Vietnam Journal of Mathematics **26**:3 (1998) 277–282

Vietnam Journal
of
MATHEMATICS

Short Communication

On the Uniqueness of Global Semiclassical Solutions of the Cauchy Problem for Weakly Coupled Systems

Le Van Hap

Hanoi Institute of Mathematics,
P.O. Box 631, Bo Ho, 10.000 Hanoi, Vietnam

Received June 15, 1997

In this note, we consider the differential inequality of the form

$$|u_t| \leq k(t)(1+|x|)|u_x| + f(t, |u|), \tag{1}$$

where $k(\cdot) \in L^1(0,T)$, $T > 0$ and $f(t,|u|)$ is the right side of a *comparison equation* satisfying the Carathéodory condition. Using the method based on the differential comparison equations and the theory of multifunctions, we integrate the differential inequality (1) and apply it to derive uniqueness results for global semiclassical solutions of the Cauchy problem:

$$\frac{\partial u_j}{\partial t} + H_j(t, x, u, \nabla_x u_j) = 0, \tag{2}$$

$$u_j(0,x) = u_j^0(x), \quad j = 1, \dots, m, \tag{3}$$

where $(t,x) \in \Omega_T = (0,T) \times \mathbf{R}^n$; $n, m \in \mathbf{N}$; H_j, $j = 1, ..., m$ are functions of $(t, x, p, q_j) \in \Omega_T \times \mathbf{R}^m \times \mathbf{R}^n$. Vectors $p = (p_1, \dots, p_m)$ and $q_j = (q_j^1, \dots, q_j^n)$ are corresponding to $u = (u_1, \dots, u_m)$ and $\nabla_x u_j = (\frac{\partial u_j}{\partial x_1}, \dots, \frac{\partial u_j}{\partial x_n})$, respectively. Systems of the form (2) are called *weakly coupled systems*.

1. Differential Equations of Comparison Type

In this section, we will give some theorems which generalize to the Carathéodory case of Lemma 14.2 and the second Comparison Theorem in [4]. They will be used to prove some results in Sec. 2. For this aim, we consider the Cauchy problem on domain $D \subset \mathbf{R}^2$

$$y' = f(t, y), \tag{1.1}$$

$$y(t_0) = y_0, \ \ (t_0, y_0) \in D. \tag{1.2}$$

We recall the definition of the Carathéodory equation (see [3]).

Definition 1.1. *Equation (1.1) is said to be the Carathéodory equation provided:*

(a) *$f(t, \cdot)$ is a continuous function in y for almost all t;*
(b) *$f(\cdot, y)$ is measurable in t for all y;*
(c) *there exists an integrable function (in Lebesgue's sense) $m = m(t)$ such that*

$$|f(t, y)| \leq m(t), \ \ \forall (t, y) \in D.$$

Here and in what follows, a function defined on an interval $I \subset \mathbf{R}$ and absolutely continuous (a.c.) on every closed subinterval of I is said to be an *absolutely continuous function on I.*

By a solution of (1.1), we mean a function $y = y(t)$ which is a.c. on interval (α, β) and satisfies (1.1) for almost all $t \in (\alpha, \beta)$. A solution $m = m(t)$ of (1.1) and (1.2) defined on (α, β) is said to be a minimal solution of the problem provided, for every solution $y = y(t)$ of (1.1) and (1.2) defined on (α', β'), we have

$$m(t) \leq y(t), \quad \forall t \in (\alpha.\beta) \cap (\alpha', \beta').$$

Theorem 1.2. *Let $\Omega_+ = (0, T) \times \mathbf{R}_+ = \{(t, y) : t \in (0, T), y \geq 0\}$ and $(t_0, y_0) \in \Omega_+$. Consider the problem (1.1)–(1.2) on the domain Ω_+, where the function f satisfies the following conditions:*

$$f(t, y) \geq 0, \quad \forall (t, y) \in \Omega_+ \ \text{and} \ f(t, 0) = 0, \ \forall t \in (0, T).$$

Then there exists a minimal solution $m = m(t, t_0, y_0)$ of the problem (1.1)–(1.2) defined on $(0, t_0]$. In particular, $m(t, t_0, 0) = 0$.

The proof of Theorem 1.2 is similar to the proof of Lemma 14.2 in [4]. For more details, refer to [4].

Definition 1.3. *A differential equation $y' = f(t, y)$ defined on $\Omega_+ = (0, T) \times \mathbf{R}_+$ is called a* comparison equation *if the following conditions are satisfied:*

(a) *$y' = f(t, y)$ is the Carathéodory equation;*
(b) *$f(t, y) \geq 0$ for all $(t, y) \in \Omega_+$ and $f(t, 0) = 0$ for all $t \in (0, T)$;*
(c) *$y = y(t) = 0$ in every subinterval $(0, \gamma) \subset (0, T)$ is the unique solution of (1.1) which satisfies $\lim_{t \to 0^+} y(t) = 0$.*

Example 1.4. Let $k(\cdot)$ be a nonnegative integrable function on $(0, T)$, then the following functions:

(a) $f(t, y) = k(t) \cdot \frac{y}{y+1}, \quad y \geq 0 \ t \in (0, T);$

(b) $f(t, y) = k(t) \cdot \frac{2y}{y^2+1}, \quad y \geq 0 \ t \in (0, T),$

are right sides of comparison equations.

Consider the Cauchy problem (1.1)–(1.2) on Ω_+. Suppose (1.1) is a comparison equation and φ is an a.c. function on $[0, T)$. Let G be a set of zero measure such that $\varphi(\cdot)$ is differentiable at every point of $[0, T) \setminus G$ and $E := \{t \in (0, T) \setminus G : \varphi(t) > 0\}$.

Theorem 1.5. *If the following conditions*

$$\varphi(0) \leq 0 \ \ and \ \ \varphi'(t) \leq f(t, \varphi(t))$$

hold for all $t \in E$, then $\varphi(t) \leq 0$ for all $t \in [0, T)$.

The proof is similar to the proof of the second Comparison Theorem in [4]. The details are left to the reader.

2. Uniqueness of Global Semiclassical Solutions of the Cauchy Problem for Weakly Coupled Differential Equation Systems of First Order

Let us denote by $|\cdot|$, $\langle\cdot\rangle$ the Euclid norm and the corresponding scalar product in $\mathbf{R}^n$, respectively. Further, let

$$\begin{aligned}
\mathrm{Lip}\,(\Omega_T) &:= \{u : \Omega_T \to \mathbf{R} : u \text{ is locally Lipschitz continuous}\},\\
\mathrm{Lip}\,([0, T) \times \mathbf{R}^n) &:= \mathrm{Lip}\,(\Omega_T) \cap \mathrm{C}\,([0, T) \times \mathbf{R}^n),\\
\mathrm{V}\,(\Omega_T) &:= \big\{u \in \mathrm{Lip}\,([0, T) \times \mathbf{R}^n) : u \text{ is differentiable for}\\
&\qquad \text{all } x \in \mathbf{R}^n \text{ and for almost everywhere } t \in (0, T)\big\}\\
\mathrm{V}_m\big(\Omega_T\big) &:= \underbrace{\mathrm{V}\,\big(\Omega_T\big) \times \cdots \times \mathrm{V}\,\big(\Omega_T\big)}_{m \text{ times}}.
\end{aligned}$$

Theorem 2.1. *Let $u = (u_1, u_2, \ldots, u_m) \in V_m(\Omega_T)$. Suppose there exist nonnegative functions $k = k(t) \in L^1(0, T)$ and $f = f(t, y)$ defined on $(0, T) \times \mathbf{R}_+$, which is the right side of a comparison equation such that the following conditions hold*

$$u(0, x) = 0, \ \forall x \in \mathbf{R}^n, \tag{2.1}$$

$$\Big|\frac{\partial u_i}{\partial t}\Big| \leq k(t)(1 + |x|)|\nabla_x u_i| + f(t, \max_{1 \leq j \leq m} |u_j|), \ i = 1, \ldots, m, \tag{2.2}$$

for all $x \in \mathbf{R}^n$ and for almost all $t \in (0, T)$. Then $u(t, x) \equiv 0$ in Ω_T.

Proof. Let $(t_0, x_0) \in \Omega_T$. It suffices to prove that $u(t_0, x_0) = 0$. Consider the multifunction $F : \Omega_T \rightsquigarrow \mathbf{R}^n$ given by $F(t, x) = \overline{B}_{mk(t)(1+|x|)}$, $(t, x) \in \Omega_T$, where $\overline{B}_r$ is a closed ball with the center at origin and radius r. By Theorem VI–13 in [2], it follows that the set of a.c. solutions defined on $I = [0, t_0]$, $\Sigma_I(t_0, x_0)$, of the differential inclusion $\dfrac{dx(t)}{dt} \in F(t, x(t))$ subject to the constraint $x(t_0) = x_0$, is a nonempty compact set in $C(I, \mathbf{R}^n)$.

Let us define a function $\varphi : I \to \mathbf{R}$ as follow:

$$\varphi(t) := \max\Big\{ \max_{1\le j\le m} |u_j(t, x(t))| : x(\cdot) \in \Sigma_I(t_0, x_0)\Big\}, \quad t \in [0, t_0]. \tag{2.3}$$

To prove $u(t_0, x_0) = 0$, it is sufficient to prove that $\varphi(t_0) = 0$.

By hypothesis of Theorem 2.1 and Lemma 3 in [5], there exists a set $G \subset (0, T)$ of zero measure, such that

- $g = g(t) := \int_0^t k(\tau)d\tau$ is differentiable at every point (a.e.p.) in $(0, t_0) \setminus G$;
- $u(\cdot, \cdot)$ is differentiable a.e.p. in $\big((0, t_0) \setminus G\big) \times \mathbf{R}^n$,
- φ is differentiable a.e.p. in $(0, t_0) \setminus G$.

Denote $E = \{t \in (0, t_0) \setminus G : \varphi(t) > 0\}$ and take $t_1 \in E$ (for the case $E = \emptyset$, immediately $\varphi(t_0) = 0$). By (2.3), there exist $j \in \{1, 2, \ldots, m\}$ and $x_1(\cdot) \in \Sigma_I(t_0, x_0)$ such that $\varphi(t_1) = |u_j(t_1, x_1(t_1))|$. Without restriction of generality, we can assume

$$\varphi(t_1) = |u_1(t_1, x_1(t_1))|. \tag{2.4}$$

We write, for short, $x^1 = x_1(t_1)$. If

$$\varphi(t_1) = u_1(t_1, x_1(t_1)) = u_1(t_1, x^1), \tag{2.5}$$

then we choose $e \in \mathbf{R}^n$ such that $|e| = 1$ and

$$\big\langle \nabla_x u_1(t_1, x^1), e\big\rangle = -|\nabla_x u_1(t_1, x^1)|. \tag{2.6}$$

Denote by $y = y(p)$ a solution continuously differentiable on $\mathbf{R}$ of the system $\dfrac{dy(p)}{dp} = (1 + |y(p)|)e$. Subject to $y(g(t_1)) = x^1$ and put

$$x_2 = x_2(t) = y(g(t)) \quad t \in [0, \mathrm{T}], \tag{2.7}$$

we have x_2 being differentiable at $t = t_1$ and the function $\tilde{x}$ given by

$$\tilde{x}(t) = \begin{cases} x_2(t) & \text{for } 0 \le t \le t_1, \\ x_1(t) & \text{for } t_1 \le t \le t_0, \end{cases}$$

belongs to $\Sigma_I(t_0, x_0)$. Moreover, $\tilde{x}(t_1) = x^1$ and $\tilde{x}(t) = x_2(t)$, $\forall t \in [0, t_1]$.

By (2.5) and the continuity of $u_1(\cdot, x_2(\cdot))$, for $\delta < 0$ small enough, we have

$$\frac{\varphi(t_1 + \delta) - \varphi(t_1)}{\delta} \le \frac{u_1(t_1 + \delta, x_2(t_1 + \delta)) - u_1(t_1, x_2(t_1))}{\delta}.$$

Letting $\delta \to 0^-$ and applying (2.6)–(2.7) yield

$$\varphi'(t_1) \le |\frac{\partial}{\partial t}u_1(t_1, x^1)| - k(t_1)\big(1 + |x^1|\big)|\nabla_x u_1(t_1, x^1)|.$$

By (2.2), we obtain $\varphi'(t_1) \le f(t_1, \max_{1\le j\le m} |u_j(t_1, x^1)|)$. From (2.4),

$$\varphi'(t_1) \le f(t, \varphi(t_1)). \tag{2.8}$$

If $\varphi(t_1) = -u_1(t_1, x^1)$, similarly to the case above, we also have (2.8). Since t_1 is arbitrary in E, we have $\varphi'(t) \le f(t, \varphi(t))$, $\forall t \in E$. Hence, by $\varphi(0) = 0$ and by Theorem 1.5, we deduce $\varphi(t) \le 0$, $\forall t \in [0, t_0]$, and consequently, $\varphi(t_0) = 0$. By the definition of φ, we conclude $u(t_0, x_0) = 0$. Since (t_0, x_0) is an arbitrary point in Ω_T, $u(t, x) \equiv 0$ in Ω_T. ■

We now apply Theorem 2.1 to prove a uniqueness criterion for global semiclassical solutions of the Cauchy problem (2)–(3), which is the main result of this paper.

First, we recall the definition of global semiclassical solutions for the problem (see [5]).

Definition 2.2. *A vector function $u \in V_m(\Omega_T)$ is called a* global semiclassical *solution of (2)–(3) if u satisfies the condition (3) for all $x \in \mathbf{R}^n$ and u satisfies the system (2) for all $x \in \mathbf{R}^n$ and for almost all $t \in (0, T)$.*

Theorem 2.3. *Suppose $H_j = H_j(t, x, p, q_j)$, $j = 1, \dots, m$, satisfies the following condition: There exist nonnegative functions $k(t) \in L^1(0, T)$ and $f(t, y)$ defined on $(0, T) \times \mathbf{R}_+$ which is the right side of a comparison equation such that the following inequality holds for all $x \in \mathbf{R}^n$ and for almost all $t \in (0, T)$:*

$$\begin{aligned}&|H_j(t, x, p, q_j) - H_j(t, x, p', q_j')|\\ &\le k(t)(1 + |x|)|q_j - q_j'| + f(t, \max_{1\le i\le m} |p_j - p_j'|),\ \ j = 1, \dots, m,\end{aligned}$$

where $p = (p_1, \dots, p_m)$, $p' = (p_1', \dots, p_m'), \in R^m$, $q_j = (q_j^1, \dots, q_j^n)$, $q_j' = (q'^1_j, \dots, q'^n_j) \in \mathbf{R}^n$. If $u = (u_1, \dots, u_m)$ and $v = (v_1, \dots, v_m)$ are global semiclassical solutions of (2)–(3), then $u \equiv v$ on Ω_T.

Proof. Put $z = u - v = (z_1, \dots, z_m)$. By hypothesis of Theorems 2.1 and 2.3, we deduce $z(t, x) = 0$ for all $(t, x) \in \Omega_T$.

For other recent results on the inequalities of type (1) and their applications, we refer to [1] and to the bibliography quoted there.

Acknowledgement. We would like to thank Professor Dr. Sc. Tran Duc Van and Dr. Nguyen Duy Thai Son for many useful discussions.

References

1. P. Brandi and C. Marcelli, Haar inequality in hereditary setting and applications, *Rend. Sem. Mat. Univ. Padova* **96** (1996) 177–194.

2. Ch. Castaing and M. Valadier, *Convex Analysis and Measurable Multifunctions*, Lecture Notes in Mathematics, Vol. 580, Springer-Verlag, New York, 1977.
3. A. F. Filippov, *Differential Equations with Discontinuous Righthand Sides*, Kluwer Academic Publishers, 1988.
4. J. Szarski, *Differential Inequalities*, Monografie Matematyczne (2nd ed.), Warszawa, 1967.
5. T. D. Van and L. V. Hap, Uniqueness of global semiclassical solutions for some systems of first-order nonlinear partial differential equations, Hanoi Institute of Mathematics, preprint 96/15.
6. T. D. Van and N. D. Thai Son, On a class of Lipschitz continuous functions of several variables, *Proc. Amer. Math. Soc.* **121** (1994) 865–870.