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Short Communication

A Remark on Analytic Pseudodifferential Operators with Singularities

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1. Introduction

Let $\Omega \subset \mathbf{C}_\zeta$ be a Runge domain. We denote by $\mathcal{O}(\Omega)$ the space of all holomorphic functions on Ω. For a function $a(\zeta) \in \mathcal{O}(\Omega)$, replacing formally by D, Dubinskii [3] has defined an analytic pseudodifferential operator (an APO for short) $A(D)$ with symbol $a(\zeta) \in \mathcal{O}(\Omega)$ and constructed the algebra of APOs on Ω. He proved that every $A(D) \in \mathcal{A}(\Omega)$ acts continuously and invariantly in $\mathcal{E}xp_\Omega(\mathbf{C}_z)$, the space of exponential functions in $\mathbf{C}_z$ growing over Ω. So, if an APO $A(D) \in \mathcal{A}(\Omega)$ has the inverse $A^{-1}(D) \in \mathcal{A}(\Omega)$, then the analytic pseudodifferential equation

$$A(D)u(z) = v(z), \quad v(z) \in \mathcal{E}xp_\Omega(\mathbf{C}_z), \tag{1}$$

has a unique solution $u(z) = A^{-1}(D)v(z) \in \mathcal{E}xp_\Omega(\mathbf{C}_z)$. We remark that the requirement $a^{-1}(\zeta) \in \mathcal{O}(\Omega)$, which guarantees the existence of $A^{-1}(D)$ in $\mathcal{A}(\Omega)$, is very strong. This requirement leads to a loss of solutions.

The purpose of this paper is to introduce a class of APOs with pole-singularities in the one-dimensional case. We will show that every APO with poles is in fact a multivalued operator acting in the space of exponential functions. Its values are described by the geometry of the operator. We give a formula for them; roughly speaking, every value of an APO $A(D)$ with pole singularities can be represented as a sum of regular and singular parts.

We denote by $\mathcal{E}xp(\mathbf{C}_z)$ the space of all exponential functions of the variable z. Let $u(z) = \sum_{i=0}^{\infty} u_i z^i \in \mathcal{E}xp(\mathbf{C}_z)$ with type $r > 0$.

$$(r \stackrel{\text{def}}{=} \inf_{r'>0} \{r' : |u(z)| < \text{const}.e^{r'|z|}, \ \forall z \in \mathbf{C}_z\}).$$

The function $Bu(\zeta) = \sum_{i=1}^{\infty} \frac{i!u_i}{\zeta^{i+1}}$ is called the Borel transform of $u(z)$.

It is well known [2] that $Bu(\zeta)$ is a holomorphic function outside the disk $\{|\zeta| \leq r\}$ if r is of the type $u(z)$. We denote by $U \subset \mathbf{C}_\zeta$ the largest open set where $Bu(\zeta)$ can be holomorphically continued. It is clear that $U \supset \{|\zeta| > r\}$.

The set $\mathbf{C}_\zeta \setminus U$ is said to be the spectrum of $u(z)$ and is denoted by K_u.

2. APO with Pole Singularities

Let $\mathcal{O}(\mathbf{C}_\zeta)$ be the space of all holomorphic functions in $\mathbf{C}_\zeta$. For $g(\zeta) \in \mathcal{O}(\mathbf{C}_z)$, we set $V(g) = \{\zeta \in \mathbf{C}_\zeta : g(\zeta) = 0\}$.

We put $\mathcal{O}_p(\mathbf{C}_\zeta) = \Big\{a(\zeta) = \frac{f(\zeta)}{g(\zeta)} : f(\zeta), g(\zeta) \in \mathcal{O}(\mathbf{C}_\zeta), g(\zeta) \neq 0$ and $V(f) \cap V(g) = \emptyset\Big\}$ and call $\mathcal{O}_p(\mathbf{C}_\zeta)$ the space of symbols with pole singularities.

Let $a(\zeta) \in \mathcal{O}_p(\mathbf{C}_\zeta)$. Replacing formally ζ by D, we obtain $A(D)$.

Definition 1. *We call $A(D)$ an APO with pole singularity and $a(\zeta)$ its symbol, respectively. We denote by $\mathcal{A}_p(\mathbf{C}_\zeta)$ the set of all APOs with pole singularities.*

Definition 2. *We call $V(g)$ the singular set of $A(D)$ and denote it by $S(A)$.*

Definition 3. *We define*

$$u(z) = A(D)v(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{g(\zeta)} Bv(\zeta)\, e^{\zeta z}\, d\zeta,$$

where $\Gamma_v(\mathbf{C} \setminus S(A)) = \{$closed simple, oriented anticlockwise contours $\gamma \subset \mathbf{C} \setminus S(A)$ enclosing $K_v\}$ and $Bv(\zeta)$ is the Borel transform of $v(z)$.

Theorem 1. *$A(D)$ acts invariantly in $\mathcal{E}xp(\mathbf{C}_z)$ as a multivalued operator if $S(A) \neq \emptyset$.*

Theorem 2. *Let $A(D) \in \mathcal{A}_p(\mathbf{C}_\zeta)$, $v(z) \in \mathcal{E}xp(\mathbf{C}_z)$. If there is a Runge domain satisfying the following conditions:*

(i) $\Gamma_0 \subset \Omega$,

(ii) $\Omega \cap S(A) = \emptyset$,

(iii) $v(z) \in \mathcal{E}xp_\Omega(\mathbf{C}_z)$,

then the following representation holds:

$$u_\gamma(z) = u_{\Gamma_0}(z) + \sum_{j \in J_\gamma} u_{\Gamma_j}(z)$$

$$= \sum_{i=1}^{k} \sum_{j=0}^{\infty} a_j^i(\lambda_i)(D - \lambda_i I)^j v_i(z) + \sum_{j \in J_\gamma} \operatorname*{res}_{\zeta=\zeta_j} [a(\zeta)\, Bv(\zeta)\, e^{\zeta z}].$$

3. Analytic Pseudodifferential AP-Equations with Symbols in $\mathcal{O}(\mathbf{C}_\zeta)$

Let us consider the AP-equation

$$A(D)u(z) = v(z), \tag{2}$$

where $A(D) \in \mathcal{A}(\mathbf{C}_\zeta)$, $v(z) \in \mathcal{E}xp(\mathbf{C}_z)$.

Theorem 3. *Equation (2) has the solutions in the form*

$$u_\gamma(z) = A^{-1}(D)_\gamma v(z) = \frac{1}{2\pi i}\int_\gamma a^{-1}(\zeta) B v(\zeta) e^{\zeta z} d\zeta,$$

where $\gamma \in \Gamma_v(\mathbf{C} \setminus S(A^{-1}))$, $A^{-1}(D) \in \mathcal{A}_p(\mathbf{C}_\zeta)$.

Corollary. *If all hypotheses of Theorem 2 are satisfied for $A^{-1}(D)$ and $v(z)$, then every solution $u_\gamma(z)$ of (2) can be written in the form:*

$$u_\gamma(z) = \sum_{i=1}^{k}\sum_{j=0}^{\infty} a_j^i(\lambda_i)(D - \lambda_i I)^j v_i(z) + \sum_{j\in J_\gamma} \operatorname*{res}_{\zeta=\zeta_j}[a(\zeta)\, B v(\zeta)\, e^{\zeta z}],$$

where $a^{-1}(\zeta) = \sum_{j=0}^{\infty} a_j^i(\lambda_i)(\zeta - \lambda_i)^j$, $\lambda_i \in \Omega$, $i = 1, \ldots, k$ *and* $v(z) = \sum_{i=1}^{k} v_i(z)$.

Example. We consider the complex shift equation:

$$A(D)u(z) = u(z+a) + u(z-a) = h(z), \quad h(z) \in \mathcal{E}xp(\mathbf{C}_z), \quad 0 \neq a \in \mathbf{C}. \tag{3}$$

We will give a representation for the solutions of (2) by using the AP-operator with pole singularities. We have $A(D)u(z) = [e^{aD} + e^{-aD}]\, u(z) = 2\mathrm{ch}(aD)\, u(z)$.

Let $\Omega \subset \mathbf{C} \setminus S\Big(\dfrac{1}{2\mathrm{ch}(aD)}\Big)$ be a Runge domain such that $0 \in \Omega$, $\Gamma^0 \subset \Omega$ and $h(z) \in \mathcal{E}xp_\Omega(\mathbf{C}_z)$ and assume that the type of $h(z)$ is less than $\dfrac{\pi}{2|a|}$. Then using the Taylor series of the function $\dfrac{1}{2\mathrm{ch}(a\zeta)}$ at zero [1], we get

$$u_{\Gamma^0}(z) = \frac{1}{2}\sum_{n=0}^{\infty} \frac{(-1)^n\, a^{2n}\, E_{2n}}{(2n)!}\, D^{2n}\, h(z),$$

where E_{2n} are Euler's numbers ($E_0 = 1$, $E_2 = -1$, $E_4 = 5, \ldots$). Because $\zeta_k = \dfrac{\pi + 2k\pi}{2ia}$ are simple zeros of $\mathrm{ch}(a)$, by the construction of Γ_k, we have

$$u_{\Gamma_k}(z) = \frac{1}{2\pi i}\int_{\Gamma_k} \frac{Bh(\zeta)e^{\zeta z}}{2\mathrm{ch}(a\zeta)}\, d\zeta = \operatorname*{res}_{\zeta=\zeta_k}\left(\frac{Bh(\zeta)e^{\zeta z}}{2\mathrm{ch}(a\zeta)}\right)$$

$$= \frac{Bh\left(\frac{\pi+2k\pi}{2ia}\right) e^{\frac{\pi+2k\pi}{2ia} z}}{2\mathrm{ch}_k\left(\frac{\pi+2k\pi}{2ia}\right)} \quad \left(\text{here, } \mathrm{ch}_k(a\zeta) = \frac{\mathrm{ch}(a\zeta)}{\zeta - \zeta_k}\right).$$

Finally, we get the following formula for $u_\gamma(z)$:

$$u_\gamma(z) = \frac{1}{2}\sum_{n=0}^{\infty} \frac{(-1)^n\, a^{2n}\, E_{2n}}{(2n)!}\, D^{2n}\, h(z) + \sum_k \frac{Bh\left(\frac{\pi+2k\pi}{2ia}\right) e^{\frac{\pi+2k\pi}{2ia} z}}{2\mathrm{ch}_k\left(\frac{\pi+2k\pi}{2ia}\right)}.$$

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