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Short Communication

Completeness of a Space of Germs of Holomorphic Functions

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1. Introduction

Let K be a compact set in a Frechet space E and X a Banach space. By $H(K, X)$, we denote the space of germs of X-valued holomorphic functions on K equipped with the inductive topology. This means that

$$H(K, X) = \lim_{U \supset K} \operatorname{ind}(H^{\infty}(U, X)),$$

where U ranges over all neighborhoods of K and $H^{\infty}(U, X)$ is the Banach space of bounded holomorphic functions on U.

The completeness of $H(K, X)$ was investigated by some authors. First, Dineen [2] proved that $H(K) = H(K, C)$ is complete, where C denotes the complex plane. Later, Bonet, Domanski and Mujica [1] showed that $H(K, X)$ is complete if either X is complemented in its bi-dual or E is quasi-normable. This result can be obtained from the proof of Dineen's result. In the present note, we shall prove the following:

Theorem. *Let K be a balanced compact set in a Frechet space E and X a Banach space. Then $H(K, X)$ is complete.*

2. Proof of Theorem

Choose an index set I such that $X \subset \ell^{\infty}(I)$. Consider the commutative diagram

$$\begin{array}{ccc} H(K,X) & \xrightarrow{\;S\;} & [H(K,X)]'' \\ {\scriptstyle R}\downarrow & & \downarrow{\scriptstyle R''} \\ H(K,\ell^\infty(I)) & \xrightarrow{\;\tilde{S}\;} & [H(K,\ell^\infty(I))]'' \end{array}$$

where R, S and $\tilde{S}$ are canonical maps.

Note that R is injective and S, $\tilde{S}$ are imbeddings because $H(K, X)$ and $H^\infty(K, \ell^\infty(I))$ are barrelled spaces.

(i) We first check that $[H(K, X)]'$ is dense in $[H(U, X)]'$ for every balanced neighborhood U of K. It suffices to show this for $[H^\infty(V, X)]'$, where V is a balanced neighborhood of K satisfying $\delta^{-1}V \subset U$ with $0 < \delta < 1$. Let $\mu \in [H^\infty(U, X)]'$ and $\epsilon > 0$. Write the Taylor expansion of every $f \in H^\infty(U, X)$

$$f(z) = \sum_{n\geq 0} P_n f(z),$$

where

$$P_n f(z) = \frac{1}{2\pi i}\int_{|\lambda|=1} \frac{f(\lambda z)}{\lambda^{n+1}}\, d\lambda.$$

Since $\delta^{-1}V \subset U$ with $0 < \delta < 1$, it follows that $\sum\limits_{n\geq 0} \|P_n f\|_V < +\infty$. Take m such that

$$\sum_{n\geq m} \|P_n f\|_V < \epsilon \quad \text{for } f \in H^\infty(U, X) \text{ with } \|f\|_U < 1.$$

Put $\mu_m(f) = \sum\limits_{0\leq n\leq m} \mu(P_n f)$. Then $\mu_m \in [H^\infty(V, X)]'$ because $[P(^nE, X); \|.\|_V] \cong [P(^nE, X); \|.\|_U]$ for $n \geq 0$, where $P(^nE, X)$ stands for the space of continuous n-homogeneous polynomials from E into X and

$$|(\mu - \mu_m)(f)| \leq \sum_{n>m} \|P_n f\|_V < \epsilon \quad \text{for } f \in H^\infty(U, X) \text{ with } \|f\|_U < 1.$$

(ii) We now show that the map R'' is injective. Since $H^\infty(U, X)$ is imbedded in $H^\infty(U, \ell^\infty(I))$ for every neighborhood U of K in E, from (i) and the relations

$$[H(K, X)]' \cong \operatorname{lim\,proj}\, [H^\infty(U, X)]'$$

$$[H(K, \ell^\infty(I))]' \cong \operatorname{lim\,proj} [H^\infty(U, \ell^\infty(I))]',$$

we infer that $R' : [H(K, \ell^\infty(I))]' \to [H^\infty(K, X)]'$ has the dense range. Hence, $R'' : [H^\infty(K, X)]'' \to [H(K, \ell^\infty(I))]''$ is injective.

Note that $[H(K, X)]''$ and $[H(K, \ell^\infty(I))]$ are complete because $[H(K, X)]''$ is the dual of the Frechet space $[H(K, X)]'$ and $\ell^\infty(I)$ is complemented in its bi-dual.

(iii) Now, let $\{f_\alpha\}_\alpha$ be a Cauchy net in $H(K, X)$. Since $[H(K, X)]''$ and $H(K, \ell^\infty(I))$ are complete, we have

$$Sf_\alpha \to \mu \text{ and } Rf_\alpha \to g$$

in $[H(K, X)]''$ and $H(K, \ell^\infty(I))$, respectively.

Let $\hat{\pi} : H(K, \ell^{\infty}(I)) \to H(K, \ell^{\infty}(I)/X)$ be the map induced by the canonical projection $\pi : \ell^{\infty}(I) \to \ell^{\infty}(I)/X$. Since $f_\alpha \in H(K, X)$, there exists a neighbourhood W containing K and a holomorphic function $\tilde{f}_\alpha \in H^{\infty}(W, X)$ such that $\tilde{f}_\alpha$ defines the germ f_α on $H(K, X)$. Because $\tilde{f}_\alpha(x) \in X$ for every $x \in W$ and $\tilde{f}_\alpha \in H^{\infty}(W, X)$ with germ f_α, we have $\hat{\pi} R \tilde{f}_\alpha(x) = 0$. Hence,

$$\hat{\pi} g = \lim_{\alpha} \hat{\pi} R f_\alpha = 0.$$

Thus, we can find $f \in H(K, X)$ for which $Rf = g$. From the relations

$$R''\mu = \lim_{\alpha} R'' S f_\alpha = \lim_{\alpha} \tilde{S} R f_\alpha = \tilde{S} g = \tilde{S} R f = R'' S f$$

and the injectivity of R'', we obtain $Sf = \mu$. Hence, $f_\alpha \to f$ in $H(K, X)$, so $H(K, X)$ is complete. ∎

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References

1. J. Bonet, P. Domanski, and J. Mujica, Complete spaces of vector-valued holomorphic germs, *Math. Scand.* **75** (1994) 150–160.
2. S. Dineen, Holomorphic germs on compact subsets of locally convex spaces, in: *Functional Analysis, Holomorphic and Approximation Theory*, Lecture Notes in Mathematics, Vol. 843, S. Machado (ed.), Springer-Verlag, (1981) pp. 247–263.