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Frechet-valued Holomorphic Functions on Compact Sets and the Properties $(\overline{DN}, LB_\infty, \Omega)$

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Abstract. It is shown that every weakly holomorphic function on a compact set of uniqueness in a Frechet space $E \in (\Omega)$ with values in a Frechet space $F \in (\overline{DN})$ is holomorphic. A characterization of a Frechet space with (LB_∞) is also established.

1. Introduction

Let E, F be locally convex spaces and X a compact set in E. A function $f : X \to F$ is called holomorphic on X if it can be extended holomorphically to a neighborhood of X in E. In the case where this request holds for all $u \circ f$, $u \in F'$, the dual space of F, we say that f is weakly holomorphic on X. By $\mathcal{H}(X, F)$ (resp. $\mathcal{H}_\omega(X, F)$), we denote the vector space of holomorphic (resp. weakly holomorphic) functions on X with values in F. Write $\mathcal{H}(X)$ for $F = \mathbf{C}$.

The aim of the present paper is to find sufficient conditions such that

$$\mathcal{H}(X, F) = \mathcal{H}_\omega(X, F). \qquad (\omega)$$

Recently, Hai [4] has proved that a Frechet space F has the property (DN) if and only if (ω) holds for every $\tilde{L}$ regular compact set X in a Frechet space E. In this paper, we shall prove the following two theorems.

Theorem A. *Let E, F be Frechet spaces and X a compact set of uniqueness in E. Then (ω) holds if $E \in (\Omega)$ and $F \in (\overline{DN})$.*

Theorem B. *Let F be a Frechet space. Then $F \in (LB_\infty)$ if and only if (ω) holds for every compact set X which is either of uniqueness in a nuclear Frechet space E isomorphic to a quotient space of the nuclear space $\Lambda_\infty(\alpha)$, or a compact set in* **C**.

2. Preliminaries

2.1. Linear Topological Invariants

Let E be a Frechet space with a fundamental system of semi-norms $\{\|\cdot\|_k\}_{k=1}^{\infty}$. For each subset B of E, define the general semi-norm $\|\cdot\|_B^* : E' \to [0, +\infty]$ on E', the dual space of E, by

$$\|u\|_B^* = \sup\{|u(x)| : x \in B\}, \quad u \in E'.$$

Instead of $\|\cdot\|_{U_q}^*$, we write $\|\cdot\|_q^*$, where

$$U_q = \{x \in E : \|x\|_q < 1\}.$$

We say that E has the property

$$(\Omega) \text{ if } \forall p\, \exists q\, \forall k\, \exists d,\, C > 0 : \|\cdot\|_q^{*1+d} \le C\|\cdot\|_k^* \|\cdot\|_p^{*d};$$
$$(DN) \text{ if } \exists p\, \forall q, d > 0\, \exists k, C > 0 : \|\cdot\|_q^{*1+d} \le C\|\cdot\|_k \|\cdot\|_p^{d};$$
$$(\overline{DN}) \text{ if } \exists p\, \forall q\, \exists k\, \forall d > 0\, \exists C > 0 : \|\cdot\|_q^{*1+d} \le C\|\cdot\|_k \|\cdot\|_p^{d}.$$

Finally, we say that E has the property

$$\text{if } \forall 0 < \rho_k \uparrow\ \exists p\, \forall q\, \exists k_q,\, C > 0\, \forall x \in E\, \exists q \le k \le k_q$$
$$\|x\|_q^{1+\rho_k} \le C\|x\|_k \|x\|_p^{\rho_k}. \qquad (LB_\infty)$$

The above properties were introduced and investigated by Vogt in the 1980s (see, e.g., [8, 9]).

2.2. Sequence spaces $\Lambda(A)$

If $A = (u_{j,k})_{j,k=1}^{\infty}$ is a Köthe matrix satisfying the conditions given by Pietsch [6], then we denote by $\Lambda(A)$ the sequence space

$$\Lambda(A) = \{(x_j) \subset \mathbf{C} : p_k(x) := \sum_{j\ge1} |x_j| a_{j,k} < \infty\ \forall k \ge 1\}.$$

Obviously, $\Lambda(A)$ is a Frechet space with natural locally convex topologies induced by the semi-norms p_k.

For $0 < R \le +\infty$, we write $\Lambda_R(\alpha)$ instead of $\Lambda(A)$ for $a_{j,k} = r_k^{\alpha_j}$, where $\alpha = (\alpha_j)$ is an increasing sequence of positive numbers with $\lim_j \alpha_j = +\infty$ and $r_k \nearrow R$. $\Lambda_R(\alpha)$ is called the power series space of finite type if $R < \infty$ and of infinite type if $R = +\infty$.

2.3. Holomorphic Functions

Let E and F be locally convex spaces and D an open set in E. A function $f : D \to F$ is said to be holomorphic if f is continuous and $u \circ f$ is Gateaux holomorphic for all $u \in F'$. By $\mathcal{H}(D, F)$ (resp. $\mathcal{H}^\infty(D, F)$), we denote the space of F-valued holomorphic (resp. bounded holomorphic) functions on D. A compact set X in E is called a set of uniqueness if

$$A(X) = \{f \in \mathcal{H}(X) : f|_X = 0\} = 0.$$

For more details concerning holomorphic functions, we refer the reader to [3].

3. The Boundedness of Continuous Linear Maps

In this section, we prove the following:

Proposition 3.1. *Let E and F be Frechet spaces with $E \in (\Omega)$ and $F \in (\overline{DN})$. Then every continuous linear map from E into F is bounded on a neighborhood of $0 \in E$.*

Proof. Given $f : E \to F$, a continuous linear map. By [8], we can find an index set I and a continuous linear map R from $l^1(I)\hat{\otimes}_\pi s$ onto E, where s is the space of rapidly decreasing sequences. Since R is open, it suffices to show that $g = f \circ R$ is bounded on a neighborhood of $0 \in l^1(I)\hat{\otimes}_\pi s$.

Note that $l^1(I)\hat{\otimes}_\pi s \in (\Omega)$ and

$$l^1(I)\hat{\otimes}_\pi s = \Big\{z = (z_{ij})_{i\in I, j\geq 1} \subset \mathbf{C} : \sum_{i\in I, j\geq 1} |z_{ij}| j^\gamma < \infty \ \forall \gamma \geq 1\Big\}.$$

Hence, every $z \in l^1(I)\hat{\otimes}_\pi s$ can be written in the form

$$z = \sum_{i\in I, j\geq 1} \delta^*_{ij}(z)\delta_{ij},$$

where

$$\delta^*_{ij}(z) = z_{ij} \quad \text{and} \quad \delta_{ij} = \big[\delta^{[i,j]}_{k,l} : I \times \mathbf{N}\big]$$

with

$$\delta^{[i,j]}_{k,l} = \begin{cases} 1 & \text{if } (k, l) = (i, j) \\ 0 & \text{if } (k, l) \neq (i, j). \end{cases}$$

It follows that

$$f(z) = \sum_{i\in I, j\geq 1} \delta^*_{ij}(z) f(\delta_{ij}) \quad \text{for} \quad z \in l^1(I)\hat{\otimes}_\pi s.$$

Take $\alpha \geq 1$ such that

$$M(\alpha, p) = \sup\{\|f(z)\|_p : \|z\|_\alpha < 1\} < \infty,$$

where $p \geq 1$ is chosen such that $(\overline{DN})$ holds.

Let $\beta \geq \alpha$ satisfy the following:

$$\forall \gamma \geq \beta\, \exists \delta_\gamma,\, C_\gamma > 0 : \|\cdot\|_\beta^{*1+\delta_\gamma} \leq C_\gamma \|\cdot\|_\gamma^* \|\cdot\|_\alpha^{*\delta_\gamma} \tag{1}$$

on $(l^1(I)\hat{\otimes}_\pi s)'$.

From the relations

$$\|\delta_{ij}\|_\beta^* = \frac{1}{\|\delta_{ij}\|_\beta},\quad i \in I,\ j,\ \beta \geq 1$$

and (1), it implies that

$$\frac{1}{\|\delta_{ij}\|_\beta^{1+\delta_\gamma}} \leq \frac{C_\gamma}{\|\delta_{ij}\|_\gamma \|\delta_{ij}\|_\alpha^{\delta_\gamma}} \text{ for } i \in I,\ j \geq 1. \tag{2}$$

We check that f is bounded on $\{z \in E : \|z\|_\beta < 1\}$.

Indeed, given $q \geq p$, choose $k_q \geq q$ such that

$$\forall d > 0\, \exists D_d > 0 : \|\cdot\|_q^{1+d} \leq D_d \|\cdot\|_{k_q} \|\cdot\|_p^d. \tag{3}$$

Let $\gamma_q \geq \beta$ such that

$$M(k_q, \gamma_q) < \infty.$$

By applying (2) and (3) to γ_q, d_{γ_q}, C_{γ_q} and $D_{d_{\gamma_q}}$, we obtain the following estimates

$$\begin{aligned}
&\sum_{i \in I, j \geq 1} |\delta_{ij}^*(z)| \|f(\delta_{ij})\|_q \\
&\leq \sum_{i \in I, j \geq 1} C_{\gamma_q}^{\frac{1}{1+d_{\gamma_q}}} D_{\delta_{\gamma_q}}^{\frac{1}{1+d_{\gamma_q}}} \left\| f\Big(\frac{\delta_{ij}}{\|\delta_{ij}\|_{\gamma_q}}\Big)\right\|_{k_q}^{\frac{1}{1+d_{\gamma_q}}} \\
&\times \left\| f\Big(\frac{\delta_{ij}}{\|\delta_{ij}\|_\alpha}\Big)\right\|_p^{\frac{\delta_{\gamma_q}}{1+\delta_{\gamma_q}}} |\delta_{ij}^*(z)| \|\delta_{ij}\|_\beta^{\frac{\delta_{\gamma_q}}{1+\delta_{\gamma_q}}} \\
&\leq C_{\delta_{\gamma_q}}^{\frac{1}{1+d_{\gamma_q}}} D_{\delta_{\gamma_q}}^{\frac{1}{1+d_{\gamma_q}}} M(k_q, \gamma_q)^{\frac{1}{1+d_{\gamma_q}}} M(p, \alpha) \sum_{i \in I, j \geq 1} |\delta_{ij}^*(z)| \|\delta_{ij}\|_\beta \\
&= C^{\frac{1}{1+d_{\gamma_q}}} D_{\delta_{\gamma_q}}^{\frac{1}{1+d_{\gamma_q}}} M(k_q, \gamma_q)^{\frac{1}{1+d_{\gamma_q}}} M(p, \alpha)^{\frac{\delta_{\gamma_q}}{1+\delta_{\gamma_q}}} \|x\|_\beta .
\end{aligned}$$

This means that f is bounded on $\{z \in E : \|z\|_\beta < 1\}$ ■

The following was proved by Vogt [9].

Proposition 3.2. *Let F be a Frechet space. Then*

(i) *every continuous linear map from $\Lambda_\infty(\alpha)$ into $F \in (LB_\infty)$ is bounded on a neighborhood of $0 \in \Lambda_\infty(\alpha)$ for every increasing sequence of positive numbers $\alpha = (\alpha_j)$ satisfying*

$$\lim \alpha_j = \infty \quad \textit{and} \quad \sup_{j \geq 1} \frac{\alpha_{j+1}}{\alpha_j} < \infty, \tag{$*$}$$

(ii) *if, for some sequence of positive numbers $\alpha = (\alpha_j)$ satisfying $(*)$, every continuous linear map from $\Lambda_\infty(\alpha)$ into F is bounded on a neighborhood of $0 \in \Lambda_\infty(\alpha)$, then $F \in (LB_\infty)$.*

4. Proof of Theorem A

We need the following

Proposition 4.1. [2] *Let E be a Frechet space having (Ω). Then $[\mathcal{H}(X)]' \in (\Omega)$ for every compact set X in E.*

Lemma 4.2. *Let F be a Frechet space with $F \in (\overline{DN})$. Then $[F'_{\text{bor}}]' \in (\overline{DN})$, where F'_{bor} is the space F' equipped with the bornological topologies associated with the strong topologies of F'.*

Proof. Since $F \in (\overline{DN})$, we have

$$\exists p\ \forall q\ \exists k\ \forall d > 0\ \exists C > 0 : \|\cdot\|_q \le r^d \|\cdot\|_p + \frac{C}{r}\|\cdot\|_k\ \forall r > 0,$$

or in an equivalent form

$$\exists p\ \forall q\ \exists k\ \forall d > 0\ \exists C > 0 : U_q^0 \subseteq r^d U_p^0 + \frac{C}{r} U_k^0\ \forall r > 0.$$

For $u \in [F'_{\text{bor}}]'$ and $r > 0$, we have

$$\begin{aligned} \|u\|_q^{**} &= \sup_{x^* \in U_q^0} |u(x^*)| \\ &\le r^d \sup_{x^* \in U_p^0} |u(x^*)| + \frac{C}{r} \sup_{x^* \in U_k^0} |u(x^*)| \\ &= r^d\, \|u\|_p^{**} + \frac{C}{r}\, \|u\|_k^{**}. \end{aligned}$$

Hence, $[F'_{\text{bor}}]' \in (\overline{DN})$. ■

Now, we are able to prove Theorem A.

Given $f \in \mathcal{H}_\omega(X, F)$, by Vogt [8], there exists for some Banach space B a continuous linear map R from $B\hat{\otimes}_\pi s$ onto E. Take a compact set Y in $B\hat{\otimes}_\pi s$ for which $X = R(Y)$. Consider the linear map $S : F'_{\text{bor}} \to \mathcal{H}(X)$ given by

$$S(u) = \widehat{uf} \quad \text{for} \quad u \in F'_{\text{bor}},$$

where $\widehat{uf}$ is a holomorphic extension of uf to a neighborhood of X in E.

It follows that S has a closed graph. By virtue of the closed graph theorem of Grothendieck [7], S is continuous. Applying Propositions 3.1 and 4.1 to $\hat{R}S : F'_{\text{bor}} \to \mathcal{H}(Y)$, we can find a neighborhood V of Y in $B\hat{\otimes}_\pi s$ such that $\hat{R}S$ continuously maps F'_{bor} into $\mathcal{H}^\infty(V)$. Hence, by Lemma 4.2, S continuously maps F'_{bor} into $\mathcal{H}^\infty(W)$ for some neighborhood W of X in E. This implies that the formula

$$\hat{f}(z)(u) = (Su)(z) \quad \text{for} \quad z \in W \quad \text{and} \quad u \in F'$$

defines a holomorphic extension of f to W. ■

5. Proof of Theorem B

Sufficiency of Theorem B in the case where X is a compact set of uniqueness in a nuclear Frechet space E which is isomorphic to a quotient space of the nuclear space $\Lambda_\infty(\alpha)$ is proved as in Theorem A.

By applying Proposition 3.2(i) and by [1, 5], we deduce that $[\mathcal{H}(X)]'$ is isomorphic to a quotient space of the nuclear space $\Lambda_\infty(\beta(\alpha))$ where $\beta(\alpha)$ is stable. Here, note that $F' \cong F'_{\text{bor}}$ by the reflexivity of F.

Now, we consider the case where X is a compact set in $\mathbf{C}$. Denote by X' the set consisting of all limit points of X. Choose a neighborhood basis $\{V_n\}$ of X' such that

$$X \cap \partial V_n = \emptyset \text{ for } n \geq 1.$$

Put

$$Y_n = X \setminus V_n.$$

Then, we obtain the exact sequence

$$0 \to \operatorname{lim\,ind} \mathcal{H}^\infty(Y_n)/A(Y_n) \to \mathcal{H}(X)/A(X) \overset{R}{\to} \mathcal{H}(X') \to 0,$$

with

$$\mathcal{H}^\infty(Y_n)/A(Y_n) \cong \mathbf{C}^{k_n},$$

where

$$k_n = \#Y_n \text{ for } n \geq 1$$

and R is the restriction map.

As in Theorem A, we consider the linear map

$$S : F' \to \mathcal{H}(X)/A(X)$$

given by

$$S(u) = \widehat{uf} + A(X) \text{ for } u \in F'.$$

It is easy to check that S has the closed graph and hence, it follows from the closed graph theorem of Grothendieck [7] that S is continuous.

Thus, $R \circ S$ is factorized through F'_ρ for some continuous semi-norm ρ on F'. Here, F'_ρ stands for the Banach space associated with ρ, i.e., there exists a continuous linear map

$$T : F'_\rho \to \mathcal{H}(X')$$

verifying

$$R \circ S = T \circ \omega_\rho,$$

where $\omega_\rho : F' \to F'_\rho$ is the canonical map.

Consider the continuous linear map

$$S - T\omega_\rho : F' \to \operatorname{Ker} R \cong \mathbf{C}^{(N)}.$$

Since F has a continuous norm, we infer that $S - T\omega_\rho$ can be factorized through F'_{ρ_1} for some continuous semi-norm $\rho_1 \geq \rho$ on F'.

Replacing ρ by ρ_1 if necessary, we may assume $\rho_1 = \rho$. We let

$$G : F'_{\rho_1} \to \mathrm{Ker} R$$

be a continuous linear map satisfying

$$S - T\omega_\rho = G\omega_\rho \text{ or equivalently, } S = (T + G)\omega_\rho .$$

This means that S can be factorized through F'_ρ.

Because R is a surjection between dual nuclear Frechet spaces, we can find a continuous linear map

$$\hat{S} : F'_\rho \to \mathcal{H}(X)$$

satisfying

$$\hat{S}\omega_\rho = S\omega_X ,$$

where $\omega_X : \mathcal{H}(X) \to \mathcal{H}(X)/A(X)$ is the canonical projection.

Choose a neighborhood V of X in $\mathbf{C}$ such that $\hat{S}$ continuously maps F'_ρ into $\mathcal{H}^\infty(V)$. This implies that the function defined by

$$\hat{f}(z)(w) = \hat{S}\omega_\rho(u)(z) \text{ for } z \in V \text{ and } u \in F'$$

is a holomorphic extension of f to V.

Conversely, by Proposition 3.2(ii), it suffices to check that every continuous linear map $T : \Lambda_\infty(j) \to F$ is compact. Choose $X = C$, the polar compact set of uniqueness in $\mathbf{C}$. Then by virtue of [10], we have

$$[\mathcal{H}(X)]' \cong \mathcal{H}(\mathbf{C}/X) \cong \mathcal{H}(\mathbf{C}) \cong \Lambda_\infty(j).$$

Define a function $f : X \to F$ by

$$f(z)(u) = T'(u)(z) \text{ for } z \in X,\ u \in F'.$$

Obviously, $f \in \mathcal{H}_\omega(X, F)$. Thus, f is extended to a bounded holomorphic function $\hat{f}$ on a neighborhood V of X in $\mathbf{C}$. It follows that T' is bounded on $[\hat{f}(V)]^\circ$, a neighborhood of $0 \in F$. ■

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