(B)} z_j \,. \tag{1}$$

We define $\chi_k \in E_\alpha^*$ given by $\chi_k(z) = \langle z, z_k\rangle_{E_\alpha}$, $z \in E_\alpha$, and obtain

$$\|\chi_k\|_\alpha^* = \sup_{\|z\|\leq 1} |\langle z, z_k\rangle_{E_\alpha}| = 1 \quad \forall k \geq 1. \tag{2}$$

Then

$$\|A^* \chi_k\|_B^* = \sup_{\|x\|\leq 1} |\chi_k A(x)| = \sup_{\|x\|\leq 1} |\langle A(x), z_k\rangle| = \sup_{\|x\|\leq 1} |\lambda_k \langle x, y_k\rangle| = \lambda_k \tag{3}$$

(by the Bessel inequality $|\langle x, y_k\rangle| \leq \|x\|$).

Let us now put $\varphi_k = \nu^* \chi_k \in E_\beta^*$ with $\nu = \omega_{\beta\alpha}$. By $(*)$, we have

$$\begin{aligned} &\forall d > 0 \;\; \exists C_d > 0 : \|\varphi_k\|_\beta^{*1+d} \\ &= \|\nu^* \chi_k\|_\beta^{*1+d} \leq C_d \|A^* \chi_k\|_B^* \, \|\chi_k\|_\alpha^{*d} \\ &\leq C_d \lambda_k \quad \text{(by (2) and (3))}. \end{aligned}$$

Hence,

$$\|\varphi_k\|_\beta^* \leq \Big(C_d \lambda_k\Big)^{\frac{1}{1+d}} \quad \forall k \geq 1. \tag{4}$$

Put $W = U_\alpha \cap E(B)$. Choose $\delta \in (0,1)$ such that, for $\mu = (\mu_j := \delta/j)_{j\geq 1}$, the set

$$\Big\{u \in E(B) \,\Big|\, u = \sum_{j=1}^{\infty} \frac{\xi_j}{\lambda_j} y_j \;\text{ and }\; |\xi_j| \leq \mu_j \;\; \forall j \geq 1\Big\}$$

is contained in W.

Without loss of generality, we may assume

$$\sup\{\|\omega_p f(u)\| \,|\, u \in W\} \leq 1$$

($\omega_p f$ is locally bounded, where $\omega_p : F \longrightarrow F_p$ is the canonical map). We put

$$M = \{m = (m_j) \in N^N \,|\, m_j \neq 0 \text{ only for finitely many } j \in \mathbf{N}\}.$$

For each $m = (m_1, m_2, ..., m_n, 0, 0...) \in M$, we define

$$a_m = \Big(\frac{1}{2\pi i}\Big)^n \int\limits_{|\rho_1|=\mu_1} \int\limits_{|\rho_2|=\mu_2} \cdots \int\limits_{|\rho_n|=\mu_n} \frac{\omega_p f\big(\frac{\rho_1}{\lambda_1} y_1 + \frac{\rho_2}{\lambda_2} y_2 + \cdots + \frac{\rho_n}{\lambda_n} y_n\big)}{\rho^{m+1}} \, d\rho,$$

where $\rho^{m+1} := \rho_1^{m_1+1}\rho_2^{m_2+1}...\rho_n^{m_n+1}$, $d\rho := d\rho_1 d\rho_2...d\rho_n$. Then

$$\|a_m\|_{F_p} \le \frac{1}{\mu^m} \text{ (where } \mu^m := \mu_1^{m_1}\mu_2^{m_2}...\mu_n^{m_n}).$$

Moreover, we have

$$a_m = \frac{1}{\lambda^m}\left(\frac{1}{2\pi i}\right)^n \int\limits_{|\theta_1|=r_1}\int\limits_{|\theta_2|=r_2}\cdots\int\limits_{|\theta_n|=r_n} \frac{\omega_p f(\theta_1 y_1 + \theta_2 y_2 + \cdots + \theta_n y_n)}{\theta^{m+1}}\, d\theta$$

$$a_m = \omega_p\left(\underbrace{\frac{1}{\lambda^m}\left(\frac{1}{2\pi i}\right)^n \int\limits_{|\theta_1|=r_1}\int\limits_{|\theta_2|=r_2}\cdots\int\limits_{|\theta_n|=r_n} \frac{f(\theta_1 y_1 + \theta_2 y_2 + \cdots + \theta_n y_n)}{\theta^{m+1}}\, d\theta}_{b_m\in F}\right).$$

We have

$$\|b_m\|_p = \|\omega_p(b_m)\|_{F_p} = \|a_m\|_{F_p} \le \frac{1}{\mu^m}. \tag{5}$$

We also obtain

$$\|b_m\|_q \le \frac{N(q,t)}{\lambda^m \mu^m t^{|m|}} \tag{6}$$

for $m \in M$, $q \ge 1$, $t > 0$, $|m| = \sum_j m_j$ and

$$N(q,t) = \sup\left\{\|f(x)\|_q \,\Big|\, x = \sum_{j=1}^{\infty}\xi_j y_j \text{ and } |\xi_j| \le t\mu_j \ \forall j \ge 1\right\}.$$

By the property (DN) of F,

$$\forall q \ge 1\ \exists k \ge 1\ \exists C, \overline{d} > 0 : \|\cdot\|_q^{1+\overline{d}} \le C\|\cdot\|_k \|\cdot\|_p^{\overline{d}}. \tag{7}$$

Now, for $d = \delta\overline{d}$, $\gamma = \dfrac{1}{2(1+d)}$ $(0 < \delta < 1)$, we have

$$T := \sum_{m\in M} r^{|m|}\|b_m\|_q \prod_{j=1}^{\infty}\|\varphi_j\|_\beta^{*m_j} \le \sum_{m\in M} r^{|m|}\|b_m\|_q \prod_{j=1}^{\infty}\left(C_d\lambda_j\right)^{\frac{m_j}{1+d}} \quad \text{(by (4))}$$

$$\le \sum_{m\in M} r^{|m|}(C_d\lambda)^{2m\gamma}\|b_m\|_q = \sum_{m\in M} r^{|m|}(\lambda^m\|b_m\|_q)^\gamma \left(C_d^2\lambda\right)^{m\gamma}\|b_m\|_q^{1-\gamma}$$

$$\le C^{\frac{1-\gamma}{1+\overline{d}}}\sum_{m\in M} r^{|m|}\left(\frac{N(q,t)}{\mu^m t^{|m|}}\right)^\gamma (C_d^2\lambda)^{m\gamma}\|b_m\|_k^{\frac{1-\gamma}{1+\overline{d}}}\|b_m\|_p^{\frac{(1-\gamma)\overline{d}}{1+\overline{d}}} \quad \text{(by (6) and (7))}$$

$$\le C^{\frac{1-\gamma}{1+\overline{d}}}\sum_{m\in M} r^{|m|}\left(\frac{N(q,t)}{\mu^m t^{|m|}}\right)^\gamma (C_d^2\lambda)^{m\gamma}\left(\frac{N(k,t)}{\lambda^m\mu^m t^{|m|}}\right)^{\frac{1-\gamma}{1+\overline{d}}}\left(\frac{1}{\mu^m}\right)^{\frac{(1-\gamma)\overline{d}}{1+\overline{d}}} \quad \text{(by (5))}$$

$$\le C^{\frac{1-\gamma}{1+\overline{d}}} N(q,t)^\gamma N(k,t)^{\frac{1-\gamma}{1+\overline{d}}}\sum_{m\in M}\left(\frac{r}{t^{\gamma+\frac{1-\gamma}{1+\overline{d}}}}\right)^{|m|}\frac{\lambda^{m\left(\gamma-\frac{1-\gamma}{1+\overline{d}}\right)}C_d^{2m\gamma}}{\mu^{m\left(\gamma+\frac{1-\gamma}{1+\overline{d}}+\frac{(1-\gamma)\overline{d}}{1+\overline{d}}\right)}},$$

$$T \le C^{\frac{1-\gamma}{1+d}} N(q,t)^{\gamma} N(k,t)^{\frac{1-\gamma}{1+d}} \sum_{m\in M} \left(\frac{r}{t^{\gamma+\frac{1-\gamma}{1+d}}}\right)^{|m|} \left(\frac{\lambda^{\gamma-\frac{1-\gamma}{1+d}} C_d^{2\gamma}}{\mu}\right)^m .$$

Put $\alpha = \gamma - \frac{1-\gamma}{1+d} > 0$ for $0 < \delta < \frac{1}{2}$. Since $\lambda = (\lambda_j) \in s$, we have $\left(\frac{\lambda_j^\alpha C_d^{2\gamma}}{\mu_j}\right) \in \ell^1$.

Hence, for $R = \sum_j \dfrac{\lambda_j^\alpha C_d^{2\gamma}}{\mu_j}$, we obtain $2R > R \ge \frac{\lambda_j^\alpha C_d^{2\gamma}}{\mu_j}$ $\forall j$. This implies

$$0 < \frac{\lambda_j^\alpha C_d^{2\gamma}}{2R\mu_j} < \frac{1}{2} \quad \forall j.$$

By choosing

$$v = \gamma + \frac{1}{1+d} > 0\,, \quad t = \sqrt[v]{2Rr}$$

we have

$$T \le C^{\frac{1-\gamma}{1+d}} N(q, \sqrt[v]{2Rr})^{\gamma} N(k, \sqrt[v]{2Rr})^{\frac{1-\gamma}{1+d}} \sum_{m\in M} \left(\frac{\lambda^\alpha C_d^{2\gamma}}{2R\mu}\right)^m$$

$$= C^{\frac{1-\gamma}{1+d}} N(q, \sqrt[v]{2Rr})^{\gamma} N(k, \sqrt[v]{2Rr})^{\frac{1-\gamma}{1+d}} \prod_{j=1}^{\infty} \frac{1}{1-\left(\frac{\lambda_j^\alpha C_d^{2\gamma}}{2R\mu_j}\right)} < \infty \ \forall r > 0.$$

This proves that the series $\sum_{m\in M} b_m \prod_{j=1}^{\infty} \varphi_j^{m_j}(x)$ defines a holomorphic function h on E_β such that $f|_{E(B)} = h \circ \omega_{\beta|E(B)}$. Since $E(B)$ is dense in E, we have $f = h \circ \omega_\beta$. ∎

Remark. The proposition for the $(\widetilde{\Omega}$, DN)-case was independently proved by Hai [5].

Proposition 1.2. *Let E be a Frechet–Montel space and F a nuclear Frechet space. Then every holomorphic function on E^* with values in F^* is factorized holomorphically through a Banach space if*

$$F \in (\overline{\overline{\Omega}}) \text{ and } \mathcal{H}(E^*) \in (\underline{\mathrm{DN}}).$$

Proof. Given $f : E^* \longrightarrow F^*$ a holomorphic function where $F \in (\overline{\overline{\Omega}})$ and $\mathcal{H}(E^*) \in (\underline{\mathrm{DN}})$. Consider the continuous linear map $\hat{f} : \mathcal{H}(F^*) \longrightarrow \mathcal{H}(E^*)$ induced by f and given by $\hat{f}(\varphi)(u) = \varphi(f(u))$ for $\varphi \in \mathcal{H}(F^*)$ and $u \in E^*$. Since $F \in (\overline{\overline{\Omega}})$ and is contained in $\mathcal{H}(F^*)$ as a subspace and $\mathcal{H}(E^*) \in (\underline{\mathrm{DN}})$, by Proposition 1.1, we can find a zero neighborhood V in F such that $\hat{f}(V)$ is bounded. Then, for every bounded set B in E^*, we have

$$\sup\{|f(u)(y)| : u \in B,\ y \in V\} = \sup\{|\hat{f}(y)(u)| : u \in B,\ y \in V\} < \infty\,.$$

Thus, $f : E^* \longrightarrow F_V^*$, where F_V is the Banach space associated with V, is bounded and Gateaux holomorphic. Hence, $f : E^* \longrightarrow F_V^*$ is holomorphic. By [1], f is factorized holomorphically through a Banach space. ∎

2. Proof of the Theorem

Given $f : E^* \longrightarrow F^*$, a meromorphic function. By $\mathcal{O}_{E^*}$ (resp. M_{E^*}), we denote the sheaf of germs of holomorphic (resp. meromorphic) functions on E^*.

Let

$$\mathcal{O}^*_{E^*} = \{\sigma \in \mathcal{O}_{E^*} : \sigma \text{ is inverse}\}$$
$$M^*_{E^*} = M_{E^*} \setminus \{0\} \text{ and } D_{E^*} = M^*_{E^*} / \mathcal{O}^*_{E^*}$$

Then we have the two exact sequences on E^*:

$$0 \longrightarrow \mathbf{Z} \longrightarrow \mathcal{O}_{E^*} \overset{\exp}{\longrightarrow} \mathcal{O}^*_{E^*} \longrightarrow 0$$

$$0 \longrightarrow \mathcal{O}^*_{E^*} \longrightarrow M_{E^*} \overset{\eta}{\longrightarrow} D_{E^*} \longrightarrow 0$$

where $\exp(\sigma) = e^{2\pi i \sigma}$ and η is the canonical map. By [2], $H^1(E^*, \mathcal{O}_{E^*}) = 0$. On the other hand, since $H^2(E^*, \mathbf{Z}) = 0$, the exact cohomology sequences associated to the above exact sheaf sequences give that, for every divisor $d \in H^0(E^*, D_{E^*})$, there exists a meromorphic function $\tau \in H^0(E^*, M^*_{E^*})$ such that $\eta(\tau) = d$.

By the meromorphicity of f, for every $z \in E^*$, we can choose a neighborhood V_1 of z and holomorphic functions $h : V_1 \longrightarrow F^*$, $\sigma : V_1 \longrightarrow \mathbf{C}$, $\sigma \neq 0$ such that $f|_{V_1} = \frac{h}{\sigma}$. Write $\sigma = \sigma_1^{m_1} \sigma_2^{m_2} ... \sigma_p^{m_p}$ in a neighborhood V_2 of z in V_1 such that the germs $\sigma_{1z}, \sigma_{2z}, ..., \sigma_{pz}$ at z are irreducible [6]. Without loss of generality, we may assume h_z cannot be divisible by $\sigma_{1z}, \sigma_{2z}, ..., \sigma_{pz}$. This yields a neighborhood U of z in V_2 such that $f|_U = \frac{h}{\sigma}$ and $\operatorname{codim} Z(h, \sigma) \geq 2$ in U (where $Z(h, \sigma) = h^{-1}(0) \cap \sigma^{-1}(0)$). Thus, we can find an open cover $\{U_j\}$ of E^* and holomorphic functions $h_j : U_j \longrightarrow F^*$, $\sigma_j : U_j \longrightarrow \mathbf{C}$ such that $f|_{U_j} = \frac{h_j}{\sigma_j}$ and $\operatorname{codim} Z(h_j, \sigma_j) \geq 2$ for $j \geq 1$.

Now, we need the following lemma.

Lemma. *Let β and σ be holomorphic functions on an open set D in a locally convex space, and g a holomorphic function with values in a locally convex space. Assume $\frac{\beta g}{\sigma}$ is holomorphic on D and* $\operatorname{codim} Z(g, \sigma) \geq 2$. *Then $\frac{\beta}{\sigma}$ is holomorphic on D.*

Proof. Given $z_0 \in D$. Since the local ring $\mathcal{O}_{z_0}$ of germs of holomorphic functions at z_0 is factorial [6], we can write $\sigma = \sigma_1^{m_1} \sigma_2^{m_2} ... \sigma_p^{m_p}$ in a neighborhood U of z_0 such that $\sigma_{1z_0}, \sigma_{2z_0}, ..., \sigma_{pz_0}$ are irreducible. By the hypothesis and the equality $\frac{\beta g}{\sigma_1} = \frac{\beta g}{\sigma} \sigma_1^{m_1 - 1} ... \sigma_p^{m_p}$, it follows that $\frac{\beta g}{\sigma_1}$ is holomorphic at z_0. On the other hand, from the hypothesis $\operatorname{codim} Z(g, \sigma) \geq 2$ and $Z(\sigma) = \bigcup_{i=1}^{p} Z(\sigma_i)$, it follows that $\operatorname{codim} Z(g, \sigma_i) \geq 2$ for $i = 1, ..., p$. Hence, by the irreducibility of σ_{1z_0}, we infer that $Z(\sigma_1)_{z_0} \subseteq Z(\beta)_{z_0}$. This again implies $\beta = \beta_1 \sigma_1$ at z_0.

Continuing the process we infer that $\frac{\beta}{\sigma}$ is holomorphic at z_0. ■

We continue the proof of the theorem.

Since $\frac{h_i}{\sigma_i} = \frac{h_j}{\sigma_j}$ on $U_i \cap U_j$ for all $i, j \geq 1$, the above lemma implies that the form $z \mapsto (\sigma_j)_z \mathcal{O}^*_{E^*, z}$ for $z \in U_j$ defines a divisor d on E^*. Thus, there exists a meromorphic function β on E^* such that $\beta \neq 0$ and $\frac{\beta_z}{d_z} \in \mathcal{O}^*_{E^*, z}$ for $z \in E^*$. These relations imply that β is holomorphic on E^* and hence $h = \beta f$ is holomorphic on E^*. From Proposition 1.2, we infer that h, β are of uniform type, and hence, so is f. The theorem is proved. ■

3. Examples

Proposition 3.1. *Let E be a nuclear Frechet space and F a Banach space. Then $\mathcal{H}_b(F\hat{\otimes}_\pi E^*) \in$ (DN) if $E \in$ (DN) and E has a Schauder basis. Here, $\mathcal{H}_b(F\hat{\otimes}_\pi E^*)$ denotes the Frechet space of holomorphic function on $F\hat{\otimes}_\pi E^*$ which are bounded on every bounded set in $F\hat{\otimes}_\pi E^*$.*

Proof. Let $\{e_j\}$ be a Schauder basis of E and $\{e_j^*\}$ the dual basis of E^*. Write the Taylor expansion of each $f \in \mathcal{H}_b(F\hat{\otimes}_\pi E^*)$ at $0 \in F\hat{\otimes}_\pi E^*$ as

$$f(\omega) = \sum_{n\geq 0} P_n f(\omega), \text{ where } P_n f(\omega) = \frac{1}{2\pi i}\int_{|\lambda|=r>0} \frac{f(\lambda\omega)}{\lambda^{n+1}} d\lambda.$$

Given $p \geq 1$. Choose $q \geq p$ such that

$$M = \sum_{j\geq 1} \|e_j^*\|_q^* \, \|e_j\|_p < \infty.$$

With $B = \{u \in F : \|u\| \leq 1\}$, it follows that

$$\begin{aligned}
&\sup\Big\{ \sum_{j_1, j_2, \ldots, j_n \geq 1} p^n |\hat{P}_n f(u_1 \otimes e_{j_1}^*, \ldots, u_n \otimes e_{j_n}^*)| \, \|e_{j_1}\|_p \ldots \|e_{j_n}\|_p : \\
&\qquad\qquad\qquad u_1, \ldots, u_n \in B,\ n \geq 0\Big\} \\
&= \sup\Big\{ \sum_{j_1, j_2, \ldots, j_n \geq 1} p^n \left| \hat{P}_n f\left(u_1 \otimes \frac{e_{j_1}^*}{\|e_{j_1}^*\|_q^*}, \ldots, u_n \otimes \frac{e_{j_n}^*}{\|e_{j_n}^*\|_q^*}\right)\right| \\
&\qquad\qquad \times \|e_{j_1}^*\|_q^* \, \|e_{j_1}\|_p \ldots \|e_{j_n}^*\|_q^* \, \|e_{j_n}\|_p \ : \ u_1, \ldots, u_n \in B,\ n \geq 0\Big\} \\
&\leq \|f\|_{\mathrm{convr}(B\otimes V_q^0)} \sup\left\{\frac{n^n p^n}{n!\, r^n} M^n : n \geq 0\right\} = C(p, M, r)\, \|f\|_{\mathrm{convr}(B\otimes V_q^0)},
\end{aligned}$$

where

$$C(p, M, r) = \sup\left\{\frac{n^n p^n}{n!\, r^n} : n \geq 0\right\} < \infty \text{ for } r > 0 \text{ sufficiently large.}$$

Thus, for each $p \geq 1$, the formula

$$\begin{aligned}
|||f|||_p = \sup\Big\{ p^n \sum_{j_1, \ldots, J_n \geq 1} |\hat{P}_n f(u_1 \otimes e_{j_1}^*, \ldots, u_n \otimes e_{j_n}^*)| \, \|e_{j_1}\|_p \ldots \|e_{j_n}\|_p : \\
u_1, \ldots, u_n \in B,\ n \geq 0\Big\}
\end{aligned}$$

defines a continuous seminorm $|||\cdot|||_p$ on $\mathcal{H}_b(F\hat{\otimes}_\pi E^*)$.

On the other hand, since

$$\|f\|_{\mathrm{convr}(B\otimes V_q^0)} = \sup\left\{\left|f\left(p\sum_{k\geq 1}\lambda_k u_k\otimes v_k\right)\right| : u_k\in B,\ v_k\in V_p^0,\ \sum_{k\geq 1}|\lambda_k|\leq 1\right\}$$

$$\leq \sup\left\{\sum_{n\geq 0}p^n \sum_{k_1,\ldots,k_n\geq 1}|\lambda_{k_1}|\ldots|\lambda_{k_n}| \sum_{j_1,\ldots,j_n\geq 1}|\hat{P}_n f(u_{k_1}\otimes e_{j_1}^*,\ldots,u_{k_n}\otimes e_{j_n}^*)| \times \|e_{j_1}\|_p\ldots\|e_{j_n}\|_p \mid u_k\in B\right\}$$

$$\leq \sum_{n\geq 0}\left(\frac{p}{q}\right)^n \sup\left\{\sum_{k_1,\ldots,k_n\geq 1}|\lambda_{k_1}|\ldots|\lambda_{k_n}| \times \sup\left\{\sum_{j_1,\ldots,j_n\geq 1} q^n|\hat{P}_n f(u_{k_1}\otimes e_{j_1}^*,\ldots,u_{k_n}\otimes e_{j_n}^*)| \times \|e_{j_1}\|_q\ldots\|e_{j_n}\|_q : u_1,\ldots,u_n\in B\right\} \mid \sum_{k\geq 1}|\lambda_k|\leq 1\right\}$$

$$\leq |||f|||_q \sum_{n\geq 0}\left(\frac{p}{q}\right)^n$$

for $f\in\mathcal{H}_b(F\hat{\otimes}_\pi E^*)$, we infer that the topology of $\mathcal{H}_b(F\hat{\otimes}_\pi E^*)$ can be defined by $\{|||\cdot|||_p\}$.

Choose $p\geq 1$ such that

$$\exists p\,\forall q\,\exists k, d, C>0 : \|e_j\|_q^{1+d}\leq \|e_j\|_k\,\|e_j\|_p^d,\quad \forall j\geq 1$$

and $q^{1+d}\leq kp^d$. We have

$$|||f|||_p^{1+d} = \sup\left\{p^n\sum_{j_1,\ldots,j_n\geq 1}|\hat{P}_n f(u_1\otimes e_{j_1}^*,\ldots,u_n\otimes e_{j_n}^*)|\,\|e_{j_1}\|_q\ldots\|e_{j_n}\|_q : u_1,\ldots,u_n\in B,\ n\geq 0\right\}^{1+d}$$

$$\leq \sup\left\{k^n p^{nd}\sum_{j_1,\ldots,j_n\geq 1}|\hat{P}_n f(u_1\otimes e_{j_1}^*,\ldots,u_n\otimes e_{j_n}^*)|\,\|e_{j_1}\|_k^{\frac{1}{1+d}}\ldots\|e_{j_n}\|_k^{\frac{1}{1+d}} \times \|e_{j_1}\|_p^{\frac{d}{1+d}}\ldots\|e_{j_n}\|_p^{\frac{d}{1+d}} \mid u_1,\ldots,u_n\in B,\ n\geq 0\right\}$$

$$\leq \sup\left\{k^n\sum_{j_1,\ldots,j_n\geq 1}|\hat{P}_n f(u_1\otimes e_{j_1}^*,\ldots,u_n\otimes e_{j_n}^*)|\,\|e_{j_1}\|_k\ldots\|e_{j_n}\|_k : u_1,\ldots,u_n\in B,\ n\geq 0\right\} \times \sup\left\{p^n\sum_{j_1,\ldots,j_n\geq 1}|\hat{P}_n f(u_1\otimes e_{j_1}^*,\ldots,u_n\otimes e_{j_n}^*)|\,\|e_{j_1}\|_p\ldots\|e_{j_n}\|_p : u_1,\ldots,u_n\in B,\ n\geq 0\right\}^d$$

$$= |||f|||_k\,|||f|||_p^d \quad \text{for all } f\in\mathcal{H}_b(F\hat{\otimes}_\pi E^*).$$

Hence, $\mathcal{H}_b(F\hat{\otimes}_\pi E^*) \in \underline{(DN)}$. ■

Let $\Lambda(A)$ be a nuclear Frechet–Köther space and B a Banach space with the unit ball V. For each $a = (a_j) \in \Lambda(A)$, $a > 0$ (i.e., $a_j \geq 0$ for all $j \geq 1$), we define an open polydisc $\mathbf{D}_a^V$ in $B\hat{\otimes}_\pi \Lambda'(A)$ by

$$\mathbf{D}_a^V = \Big\{\omega = \sum_{j\geq 1} x_j \otimes \xi_j e_j^* : (x_j) \subset V,\ (\xi_j) \in \mathbf{D}_a\Big\},$$

where $\mathbf{D}_a = \big\{(\xi_j) \in \Lambda'(A) : \sup_j |\xi_j| a_j < 1\big\}$.

By $\mathcal{H}_b(\mathbf{D}_a^V)$, we denote the Frechet space of holomorphic functions on $\mathbf{D}_a^V$ which are bounded on the subsets $W(r, K)$ of $\mathbf{D}_a^V$ given by

$$W(r,K) = \Big\{\omega = \sum_{j\geq 1} x_j \otimes \xi_j e_j^* : \sup_{j\geq 1} \|x_j\| \leq r,\ (\xi_j) \in K\Big\},$$

where $K \subset\subset \mathbf{D}_a$ and $0 < r < 1$.

Proposition 3.2. *If $\Lambda(A)$ has $\underline{(DN)}$, then $\mathcal{H}_b(\mathbf{D})_a^V$ has also $\underline{(DN)}$.*

Proof. It is known in [9] that $\mathcal{H}(\mathbf{D}_a) \in \underline{(DN)}$. Thus, we can find an increasing exhaustion sequence of compact sets (K_q) in $\mathbf{D}_a$ such that $\underline{(DN)}$ holds on $\mathcal{H}(\mathbf{D}_a)$ for the system of sup-seminorms on K_q. It is easy to check that the topology of $\mathcal{H}_b(\mathbf{D}_a^V)$ can be defined by the system of semi-norms $\big\{|||\cdot|||_{(r,q)}\big\}_{0<r<1, q\geq 1}$ with

$$|||f|||_{(r,q)} = \sup\Big\{r^n\Big|P_n f\Big(\sum_{i\geq 1} x_i \otimes \xi_i e_i^*\Big)\Big| : (x_i) \subset V,\ (\xi_i) \in K_q,\ n \geq 0\Big\}.$$

Given $q \geq 1$. Choose $k, d, C > 0$ such that $\underline{(DN)}$ holds on $\mathcal{H}(\mathbf{D}_a)$. We have

$$\begin{aligned}
|||f|||_{(r,q)}^{1+d} &= \sup\Big\{r^{n(1+d)}\Big|P_n f\Big(\sum_{i\geq 1} x_i \otimes \xi_i e_i^*\Big)\Big|^{1+d} : (x_i) \subset V,\ (\xi_i) \in K_q,\ n \geq 0\Big\} \\
&= \sup\Big\{r^{n(1+d)} \sup\Big\{\Big|P_n f\Big(\sum_{i\geq 1} x_i \otimes \xi_i e_i^*\Big)\Big|^{1+d} : (\xi_i) \in K_q\Big\},\ (x_i) \subset V,\ n \geq 0\Big\} \\
&\leq \sup\Big\{r^{n(1+d)} C\Big\|P_n f\Big(\sum_{i\geq 1} x_i \otimes \cdot\Big)\Big\|_k \Big\|P_n f\Big(\sum_{i\geq 1} x_i \otimes \cdot\Big)\Big\|_p^d : (x_i) \subset V,\ n \geq 0\Big\} \\
&\leq C|||f|||_{(r,q)}\, |||f|||_{(r,q)}^d \quad \text{for all } f \in \mathcal{H}_b(\mathbf{D}_a^V).
\end{aligned}$$

Consequently, $\mathcal{H}_b(\mathbf{D}_a^V) \in \underline{(DN)}$. ■

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