Vietnam Journal of Mathematics **26**:3 (1998) 217–228

Vietnam Journal
of
MATHEMATICS

Sufficient Conditions for the Existence of a Hamilton Cycle in Cubic (6,n)-metacirculant Graphs II*

Ngo Dac Tan
Institute of Mathematics, P.O. Box 631, Bo Ho, 10.000 Hanoi, Vietnam

Received September 26, 1995
Revised November 28, 1996

Abstract. The smallest value of m for which we are still unsure if all connected cubic (m, n)-metacirculant graphs have a Hamilton cycle is $m = 6$. In this paper, we shall prove that a connected cubic (6,n)-metacirculant graph $G = \mathrm{MC}(6, n, \alpha, S_0, S_1, S_2, S_3)$ has a Hamilton cycle if either one of the numbers $\alpha + 1$, $\alpha - 1$, or $1 - \alpha + \alpha^2$ is relatively prime to n, or the order of α in Z_n^* is not equal to 6. As an application of these results, we shall show that every connected cubic (6,n)-metacirculant graph has a Hamilton cycle if either $n = p^a q^b$, where p and q are distinct primes, $a \geq 0$ and $b \geq 0$, or n is such that $\varphi(n)$ is not divisible by 3 where $\varphi(n)$ is the number of integers z satisfying $0 \leq z < n$ and $\gcd(z, n) = 1$.

1. Introduction

This paper is a sequel to the first paper [12] in which it was shown that a connected cubic $(6, n)$-metacirculant graph $G = \mathrm{MC}(6, n, \alpha, S_0, S_1, S_2, S_3)$ has a Hamilton cycle if $\emptyset \neq S_1 = \{s\}$ and $(1 + \alpha + \alpha^2 + \alpha^3 + \alpha^4 + \alpha^5)s \equiv 0 \pmod{n}$. As in [12], we consider here only finite undirected graphs without loops or multiple edges. If G is a graph, then $V(G)$ and $E(G)$ denote its vertex-set and its edge-set, respectively. If n is a positive integer, then we write Z_n for the ring of integers modulo n and Z_n^* for the multiplicative group of units in Z_n.

Let m and n be two positive integers, $\alpha \in Z_n^*$, $\mu = [m/2]$ and let $S_0, S_1, \ldots, S_\mu$ be subsets of Z_n satisfying the following conditions:

(1) $0 \notin S_0 = -S_0$;
(2) $\alpha^m S_r = S_r$ for $0 \leq r \leq \mu$;
(3) if m even, then $\alpha^\mu S_\mu = -S_\mu$.

Then we define the (m, n)-metacirculant graph $G = \mathrm{MC}(m, n, \alpha, S_0, S_1, \ldots, S_\mu)$ to be the graph with vertex set $V(G) = \{v_j^i : i \in Z_m, j \in Z_n\}$ and edge set

*This work was supported in part by the National Basic Research Program in Natural Sciences, Vietnam.

$E(G) = \{v_j^i v_h^{i+r} : 0 \leq r \leq \mu;\ i \in Z_m;\ h, j \in Z_n;\ (h - j) \in \alpha^i S_r\}$, where superscripts and subscripts are reduced modulo m and modulo n, respectively.

The concept of (m, n)-metacirculant graphs was introduced in [1]. It was asked if all connected (m, n)-metacirculant graphs, other than the Petersen graph, have a Hamilton cycle. For $n = p^t$ with p a prime, an affirmative answer was obtained in [2]. Connected cubic (m, n)-metacirculant graphs, other than the Petersen graph, are also proved to be Hamiltonian for m odd [7], $m = 2$ [4, 7], and m divisible by 4 [8, 10]. Thus, the smallest value of m, for which we are unsure if all connected cubic (m, n)-metacirculant graphs have a Hamilton cycle, is $m = 6$.

This paper is a continuation of the first paper [12] in this series and is geared towards the resolution of the problem of the existence of a Hamilton cycle in connected cubic $(6, n)$-metacirculant graphs. Using the results obtained in [12], we will prove in Sec. 3 two sufficient conditions for connected cubic $(6, n)$-metacirculant graphs to be hamiltonian, namely, we will prove that a connected cubic $(6, n)$-metacirculant graph $G = \mathrm{MC}(6, n, \alpha, S_0, S_1, S_2, S_3)$ has a Hamilton cycle if either one of the numbers $\alpha + 1$, $\alpha - 1$, or $1 - \alpha + \alpha^2$ is relatively prime to n or the order of α in Z_n^* is not equal to 6. As an application of these results, we will obtain in Sec. 4 a partial affirmative answer to the question whether all connected cubic $(6, n)$-metacirculant graphs have a Hamilton cycle, proving that every connected cubic $(6, n)$-metacirculant graph has a Hamilton cycle if either $n = p^a q^b$, where p and q are distinct primes, $a \geq 0$ and $b \geq 0$, or n is such that $\varphi(n)$ is not divisible by 3 where $\varphi(n)$ is the number of integers z satisfying $0 \leq z < n$ and $\gcd(z, n) = 1$.

2. Preliminaries

First, we recall a method used in [10, 11] for lifting a Hamilton cycle in a quotient graph $\overline{G}$ of a graph G to a Hamilton cycle in G. This method will be used in Sec. 3 to prove Theorem 1.

A permutation β is said to be semiregular if all cycles in the disjoint cycle decomposition of β have the same length. If a graph G has a semiregular automorphism β, then the quotient graph G/β with respect to β is defined as follows [3]. The vertices of G/β are the orbits of the subgroup $\langle\beta\rangle$ generated by β and two such vertices are adjacent if and only if there is an edge in G joining a vertex of one corresponding orbit to a vertex in the other orbit.

Let β be of order t and $G^0, G^1, \ldots, G^h$ the subgraphs induced by G on the orbits of $\langle\beta\rangle$. Let $v_0^i, v_1^i, \ldots, v_{t-1}^i$ be a cyclic labeling of the vertices of G^i under the action of β and let $C = G^0 G^i G^j \ldots G^r G^0$ be a cycle of G/β. Consider a path P of G arising from a lifting of C, namely, start at v_0^0 and choose an edge from v_0^0 to a vertex v_a^i of G^i. Then take an edge from v_a^i to a vertex v_b^j of G^j following G^i in C. Continue in this way until returning to a vertex v_d^0 of G^0. The set of all paths that can be constructed in this way using C is called in [3] the coil of C and is denoted by coil (C).

The following lemma is easy to prove. However, it has been proved in [8].

Lemma 1. [8] *Let t be the order of a semiregular automorphism β of a graph G and G^0 the subgraph induced by G on an orbit of $\langle\beta\rangle$. If there exists a Hamilton cycle C in G/β such that coil(C) contains a path P whose terminal vertices are distance d apart in the G^0 where P starts and terminates and $\gcd(d, t) = 1$, then G has a Hamilton cycle.*

The following lemmas are particular cases of Theorem 2 in [9] and Lemmas 5 and 6 in [11], respectively. Therefore, we omit their proofs here.

Lemma 2. [9] *Let $G = MC(6, n, \alpha, S_0, S_1, S_2, S_3)$ be a cubic $(6, n)$-metacirculant graph with $S_0 = \emptyset$. Then G is connected if and only if one of the following conditions holds:*

(i) $S_1 = \{s\}$, $S_2 = \emptyset$, $S_3 = \{k\}$ *and* $\gcd(e, n) = 1$ *where e is* $[k - s(1 + \alpha + \alpha^2)]$ *reduced modulo n;*

(ii) $S_1 = \emptyset$, $S_2 = \{s\}$, $S_3 = \{k\}$ *and* $\gcd(g, n) = 1$ *where g is* $[k(1+\alpha) - s(1+\alpha+\alpha^2)]$ *reduced modulo n.*

Lemma 3. [11] *Let $G = MC(6, n, \alpha, S_0, S_1, S_2, S_3)$ be a connected cubic $(6, n)$-metacirculant graph such that $S_0 = S_1 = \emptyset$, $S_2 = \{s\}$ and $S_3 = \{k\}$. Let $\overline{n} = gcd(\alpha - 1, n)$ and $\overline{\overline{n}} = gcd(1 - \alpha + \alpha^2, n)$. Then G has a Hamilton cycle if any one of the following conditions holds:*

(i) *Either* $gcd(n / (\overline{n}\,\overline{\overline{n}}), 3\overline{n} - 1) = 1$;

(ii) $\overline{\overline{n}} = 1$.

Lemma 4. [11] *Let $G = MC(6, n, \alpha, S_0, S_1, S_2, S_3)$ be a connected cubic $(6, n)$-metacirculant graph such that $S_0 = \emptyset$, $S_1 = \{s\}$, $S_2 = \emptyset$ and $S_3 = \{k\}$. Then G has a Hamilton cycle if n is even.*

We now recall the definition of a brick product of a cycle with a path defined in [4]. This product plays a role in the proof of Theorem 2 in the next section.

Let C_n with $n \geq 3$ and P_m with $m \geq 1$ be the graphs with vertex sets $V(C_n) = \{u_1, u_2, ..., u_n\}$, $V(P_m) = \{v_1, v_2, ..., v_{m+1}\}$ and edge sets $E(C_n) = \{u_1u_2, u_2u_3, ..., u_nu_1\}$, $E(P_m) = \{v_1v_2, v_2v_3, ..., v_mv_{m+1}\}$, respectively. The brick product $C_n^{[m+1]}$ of C_n with P_m is defined in [4] as follows. The vertex set of $C_n^{[m+1]}$ is the cartesian product $V(C_n) \times V(P_m)$. The edge set of $C_n^{[m+1]}$ consists of all pairs of the form $(u_i, v_h)(u_{i+1}, v_h)$ and $(u_1, v_h)(u_n, v_h)$, where $i = 1, 2, ..., n - 1$ and $h = 1, 2, ..., m + 1$, together with all pairs of the form $(u_i, v_h)(u_i, v_{h+1})$, where $i + h \equiv O \pmod 2$, $i = 1, 2, ..., n$ and $h = 1, 2, ..., m$.

The following result has been proved in [4].

Lemma 5. [4] *Consider the brick product $C_n^{[m]}$ with n even. Let $C_{n,1}$ and $C_{n,m}$ denote the two n-cycles in $C_n^{[m]}$ on the vertex-sets $\{(u_i, v_1) : i = 1, 2, ..., n\}$ and $\{(u_i, v_m) : i = 1, 2, ..., n\}$, respectively. Let F denote an arbitrary perfect matching joining the vertices of degree 2 in $C_{n,1}$ with the vertices of degree 2 in $C_{n,m}$. If X is a graph obtained by adding the edges of F to $C_n^{[m]}$, then X has a Hamilton cycle.*

3. Sufficient Conditions

Using results obtained in [12], we will prove in this section two sufficient conditions for connected cubic $(6, n)$-metacirculant graphs to be hamiltonian which are expected to be helpful in further investigation of the problem of the existence of a Hamilton cycle in connected cubic $(6, n)$-metacirculant graphs. As an application of these conditions, we will prove in Sec. 4 that every connected cubic $(6, n)$-metacirculant graph has a Hamilton cycle if either $n = p^a q^b$ where p and q are distinct primes, $a \geq 0$ and $b \geq 0$ or n is such that $\varphi(n)$ is not divisible by 3, where $\varphi(n)$ is the number of integers z satisfying $0 \leq z < n$ and $\gcd(z, n) = 1$.

Theorem 1. *Let $G = MC(6, n, \alpha, S_0, S_1, S_2, S_3)$ be a connected cubic $(6, n)$-metacirculant graph. If one of the numbers $\alpha + 1, \alpha - 1$ or $1 - \alpha + \alpha^2$ is relatively prime to n, then G possesses a Hamilton cycle.*

Proof. Let $G = MC(6, n, \alpha, S_0, S_1, S_2, S_3)$ be a connected cubic $(6, n)$-metacirculant graph, $\overline{n} = \gcd(\alpha - 1, n)$, $\overline{\overline{n}} = \gcd(1 - \alpha + \alpha^2, n)$ and $\tilde{n} = \gcd(\alpha + 1, n)$. If $S_0 \neq \emptyset$, then by [7], G has a Hamilton cycle. Therefore, we may assume from now on that $S_0 = \emptyset$. Since G is a cubic $(6, n)$-metacirculant graph, only the following cases may happen:

Case 1. $|S_1| = 1$, $S_2 = \emptyset$ and $|S_3| = 1$.

Case 2. $S_1 = \emptyset$, $|S_2| = 1$ and $|S_3| = 1$.

Case 3. $S_1 = S_2 = \emptyset$ and $|S_3| = 3$.

Since G is connected, Case 3 does not occur. Now, consider Cases 1 and 2 in turn.

Case 1. $|S_1| = 1$, $S_2 = \emptyset$ and $|S_3| = 1$.

Let $S_1 = \{s\}$ with $0 \leq s < n$ and $S_3 = \{k\}$ with $0 \leq k < n$. By the definition of $(6, n)$-metacirculant graphs, we have

(1) $\alpha^6 s \equiv s \pmod{n}$

$\Leftrightarrow \quad (\alpha^3 + 1)(\alpha - 1)(1 + \alpha + \alpha^2)s \equiv 0 \pmod{n}$, and (3.1)

(2) $\alpha^3 k \equiv -k \pmod{n}$

$\Leftrightarrow \quad (\alpha^3 + 1)k \equiv 0 \pmod{n}$. (3.2)

Let $z = n/\gcd(\alpha^3 + 1, n)$. From (3.1) and (3.2), it follows that z is a divisor of both k and $(\alpha - 1)(1 + \alpha + \alpha^2)s$. Since G is connected, by Lemma 2(i), $\gcd(k, (1 + \alpha + \alpha^2)s, n) = 1$. Therefore, z must be a divisor of $\alpha - 1$. Thus, we have

$$(\alpha^3 + 1)(\alpha - 1) \equiv 0 \pmod{n}. \tag{3.3}$$

Assume first that $\overline{n} = \gcd(\alpha - 1, n) = 1$. Then (3.3) implies that $(\alpha^3 + 1) \equiv 0 \pmod{n}$. By [12], G has a Hamilton cycle.

Assume next that $\tilde{n} = \gcd(\alpha + 1, n) = 1$. Let $\rho : V(G) \to V(G) : v_j^i \mapsto v_{j+1}^i$. Then $\rho^{\alpha - 1}$ is a semiregular automorphism of G and therefore, we can construct the quotient graph $G/\rho^{\alpha - 1}$ which is isomorphic to the cubic $(6, \overline{n})$-metacirculant graph $\overline{G} = MC(6, \overline{n}, \overline{\alpha}, \overline{S}_0, \overline{S}_1, \overline{S}_2, \overline{S}_3)$, where $\overline{n} = \gcd(\alpha - 1, n)$, $1 = \overline{\alpha} \equiv \alpha \pmod{\overline{n}}$, $\overline{S}_0 = \emptyset$, $\overline{S}_1 = \{\overline{s}\}$ with $0 \leq \overline{s} < \overline{n}$ and $\overline{s} \equiv s \pmod{\overline{n}}$, $\overline{S}_2 = \emptyset$ and $\overline{S}_3 = \{\overline{k}\}$ with $0 \leq \overline{k} < \overline{n}$ and $\overline{k} \equiv k \pmod{\overline{n}}$. We identify $G/\rho^{\alpha - 1}$ with $\overline{G}$ and in order to avoid the confusion between vertices of G and $\overline{G}$, we assume $V(\overline{G}) = \{w_j^i : i \in Z_6, j \in Z_{\overline{n}}\}$.

If n is even, then by Lemma 4, G has a Hamilton cycle. If n is odd, then we can repeat here the proof of the main theorem in [10] for the case of n odd in order to construct a Hamilton cycle of $\overline{G}$ such that the path P of coil(C), which starts at v_0^0, terminates at v_f^0 with $f \equiv (\alpha - 1)d \pmod n$, where

$$\begin{aligned} d &= -[k - s(1+\alpha+\alpha^2)](1+\alpha+\alpha^2+\alpha^3) \\ &= -[k - s(1+\alpha+\alpha^2)](\alpha+1)(1+\alpha^2). \end{aligned}$$

Let t be the order of the automorphism $\rho^{\alpha-1}$. It is not difficult to see that $t = n/\overline{n}$. Since (3.3) holds, it follows that t is a divisor of $\gcd(\alpha^3+1, n) = \gcd((\alpha+1)(1-\alpha+\alpha^2), n)$. By our assumption, $\gcd(\alpha+1, n) = 1$. Therefore, t must be a divisor of $\gcd(1-\alpha+\alpha^2, n)$. Since G is connected, by Lemma 2(i),

$$\gcd([k - s(1+\alpha+\alpha^2)], n) = 1.$$

Therefore, $\gcd([k - s(1+\alpha+\alpha^2)], t) = 1$. Since $\gcd(\alpha, n) = 1$, it is also clear that $\gcd(1+\alpha^2, 1-\alpha+\alpha^2, n) = 1$. So $\gcd(1+\alpha^2, t) = 1$ because t is a divisor of $\gcd(1-\alpha+\alpha^2, n)$ as we have shown in the preceding paragraph. Further, $\gcd(\alpha+1, n) = 1$ by our assumption. Thus, $\gcd(d, t) = 1$. By Lemma 1, G has a Hamilton cycle.

Finally, assume $\overline{\overline{n}} = \gcd(1-\alpha+\alpha^2, n) = 1$. Since the automorphism ρ of G with $\rho(v_j^i) = v_{j+1}^i$ is semiregular, we can construct the quotient graph G/ρ. It is easy to see that G/ρ is isomorphic to the circulant graph $\overline{\overline{G}} = C(6, \{1, 3, 5\})$, the vertex set and the edge set of which are

$$\begin{aligned} V(\overline{\overline{G}}) &= \{w_j \ : \ j \in Z_6\} \text{ and} \\ E(\overline{\overline{G}}) &= \{w_j w_h \ : \ j, h \in Z_6;\ (h - j) = 1 \text{ or } 3 \text{ or } 5 \pmod 6\}, \end{aligned}$$

respectively. Therefore, we can identify G/ρ with $\overline{\overline{G}}$. It is also clear that $\overline{\overline{G}}$ possesses the following Hamilton cycle D:

$$D = w_0 w_3 w_2 w_5 w_4 w_1 w_0.$$

Let P be the path of coil(D) which starts at v_0^0. This path terminates at v_f^0 with

$$\begin{aligned} f &\equiv k - \alpha^2 s + \alpha^2 k - \alpha^4 s + \alpha^4 k - s \\ &\equiv (1-\alpha+\alpha^2)k - s(1-\alpha+\alpha^2)(1+\alpha+\alpha^2) \\ &\equiv (1-\alpha+\alpha^2)[k - s(1+\alpha+\alpha^2)] \pmod n. \end{aligned}$$

It is clear that ρ has order $t = n$ and terminal vertices of P in G^0 are v_0^0 and v_f^0 which are distance $d = f$ apart in G^0. Since G is connected, by Lemma 2(i), $\gcd([k - s(1+\alpha+\alpha^2)], n) = 1$. By our assumption, $\gcd(1-\alpha+\alpha^2, n) = 1$. Therefore, $\gcd(d, t) = \gcd(f, n) = 1$. By Lemma 1, G has a Hamilton cycle.

Case 2. $S_1=\emptyset, |S_2|=1$ and $|S_3|=1$.

Let $S_2=\{s\}$ with $0\le s<n$ and $S_3=\{k\}$ with $0\le k<n$. If $\overline{\overline{n}}=\gcd(1-\alpha+\alpha^2,n)$ $=1$, then G has a Hamilton cycle by Lemma 3. Let

$$\overline{n}=\gcd(\alpha-1,n)=1. \tag{3.4}$$

Since $\gcd(\alpha,n)=1$, equality (3.4) holds only if n is odd. Therefore, $n/(\overline{n}\,\overline{\overline{n}})$ is odd. This implies that $\gcd(n/(\overline{n}\,\overline{\overline{n}}),3\overline{n}-1)=\gcd(n/(\overline{n}\,\overline{\overline{n}}),2)=1$. By Lemma 3, G again has a Hamilton cycle. Finally, let $\tilde{n}=\gcd(\alpha+1,n)=1$. As in Case 1 but using Lemma 2(ii), we can show that, for the graph G,

$$(\alpha^3+1)(\alpha-1)\equiv(\alpha+1)(1-\alpha+\alpha^2)(\alpha-1)\equiv 0\ (\mathrm{mod}\ n). \tag{3.5}$$

Since $\gcd(\alpha+1,n)=1$, this implies that $(1-\alpha+\alpha^2)(\alpha-1)\equiv 0\ (\mathrm{mod}\ n)$. Therefore, $n/(\overline{n}\,\overline{\overline{n}})=1$ and $\gcd(n/(\overline{n}\,\overline{\overline{n}}),3\overline{n}-1)=\gcd(1,3\overline{n}-1)=1$. Again, by Lemma 3, G has a Hamilton cycle.

The proof of Theorem 1 is complete. ■

Theorem 2. *Let $G=MC(6,n,\alpha,S_0,S_1,S_2,S_3)$ be a connected cubic $(6,n)$-metacirculant graph. Then G possesses a Hamilton cycle if the order of α in Z_n^* is not equal to 6.*

Proof. Let $G=\mathrm{MC}(6,n,\alpha,S_0,S_1,S_2,S_3)$ be a connected cubic $(6,n)$-metacirculant graph. If $S_0\ne\emptyset$, then by [7], G has a Hamilton cycle. Therefore, we may assume from now on that $S_0=\emptyset$. Since G is a cubic $(6,n)$-metacirculant graph, only the following cases may happen:

Case 1. $|S_1|=1$, $S_2=\emptyset$ and $|S_3|=1$.

Case 2. $S_1=\emptyset$, $|S_2|=1$ and $|S_3|=1$.

Case 3. $S_1=S_2=\emptyset$ and $|S_3|=3$.

Since G is connected, Case 3 does not occur. Further, since (3.3) and (3.5) hold, we have $\alpha^6\equiv 1\ (\mathrm{mod}\ n)$. This means that the order of α in Z_n^* is a divisor of 6. Therefore, it is equal to one of the numbers 1, 2, 3 or 6. Thus, to prove Theorem 2, we need only to consider the possibilities where the order of α in Z_n^* is equal to 1, 2 or 3. We consider these possibilities in turn.

(i) The order of α in Z_n^* is 1, i.e., $\alpha=1$. Then $1-\alpha+\alpha^2=1$ and $\gcd(1-\alpha+\alpha^2,n)=1$. By Theorem 1, G has a Hamilton cycle.

(ii) The order of α in Z_n^* is 2.

Assume first that G is a connected cubic $(6,n)$-metacirculant graph of Case 1. Let $S_1=\{s\}$ and $S_3=\{k\}$. An edge of G of the type $v_j^iv_{j+\alpha^is}^{i+1}$ is called an S_1-edge, and of the type $v_j^iv_{j+\alpha^ik}^{i+3}$ an S_3-edge. A cycle C in G is called an S_1-cycle if every edge of C is an S_1-edge. Consider S_1-cycles in G. Since every vertex of G is incident with just two S_1-edges, any S_1-cycle B_j in G can be represented in the form $B_j=P(v_y^0)P(v_{y+z}^0)P(v_{y+2z}^0)\ldots$, where

$P(v_h^0) = v_h^0 v_{h+s}^1 v_{h+s+\alpha s}^2 v_{h+2s+\alpha s}^3 v_{h+2s+2\alpha s}^4 v_{h+3s+2\alpha s}^5$, and z is $3s+3\alpha s$. Further, it is clear that all S_1-cycles in G are isomorphic to each other and have an even length l. Moreover, two vertices v_f^i and v_g^{i+2} of G are vertices distance 2 apart in the same S_1-cycle B_j if and only if $g = f + s + \alpha s$ in Z_n.

If G has only one S_1-cycle, then this cycle is trivially a Hamilton cycle of G. Therefore, we assume G has at least two S_1-cycles. Let v_f^i and v_g^{i+2}, with i even being two vertices distance 2 apart in the same S_1-cycle B_j. Then the vertices of G adjacent to v_f^i and v_g^{i+2} by S_3-edges are $v_{f'}^{i+3}$ and $v_{g'}^{i+5}$, respectively, where $f' = f + \alpha^i k = f + k$ and $g' = g + \alpha^{i+2}k = g + k$. Since $g = f + s + \alpha s$, we have $g' = g + k = f + s + \alpha s + k = f' + s + \alpha s$. Thus, $v_{f'}^{i+3}$ and $v_{g'}^{i+5}$ are vertices distance 2 apart in the same S_1-cycle $B_{j'}$. Moreover, the superscripts $i+3$ and $i+5$ of respectively $v_{f'}^{i+3}$ and $v_{g'}^{i+5}$ are odd. Using this property and the fact that G is a connected cubic graph, it is not difficult to see that G is isomorphic to the graph X obtained from a brick product $C_l^{[r]}$ by adding the edges of a perfect matching joining the vertices of degree 2 in $C_{l,1}$ with the vertices of degree 2 in $C_{l,r}$ of $C_l^{[r]}$, where C_l is isomorphic to an S_1-cycle B_j, r is the number of distinct S_1-cycles in G, and $C_{l,1}$ and $C_{l,r}$ are two l-cycles in $C_l^{[r]}$ on the vertex sets $\{(u_i, v_1) : i = 1, 2, ..., l\}$ and $\{(u_i, v_r) : i = 1, 2, ..., l\}$, respectively. By Lemma 5, X has a Hamilton cycle. Therefore, G has a Hamilton cycle.

Assume next that G is a connected cubic $(6, n)$-metacirculant graph of Case 2. Let $S_2 = \{s\}$ and $S_3 = \{k\}$. An edge of G of the type $v_j^i v_{j+\alpha^i s}^{i+2}$ is called an S_2-edge, and of the type $v_j^i v_{j+\alpha^i k}^{i+3}$ an S_3-edge. A cycle C in G is called an S_2-cycle if every edge of C is an S_2-edge.

Since the order of α in Z_n^* is 2, we have $\alpha^2 - 1 \equiv 0 \pmod n \Leftrightarrow (\alpha+1)(\alpha-1) \equiv 0 \pmod n$. On the other hand, $\gcd(1-\alpha+\alpha^2, \alpha-1, n) = 1$ because $\gcd(\alpha, n) = 1$. Therefore, $\bar{\bar{n}} = \gcd(1-\alpha+\alpha^2, n)$ is a divisor of $\gcd(\alpha+1, n)$. Since $1-\alpha+\alpha^2 = t(\alpha+1)+3$ for some integer t, it follows that $\bar{\bar{n}}$ is a divisor of 3. Thus, $\bar{\bar{n}} = 1$ or 3.

If $\bar{\bar{n}} = 1$, then G has a Hamilton cycle by Theorem 1.

If $\bar{\bar{n}} = 3$, then $n = 3^a x$ and $\alpha + 1 = 3^a y$ with $a \geq 1$. Since G is connected, by Lemma 2, $\gcd([k(1+\alpha) - s(1+\alpha+\alpha^2)], n) = 1$. On the other hand, by the definition of $(6, n)$-metacirculant graphs, $(\alpha^3+1)k \equiv (\alpha+1)k \equiv 0 \pmod n$. Therefore, $\gcd(s, n) = 1$. Let $G' = \mathrm{MC}(6, n, \alpha', S_0', S_1', S_2', S_3')$ be a cubic $(6, n)$-metacirculant graph such that $\alpha' = \alpha$, $S_0' = S_1' = \emptyset$, $S_2' = \{1\}$, $S_3' = \{0\}$ and $V(G') = \{x_j^i : i \in Z_6,\ j \in Z_n\}$. Then it is not difficult to verify that the mapping

$$\Psi : V(G') \to V(G) : \begin{cases} x_j^i \mapsto v_{js}^i & \text{if } i = 0, 2, 4 \\ x_j^i \mapsto v_{js+k}^i & \text{if } i = 1, 3, 5 \end{cases}$$

is an isomorphism of G' and G. Therefore, without loss of generality, we may assume G is a cubic $(6, n)$-metacirculant graph $\mathrm{MC}(6, n, \alpha, S_0, S_1, S_2, S_3)$ such that $n = 3^a x$, $\alpha + 1 = 3^a y$ with $a \geq 1$, $S_0 = S_1 = \emptyset$, $S_2 = \{1\}$ and $S_3 = \{0\}$. Such a graph has six disjoint S_2-cycles, namely, C^0, C^1, C^2, D^0, D^1 and D^2 which contain v_0^0, v_0^2, v_0^4, v_0^3, v_0^5 and v_0^1, respectively. It is not difficult to see that, for each S_2-cycle C^t or D^t, $(t = 0, 1, 2)$, each element of Z_n appears as a subscript of one and only one vertex of this cycle.

Let ρ and τ be the automorphisms of G defined by $\rho(v_j^i) = v_{j+1}^i$ and $\tau(v_j^i) = v_{\alpha j}^{i+1}$. Set $\beta = \rho\tau^2$. Then

$$\beta(v_j^i) = \rho\tau^2(v_j^i) = \rho(v_{\alpha^2 j}^{i+2}) = \rho(v_j^{i+2}) = v_{j+1}^{i+2}. \tag{3.6}$$

So, β maps every vertex of C^t, $t = 0, 1, 2$, to the vertex following it in C^t. Further, since $\alpha + 1 = 3^a y$ with $a \geq 1$, $\alpha \equiv 2 \pmod 3$. Therefore,

$$\beta(D^0) = D^2, \quad \beta(D^2) = D^1, \quad \text{and } \beta(D^1) = D^0. \tag{3.7}$$

From (3.6) and (3.7), it is not difficult to see that G is isomorphic to the graph H such that

$$V(H) = \{u_j^i, w_j^i : i \in Z_3,\ j \in Z_n\} \text{ and}$$
$$E(H) = E_1 \cup E_2 \cup E_3 \cup E_4,$$

where

$$E_1 = \{u_j^i u_{j+1}^i,\ w_j^i w_{j+\alpha}^i : i \in Z_3,\ j \in Z_n\},$$
$$E_2 = \{u_j^i w_j^i : i \in Z_3,\ j \in Z_n \text{ and } j \equiv 0 \pmod 3)\},$$
$$E_3 = \{u_j^i w_j^{i+2} : i \in Z_3,\ j \in Z_n \text{ and } j \equiv 1 \pmod 3)\}, \text{ and}$$
$$E_4 = \{u_j^i w_j^{i+1} : i \in Z_3,\ j \in Z_n \text{ and } j \equiv 2 \pmod 3)\}.$$

We now show that H possesses a Hamilton cycle. Let U^i and W^i, where $i = 0, 1, 2$, be the subgraphs induced by H on $\{u_j^i : j \in Z_n\}$ and $\{w_j^i : j \in Z_n\}$, respectively. By the definition of H, it is clear that U^i and W^i, where $i = 0, 1, 2$, are cycles of length n. First, assume w_0^0, $w_{3\alpha}^0$ and w_3^0 of W^0 are pairwise distinct (Fig. 1). This implies that the vertices u_α^2, $u_{4\alpha}^2$ and $u_{\alpha+3}^2$ of U^2 are also pairwise distinct. Further, the edge $w_{4\alpha}^0 w_{5\alpha}^0$ is an edge of the subpath P of W^0 not containing w_0^0 and connecting w_α^0 with w_3^0. Moreover, $w_{4\alpha}^0$ and $w_{5\alpha}^0$ are not the endvertices of P. Such a graph H possesses a Hamilton cycle shown in Fig. 1.

Next, assume $w_{3\alpha}^0 = w_3^0$ but $w_{3\alpha}^0 \neq w_0^0$ (Fig. 2). If $w_0^0 \neq w_6^0$, then since $3\alpha \equiv 3 \pmod n$, $4\alpha = 3\alpha + \alpha \equiv 3 + \alpha \pmod n$ and $4\alpha + 1 \equiv 4 + \alpha \pmod n$. Therefore, $w_{4\alpha}^0 = w_{3+\alpha}^0$ and $w_{4\alpha+1}^2 = w_{4+\alpha}^2$. Further, the edge $w_{4\alpha}^0 w_{5\alpha}^0$ is an edge of the subpath P of W^0 not containing w_0^0 and connecting w_α^0 with $w_6^0 = w_{6\alpha}^0$. Moreover, $w_{4\alpha}^0$ and $w_{5\alpha}^0$ are not the endvertices of P. Such a graph H possesses a Hamilton cycle shown in Fig. 2. If $w_0^0 = w_6^0$, then $6 \equiv 0 \pmod n$. So $n = 3$ or 6. But $w_{3\alpha}^0 \neq w_0^0$ by our assumption. Hence, $3\alpha \not\equiv 0 \pmod n \Leftrightarrow 3 \not\equiv 0 \pmod n$. If follows that $n \neq 3$, whence $n = 6$. We leave it to the reader to verify that, for this value of n, the graph H also has a Hamilton cycle.

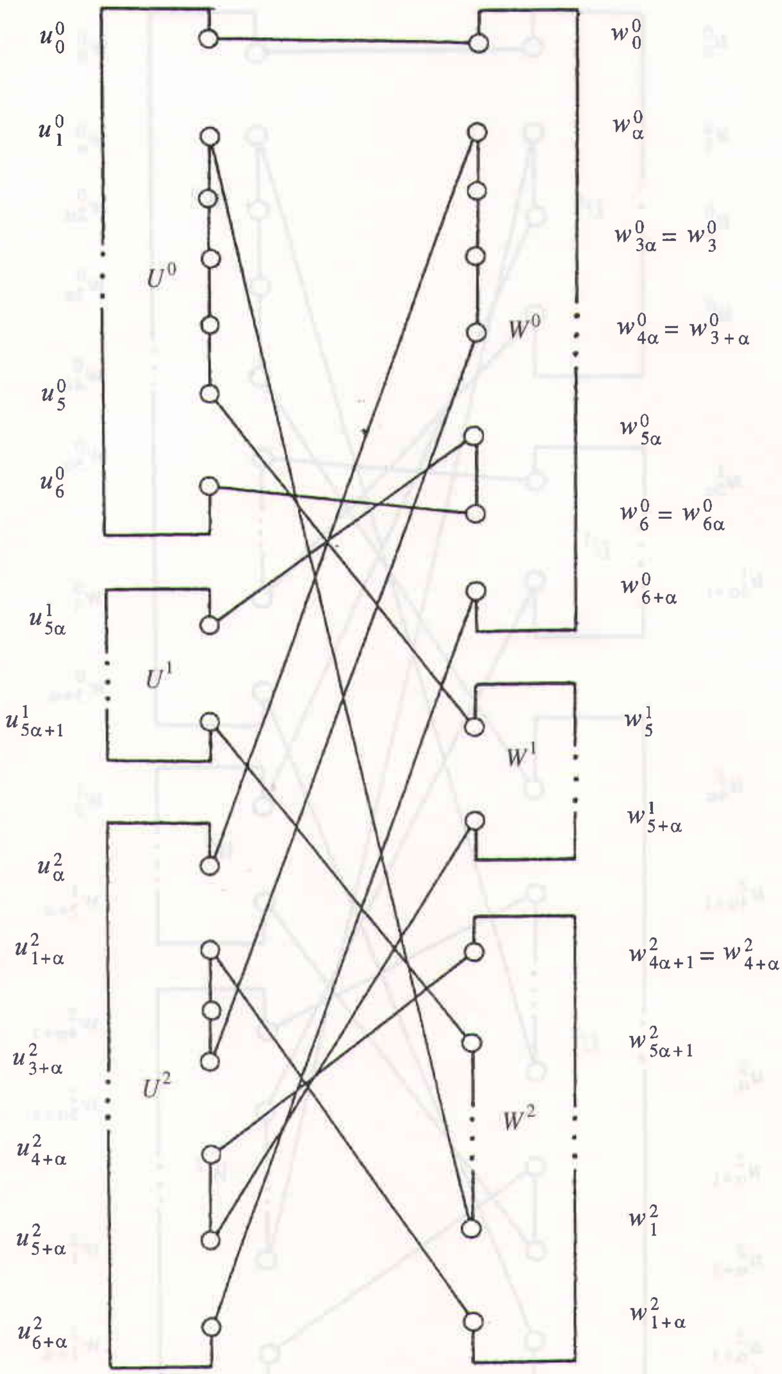
Finally, assume $w_0^0 = w_{3\alpha}^0$ or $w_0^0 = w_3^0$. It follows in both cases that $3 \equiv 0 \pmod n$. So $n = 3$. We again leave it to the reader to verify that for this value of n, H also has a Hamilton cycle.

Thus, the graph H possesses a Hamilton cycle in any of the cases. Since G is isomorphic to H, the graph G also has a Hamilton cycle.

(iii) The order of α in Z_n^* is 3.

By (3.3) and (3.5), we have $(\alpha^3 + 1)(\alpha - 1) = 2(\alpha - 1) \equiv 0 \pmod n$. If n is odd, then this implies that $\alpha - 1 \equiv 0 \pmod n \Leftrightarrow \alpha = 1$, contradicting the fact that α has order 3. If n is even, then $\alpha - 1 = t(n/2)$ for some integer t. Therefore, $\alpha = 1$ or $\alpha = n/2 + 1$. The case $\alpha = 1$ cannot occur as before. Suppose $\alpha = n/2 + 1$. Since n is even and $\gcd(\alpha, n) = 1$, α must be odd. So $n/2$ must be even. We have

$$\alpha^3 = (n/2 + 1)^3 = n^3/8 + 3n^2/4 + 3n/2 + 1$$
$$= (n/2)(n^2/4 + 3n/2 + 3) + 1.$$



u_0^0
u_1^0
U^0
u_5^0
u_6^0
$u_{5\alpha}^1$
U^1
$u_{5\alpha+1}^1$
u_α^2
$u_{1+\alpha}^2$
$u_{3+\alpha}^2$
U^2
$u_{4+\alpha}^2$
$u_{5+\alpha}^2$
$u_{6+\alpha}^2$
w_0^0
w_α^0
$w_{3\alpha}^0 = w_3^0$
W^0
$w_{4\alpha}^0 = w_{3+\alpha}^0$
$w_{5\alpha}^0$
$w_6^0 = w_{6\alpha}^0$
$w_{6+\alpha}^0$
w_5^1
W^1
$w_{5+\alpha}^1$
$w_{4\alpha+1}^2 = w_{4+\alpha}^2$
$w_{5\alpha+1}^2$
W^2
w_1^2
$w_{1+\alpha}^2$

Fig. 2.

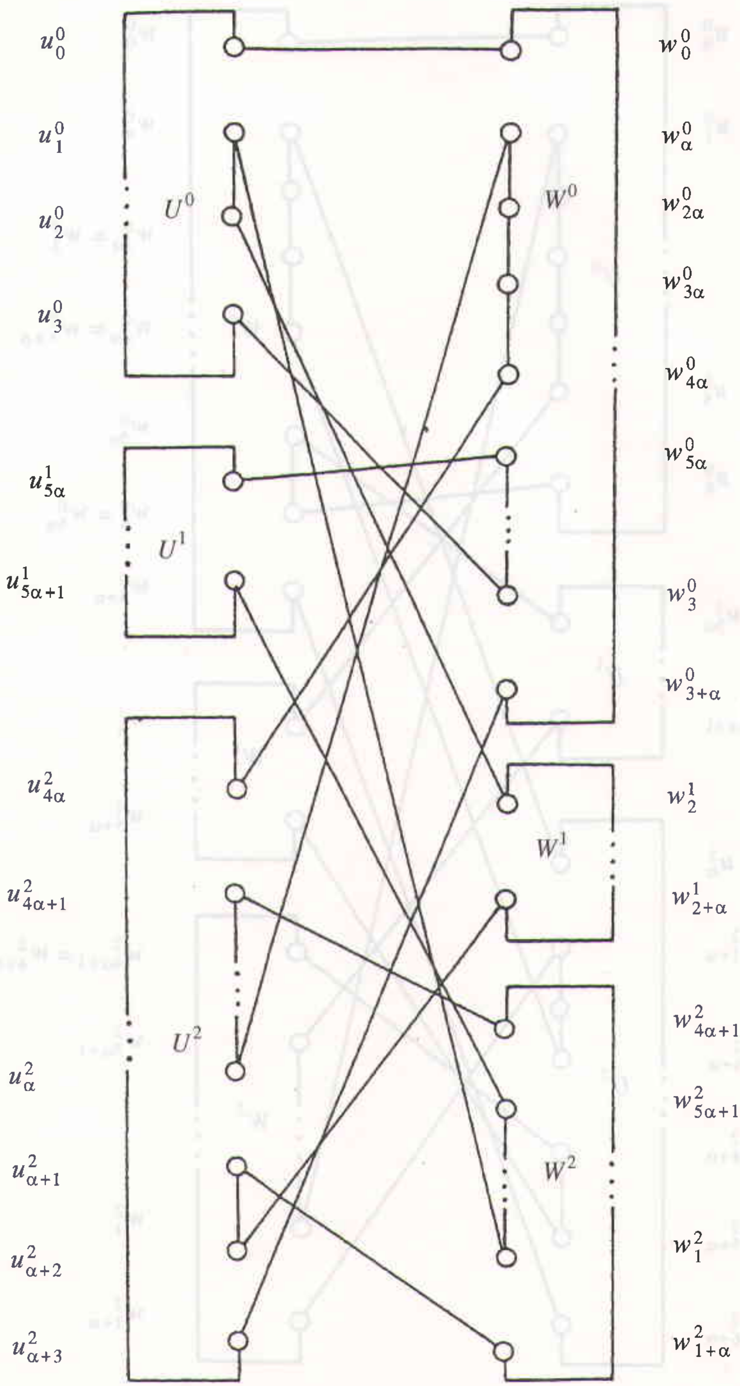

u^0_0
u^0_1
u^0_2
u^0_3
U^0
$u^1_{5\alpha}$
$u^1_{5\alpha+1}$
U^1
$u^2_{4\alpha}$
$u^2_{4\alpha+1}$
U^2
u^2_α
$u^2_{\alpha+1}$
$u^2_{\alpha+2}$
$u^2_{\alpha+3}$
w^0_0
w^0_α
W^0
$w^0_{2\alpha}$
$w^0_{3\alpha}$
$w^0_{4\alpha}$
$w^0_{5\alpha}$
w^0_3
$w^0_{3+\alpha}$
w^1_2
W^1
$w^1_{2+\alpha}$
$w^2_{4\alpha+1}$
$w^2_{5\alpha+1}$
W^2
w^2_1
$w^2_{1+\alpha}$

Fig. 1.

Since $n/2$ is even, $n^2/4+3n/2+3$ is odd. Hence, $\alpha^3 = (n/2)(n^2/4+3n/2+3)+1 \equiv n/2+1 \not\equiv 1 \pmod n$, contradicting again the fact that α has order 3. Thus, the possibility (iii) never occurs. This completes the proof of Theorem 2. ∎

4. Applications

In this section, we will use the results obtained in Sec. 3 in order to obtain a partial affirmative answer to the question: Do all connected cubic $(6, n)$-metacirculant graphs have a Hamilton cycle? Namely, we will prove the following result.

Theorem 3. *Let $G = MC(6, n, \alpha, S_0, S_1, S_2, S_3)$ be a connected cubic $(6, n)$-metacirculant graph. Then G possesses a Hamilton cycle if either $n = p^a q^b$, where p and q are distinct primes, $a \geq 0$ and $b \geq 0$ or n is such that $\varphi(n)$ is not divisible by 3, where $\varphi(n)$ is the number of integers z satisfying $0 \leq z < n$ and $\gcd(z, n) = 1$.*

Proof. Let $G = \mathrm{MC}(6, n, \alpha, S_0, S_1, S_2, S_3)$ be a connected cubic $(6, n)$-metacirculant graph. If $S_0 \neq \emptyset$, then by [7], G has a Hamilton cycle. Therefore, we may assume from now on that $S_0 = \emptyset$. Since G is a cubic $(6, n)$-metacirculant graph, only the following cases may happen:

Case 1. $|S_1| = 1$, $S_2 = \emptyset$ and $|S_3| = 1$.

Case 2. $S_1 = \emptyset$, $|S_2| = 1$ and $|S_3| = 1$.

Case 3. $S_1 = S_2 = \emptyset$ and $|S_3| = 3$.

Since G is connected, Case 3 does not occur. Further, since (3.3) and (3.5) hold, we have $\alpha^6 \equiv 1 \pmod n$. This means that the order of α in Z_n^* is a divisor of 6.

Assume first that $n = p^a q^b$, where p and q are distinct primes, $a \geq 0$ and $b \geq 0$. If either p or q is equal to 2, then by [2, 11], G has a Hamilton cycle. Therefore, we may assume $p \neq 2$ and $q \neq 2$. Since the order of α in Z_n^* is a divisor of 6, by [1], G is a Cayley graph of the group

$$\mathcal{G} = \langle \rho, \tau \,:\, \rho^n = \tau^6 = 1,\ \tau\rho\tau^{-1} = \rho^\alpha \rangle,$$

where ρ and τ are automorphisms of G with $\rho(v_j^i) = v_{j+1}^i$ and $\tau(v_j^i) = v_{\alpha j}^{i+1}$. If $\gcd(\alpha - 1, n) = 1$, then by Theorem 1, G has a Hamilton cycle. Since n is odd, we have $\gcd(\alpha^3 + 1, \alpha - 1, n) = 1$. Therefore, if $\gcd(\alpha - 1, n) \neq 1$, then (3.3) and (3.5) imply that $\gcd(\alpha - 1, n)$ is equal to either $p^a q^b$ or p^a or q^b. It is not difficult to verify that the commutator subgroup $[\mathcal{G}, \mathcal{G}]$ of $\mathcal{G}$ is the subgroup $\langle \rho^{\alpha-1} \rangle$ generated by $\rho^{\alpha-1}$. So, the order of $[\mathcal{G}, \mathcal{G}]$ is 1 or q^b or p^a depending on whether $\gcd(\alpha - 1, n)$ is equal to $p^a q^b$ or p^a or q^b. In any cases, by [6], G has a Hamilton cycle.

Assume now that n is such that $\varphi(n)$ is not divisible by 3, where $\varphi(n)$ is the number of integers z satisfying $0 \leq z < n$ and $\gcd(z, n) = 1$. Since $|Z_n^*| = \varphi(n)$ and the order of α in Z_n^* is a divisor of 6, our assumption implies that the order of α in Z_n^* is 1 or 2. By Theorem 2, G has a Hamilton cycle. This completes the proof of Theorem 3. ∎

References

1. B. Alspach and T. D. Parsons, A construction for vertex-transitive graphs, *Canad. J. Math.* **34** (1982) 307–318.
2. B. Alspach, Hamilton cycles in metacirculant graphs with prime power cardinal blocks, *Annals of Discrete Math.* **41** (1989) 7–16.
3. B. Alspach, Lifting Hamilton cycles of quotient graphs, *Discrete Math.* **78** (1989) 25–36.
4. B. Alspach and C. Q. Zhang, Hamilton cycles in cubic Cayley graphs on dihedral groups, *Ars Combin.* **28** (1989) 101–108.
5. K. Bannai, Hamilton cycles in generalized Petersen graphs, *J. Combin. Theory* **B24** (1978) 181–188.
6. K. Keating and D. Witte, On Hamilton cycles in Cayley graphs in groups with cyclic commutator subgroup, *Annals of Discrete Math.* **27** (1985) 89–102.
7. N. D. Tan, On cubic metacirculant graphs, *Acta Math. Vietnam.* **15**(2) (1990) 57–71.
8. N. D. Tan, Hamilton cycles in cubic (4,n)-metacirculant graphs, *Acta Math. Vietnam.* **17**(2) (1992) 83–93.
9. N. D. Tan, Connectedness of cubic metacirculant graphs, *Acta Math. Vietnam.* **18**(1) (1993) 3–17.
10. N. D. Tan, Hamilton cycles in cubic (m, n)-metacirculant graphs with m divisible by 4, *Graphs Combin.* **10** (1994) 67–73.
11. N. D. Tan, On Hamilton cycles in cubic (m, n)-metacirculant graphs, *Austral. J. Combin.* **8** (1993) 211–232.
12. N. D. Tan, Sufficient conditions for the existence of a Hamilton cycle in cubic (6,n)-metacirculant graphs, *Vietnam J. Math.* **25**(1) (1997) 41–52.