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Short Communication

Berry–Esseen Theorem for Stationary Strong Mixing Sequences

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Let (X_n), $n = 1, 2, ...,$ be an i.i.d. sequence of random variables with $EX_n = 0$ and $EX_n^2 = 1$ and let $B \in \sigma(X_1, X_2, ..., X_n, ...)$ with $P(B) > 0$. Denote

$$\Delta_n(B) = \sup_{t \in R} |P(S_n.n^{-1/2} < t|B) - \Phi(t)|.$$

Here, $\Phi(t)$ is a standard normal distribution function.

The classical theorem of Berry–Esseen gives an estimation of the rate of convergence to the normal law

$$\Delta_n(\Omega) = \sup_{t \in R} |P(S_n.n^{-1/2} < t) - \Phi(t)| = O(n^{-1/2}).$$

In [2], it was first shown that $\Delta_n(B) \to 0$ as $n \to \infty$ for arbitrary subset B of Ω with $P(B) > 0$.

This result (which is called the conditional central limit theorem) plays an important role in the theory of random summation, in problems of "random walk", in the sequential estimation, etc. Later, Landers and Rogge [1] proved that

$$\Delta_n(B) = O(n^{-1/2})$$

if $E|X_1|^p < \infty$ for some $p > 3$, and

$$\begin{aligned} d(B, \sigma(X_1, X_2, ..., X_n)) &= \inf\{P(B \Delta A) : A \in \sigma(X_1, X_2, ..., X_n)\} \\ &= O\Big(\frac{1}{n^{1/2}(\log n)^{3/2}}\Big). \end{aligned}$$

Using the method of differential equation, Stein has shown in [3] that if (X_n) is a stationary, uniform mixing (ϕ - mixing) sequence of random variables with $E|X_1|^8 < \infty$, then we have

$$\Delta_n(\Omega) = O(n^{-1/2}).$$

In extending the result of Stein to the case of stationary, strong mixing sequences of random variables, one unfortunately cannot obtain the same rate of convergence as in the uniform mixing case, as shown by the following:

Theorem 1. [4, p.636] *Let* (X_n), $n = 1, 2, ...$, *be a stationary, strong mixing sequence of random variables with mixing coefficient*

$$\varrho(n) < K.n^{-\theta}, \quad \theta > 0,$$

and

$$EX_1 = 0, \quad E|X_1|^s < \infty,$$

where

$$2 < s < s_0(\theta) = \frac{(\theta - 1)}{\theta} + \sqrt{\frac{(\theta - 1)^2}{\theta} + \frac{4 + 2\theta}{\theta}}. \tag{1}$$

If

$$ES_n^2 \geq \mu n EX_1^2, \quad \mu > 0,$$

then there exists a constant $C(s, \theta, K, \mu)$ *depending only on* $s, \theta, K,$ *and* μ *such that*

$$\Delta_n(\Omega) = \sup_{t \in R} |P(S_n.n^{-1/2} < t) - \Phi(t)| \leq C(s, \theta, K, \mu) \frac{\beta_s}{n^{\frac{s-2}{2}}}, \tag{2}$$

where $\beta_s = \dfrac{E|X_1|^s}{(EX_1^2)^{s/2}}$.

Note that from condition (1), we have

$$s < s_0(\theta) < 1 + \sqrt{3}.$$

This means that the best rate in (2) is

$$n^{-\frac{s-2}{2}} \quad (\leq n^{-0,36}).$$

In the theorem below, we obtain "the conditional Berry–Esseen estimation" for stationary, strong mixing sequences of random variables.

Theorem 2. *Let $(X_n)_{n\geq 1}$ be a strictly stationary, strong mixing sequence of random variables with mixing coefficient $\varrho(n) < K.n^{-\theta}$, where $K > 0$, $\theta > \frac{3}{2}$ and $EX_1 = 0$, $EX_1^2 = 1$, $E|X_1|^s < \infty$,*

$$2 < s < \min\left\{\frac{5}{2},\ s_0(\theta) = \frac{\theta-1}{\theta} + \sqrt{\frac{\theta-1}{\theta}^{2} + \frac{4+2\theta}{\theta}}\right\}. \tag{3}$$

Assume

$$ES_n^2 \geq \mu n E X_1^2, \quad \mu > 0\,.$$

Let $B \in \sigma(X_1, X_2, ..., X_n, ...)$ with $P(B) > 0$ such that

$$d(B, \sigma(X_1, X_2, ..., X_n)) = \inf\{P(A\Delta B) : A \in \sigma(X_1, ..., X_n)\}$$
$$= O\Big(\frac{1}{n^{\frac{1}{2}+\delta}(\log n)^r}\Big), \quad \delta > \frac{s-2}{s(4-s)}.$$

Then we have

$$\Delta_n(B) = \sup_{t\in R} |P(S_n(ES_n^2)^{-1/2} < t|B) - \Phi(t)| = O\Big(\frac{\log n}{n^{\frac{s-2}{2}}}\Big)\,.$$

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