Vietnam Journal of Mathematics **26**:1 (1998) 71–85

Vietnam Journal of MATHEMATICS

Convergence of the Rademacher Series in a Banach Space

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Received October 24, 1996

Abstract. Let T be a linear continuous operator from a Hilbert space H into a Banach space X. In this paper necessary and sufficient conditions for the random series $\sum_n r_n T e_n$ to be convergent almost surely (a.s.) in X are given, where (r_n) is the sequence of Rademacher random variables and (e_n) is a fixed orthonormal basis in H.

1. Introduction

Let H be a separable real Hilbert space with scalar product $\langle , \rangle$. Let (e_n) be an orthonormal basis in H; once chosen, fix it. Consider the formal random series:

$$\sum_n r_n(t)e_n \tag{1}$$

where $(r_n(t))$ are the Rademacher functions defined on $[0, 1]$, i.e.,

$$r_n(t) = \operatorname{sign}\sin(2^n \pi t), \ n = 1, 2, ..., \ t \in [0, 1].$$

Clearly, for each $h \in H$, the real random series

$$\sum_n r_n \langle e_n, h\rangle$$

converges a.s. since $\sum_n |\langle e_n, h\rangle|^2 < \infty$. However, if $\dim H = \infty$, the series (1) does not converge a.s. in H because all the Rademacher functions $(r_n(t))$ take only the two values $+1$ and -1 (with equal probability), so that for almost all $t \in [0, 1]$ we have

$$\sum_n |r_n(t)|^2 = \infty .$$

The aim of this paper is to give conditions such that the random series

$$\sum_n r_n(t) T e_n \tag{2}$$

converges a.s. in X where X is a real Banach space and T is a linear continuous operator from the Hilbert space H into the Banach space X. An analogous problem for the standard Gaussian or stable sequences was studied in [1–4, 6–9]. The above problem was also investigated in [4]. Here, first we point out that a statement in [4, p. 234] is not correct and we give a correction of it (see Theorem 5 below). Then, we give some necessary and sufficient conditions for the a.s. convergence of series (2). It should be noted that, on one side there are analogous results for the Gaussian and Rademacher cases with the same proofs (see Theorems 6 and 7), on the other side there are analogous results for these cases but with different proofs (see Theorems 5 and 8). Moreover, there are results which are true for the Gaussian case, but not true for the Rademacher one (see the remark after Theorem 4 below). For the completeness of the note, all the proofs are given in detail. The ideas of Theorems 6 and 7 are taken from [1].

2. Definitions

In this section we recall some basic facts about absolutely p-summing operators between Banach spaces and the definitions of type and cotype of Banach spaces.

Throughout this paper, probability space means the interval $[0, 1]$ with the Lebesgue measure. It is well known that in this probability space, the above Rademacher functions $(r_n(t))$ form a sequence of i.i.d. random variables taking the only two values $+1$ and -1 with equal probability.

Definition 1. *Let X, Y be two Banach spaces and T a linear continuous operator from X into Y. T is said to be an absolutely (q, p)-summing operator, $1 \leq p < \infty$, if $\sum_n \|Tx_n\|^q < \infty$ for any sequence (x_n) in X such that $\sum_n |\langle x_n, x^* \rangle|^p < \infty$, for all $x^* \in X^*$ (X^* is the dual space of X). T is said to be an absolutely p-summing operator if it is (p, p)-summing.*

Denote by $\Pi_p(X, Y)$ the class of an absolutely p-summing operators from X into Y. It is an easy consequence of the closed graph theorem that a linear continuous operator T from X into Y is an absolutely p-summing operator if and only if there exists a $\rho > 0$ such that, given any $n = 1, 2, \ldots$ and $x_1, x_2, \ldots, x_n \in X$, we have

$$\sum_{i=1}^{n} \|Tx_i\|^p \leq \rho^p \sup\Big\{\sum_{i=1}^{n} |\langle x_i, x^* \rangle|^p : \|x^*\| \leq 1\Big\}. \tag{3}$$

Put

$\pi_p(T) = \inf\{p > 0 : (3) \text{ holds for all } n = 1, 2, \ldots \text{ and for all } x_1, x_2, \ldots, x_n \in X\}.$

The following inequality is a fundamental result in the theory of p-summing operators.

Theorem 1. (Pietsch, see [16]) *Let $T \in \Pi_p(X, Y)$. Then there exists a regular Borel probability measure μ defined on the unit ball $(B_{X^*}, \sigma(X^*, X))$ for which*

$$\|Tx\| \leq \pi_p(T)\Big(\int_{B_{X^*}} |\langle x, x^*\rangle|^p \mu(dx^*)\Big)^{1/p} \tag{4}$$

holds for each $x \in X$, where $\sigma(X^*, X)$ is the Borel σ-algebra in the weak topology.*

It is well known that $\Pi_r(X, Y) \subset \Pi_p(X, Y)$, for $1 \leq r \leq p$.

Definition 2. *A Banach space X is said to be of type p $(1 \leq p \leq 2)$ if the random series*

$$\sum_n r_n x_n \tag{5}$$

converges a.s. for any sequence (x_n) in X with $\sum_n \|x_n\|^p < \infty$. A Banach space X is said to be of cotype q $(2 \leq q < \infty)$ if the a.s. convergence of series (5) implies that $\sum_n \|x_n\|^q < \infty$.

It is known that if a Banach space is of type p, then the dual space X^* is of cotype q with $1/p + 1/q = 1$.

Our objective is the following important results.

Theorem 2. (The Khinchin inequality) *For any $p, 0 < p < \infty$, there exist constants A_p and B_p such that*

$$A_p\Big(\sum_n |a_n|^2\Big)^{1/2} \leq \Big(\int_0^1 \Big|\sum_n r_n(t)a_n\Big|^p dt\Big)^{1/p} \leq B_p\Big(\sum_n |a_n|^2\Big)^{1/2}$$

for all finite sequences of real numbers (a_n).

It is known that $A_1 = \frac{1}{\sqrt{2}}$ (due to Szarek [4, pp. 227–228] for a short proof).

Theorem 3. [12, 13]

(i) *If X is of cotype 2 and Y is an arbitrary Banach space, then $\Pi_1(X, Y) = \Pi_p(X, Y)$ for all $1 \leq p < 2$.*

(ii) *If X is an arbitrary Banach space and Y is of cotype 2, then $\Pi_2(X, Y) = \Pi_p(X, Y)$ for all $2 \leq p < \infty$.*

(iii) *If X is of cotype q, $2 \leq q < \infty$ and Y is an arbitrary Banach space, then $\Pi_1(X, Y) = \Pi_r(X, Y)$ for all $1 \leq r < \frac{q}{q-1}$.*

Theorem 4. (See [14]) *The following are equivalent:*

(a) *X is of a finite cotype.*

(b) *If (x_n) and (y_n) are two arbitrary sequences in X such that $\sum_n r_n x_n$ converges a.s. in X and $\sum_n |\langle y_n, x^*\rangle|^2 \leq \sum_n |\langle x_n, x^*\rangle|^2$ for all $x^* \in X$, then $\sum_n r_n y_n$ converges a.s. in X as well.*

Remark. It is known (see [14]) that for any Banach space X, the following is always true: *If* (x_n) *and* (y_n) *are two arbitrary sequences in* X *such that* $\sum_n \gamma_n x_n$ *converges a.s. in* X *and* $\sum_n |\langle y_n, x^*\rangle|^2 \le \sum_n |\langle x_n, x^*\rangle|^2$ *for all* $x^* \in X$, *then* $\sum_n \gamma_n y_n$ *converges a.s. in* X *as well*, where (γ_n) is a standard Gaussian sequence.

3. Main Results

Let us begin with the following observation.

Proposition 1. *The following are equivalent:*
(a) *X is of a finite cotype.*
(b) *If T and U are two arbitrary operators from H into X such that the series (2) converges a.s. in X and $\|U^*x^*\| \le \|T^*x^*\|$ for all $x^* \in X^*$, then the series $\sum_n r_n U e_n$ also converges a.s. in X.*

Proof. (a) $\Rightarrow$ (b). Put $x_n = Te_n$, $y_n = Ue_n$ and note that these sequences hold the conditions of (b) in Theorem 4. It follows that the series $\sum_n r_n U e_n$ also converges a.s. in X.

(b) $\Rightarrow$ (a). Let (x_n) and (y_n) be two arbitrary sequences in X such that $\sum_n r_n x_n$ converges a.s. in X and $\sum_n |\langle y_n, x^*\rangle|^2 \le \sum_n |\langle x_n, x^*\rangle|^2$ for all $x^* \in X^*$.
Consider

$$T^*x^* = \sum_n \langle x_n, x^*\rangle e_n,$$
$$U^*x^* = \sum_n \langle y_n, x^*\rangle e_n$$

and note that these operators are from X^* into H and are the adjoint operators, respectively, of the operators T, U from H into X defined by the formulas:

$$Th = \sum_n \langle h, e_n\rangle x_n,$$
$$Uh = \sum_n \langle h, e_n\rangle y_n .$$

Clearly, the series (2) converges a.s. in X and $\|U^*x^*\| \le \|T^*x^*\|$ for all $x^* \in X^*$, so the series $\sum_n r_n y_n = \sum_n r_n U e_n$ also converges a.s. in X. This ends the proof by Theorem 4. ■

Proposition 2. *The following are equivalent:*
(a) *X is of a finite cotype.*
(b) *For any T the series (2) converges a.s. in X if and only if $\sum_n r_n T h_n$ converges a.s. in X for all sequences (h_n) in H with $\sum_n |\langle h_n, h\rangle|^2 < \infty$ for all $h \in H$.*

Proof. (a) $\Rightarrow$ (b). Let the series (2) converge a.s. and (h_n) be a sequence in H with $\sum_n |\langle h_n, h\rangle|^2 < \infty$ for all $h \in H$. Consider the following operator:

$$B : H \to H$$

$$Bh = \sum_n \langle h_n, h \rangle e_n .$$

Clearly, B is a linear continuous operator from H into H. Moreover,

$$\langle h, B^* h^* \rangle = \langle Bh, h^* \rangle = \sum_n \langle h_n, h \rangle \langle e_n, h^* \rangle .$$

In particular,

$$\langle h, B^* e_n \rangle = \langle h, h_n \rangle, \ \forall h \in H.$$

It implies that

$$B^* e_n = h_n, \ \forall n = 1, 2, \ldots .$$

Next, consider the random series

$$\sum_n r_n T h_n = \sum_n r_n T B^* e_n . \tag{6}$$

Observe that

$$\|BT^* x^*\| \leq \|B\| \, \|T^* x^*\|.$$

By Proposition 1, the series (6) is convergent a.s.

(b) $\Rightarrow$ (a). Let (x_n) and (y_n) be two arbitrary sequences in X such that $\sum_n r_n x_n$ converges a.s. in X and $\sum_n |\langle y_n, x^* \rangle|^2 \leq \sum_n |\langle x_n, x^* \rangle|^2$ for all $x^* \in X$. Let T and U be the operators defined in the proof of Proposition 1. Let $[M]$ be the closure of $M = T^* X^*$ in the Hilbert space H. Clearly, $[M]$ is also a Hilbert space. Consider now the following operator:

$$V : M \to H$$

$$V(T^* x^*) = U^* x^* .$$

It is easy to see that

$$\|V(T^* x^* - T^* y^*)\| = \|U^*(x^* - y^*)\| \leq \|T^* x^* - T^* y^*\|.$$

This shows that V is well defined and is a linear continuous operator from M into H. Hence, V can be extended into a linear continuous operator from $[M]$ into H. We have $U^* = V T^*$, so $U = T V^*$ where V^* is an operator from H into $[M]$. Choose $h_n = V^* e_n$, $n = 1, 2, \ldots$ and note that

$$h_n \in [M], \ n = 1, 2, \ldots$$

$$y_n = U e_n = T(V^* e_n) = T h_n, \ n = 1, 2, \ldots$$

$$\sum_n |\langle h_n, T^* x^* \rangle|^2 = \sum_n |\langle T h_n, x^* \rangle|^2 = \sum_n |\langle y_n, x^* \rangle|^2 = \|U^* x^*\|^2 \leq \|T^* x^*\|^2 .$$

This implies that $\sum_n |\langle h_n, h \rangle|^2 < \infty$ for all $h \in H$. By assumption, the series

$$\sum_n r_n T h_n = \sum_n r_n y_n$$

converges a.s. and this ends the proof. ∎

Remark. In [4] an operator T from H into X is called *almost summing* if the random series $\sum_n r_n T h_n$ converges a.s. in X for all sequences (h_n) in H with $\sum_n |\langle h_n, h\rangle|^2 < \infty$ for all $h \in H$. The above proposition shows that the following statements are equivalent:
(a) X is of a finite cotype.
(b) The series (2) converges a.s. in X if and only if T is almost summing.

It should be noted that

Theorem 5. *The following are equivalent:*
(a) *X contains no subspace isomorphic to c_0.*
(b) *Let T be an arbitrary operator from H into X. T is almost summing if and only if there is a number $K > 0$ such that for all $n = 1, 2, \ldots$ and for all $h_1, h_2, \ldots, h_n \in H$, the following inequality holds true:*

$$\mathbf{E}\Big\|\sum_{k=1}^{n} r_k T h_k\Big\|^2 \leq K^2 \sup\Big\{\sum_{k=1}^{n} |\langle h_k, h\rangle|^2 \,\Big|\, \|h\| \leq 1\Big\}.$$

Proof. (a) $\Rightarrow$ (b) is a consequence of Hoffman–Kwapien's Theorem: if X contains no subspace isomorphic to c_0, then the boundedness and the a.s. convergence of the partial sums of the series (2) are equivalent (see [20, Theorem 6.1, pp. 347–348]).

(b) $\Rightarrow$ (a). Let X be a Banach space which contains a subspace isomorphic to c_0. Then there is a sequence (x_n) in X such that:

$$P\Big(\sup_n \Big\|\sum_{k=1}^{n} \gamma_k x_k\Big\| < \infty\Big) = 1, \tag{7}$$

where (γ_n) is a standard Gaussian sequence, and the series

$$\sum_{k=1}^{\infty} r_k x_k \tag{8}$$

(see the proof of Proposition 6.1 in [20, p. 352]; see also [19]).

Consider now the operator $T : H \to X$ defined by the formula $T e_k = x_k$, $k \in N$. By (7), we have

$$P\Big(\sup_n \Big\|\sum_{k=1}^{n} \gamma_k T e_k\Big\| < \infty\Big) = 1. \tag{9}$$

Let (h_n) be a sequence in H such that

$$\sum_{k=1}^{n} |\langle h_k, h\rangle|^2 < \infty$$

for all $h \in H$. Then we know that

$$\sup_{\|h\|\leq 1} \sum_{k=1}^{n} |\langle h_k, h\rangle|^2 < \infty\,. \tag{10}$$

Hence

$$\sum_{k=1}^{n} |\langle Th_k, x^*\rangle|^2 \le \sup_{\|h\|\le 1} \sum_{k=1}^{n} |\langle h_k, h\rangle|^2 \sum_{k=1}^{n} |\langle Te_k, x^*\rangle|^2$$

for all $x^* \in X^*$. This, along with (9), implies that (see [20, Proposition 2.5, p. 273])

$$P\Big(\sup_n \|\sum_{k=1}^{n} \gamma_k T h_k\| < \infty\Big) = 1.$$

This implies (see [18, Corollary 2, p. 287]) that

$$\sup_n \mathbf{E}\|\sum_{k=1}^{n} \gamma_k T h_k\|^2 < \infty. \tag{11}$$

Hence, we have (10) $\Rightarrow$ (11). By the closed graph theorem, there exists a number $K > 0$ such that

$$\sup_n \mathbf{E}\|\sum_{k=1}^{n} \gamma_k T h_k\|^2 \le K^2 \sup_{\|h\|\le 1} \sum_{k=1}^{n} |\langle h_k, h\rangle|^2.$$

Meanwhile, by (8),

$$\sum_{k=1}^{\infty} r_k T e_k$$

diverges a.s. ■

Remark. It was stated wrongly in [4] that (b) is true for any Banach space.

We now give a necessary condition for the a.s. convergence of the series (2).

Lemma 1. *If the series (2) is convergent a.s. in X, then the adjoint operator T^* is absolutely 1-summing from X^* into H.*

Proof. Suppose that the series (2) is convergent a.s. in X. Denote by S its sum: $S = \sum_n r_n T e_n$. In this case S is an X-valued random variable. It is known that $\|S\| \in L_p$, for $p > 0$, in particular, $\mathbf{E}\|S\| = \int_0^1 \|S(t)\| dt < \infty$. By the definition of the adjoint operator, we have

$$T^* : X^* \to H$$

$$\langle h, T^* x^*\rangle = \langle Th, x^*\rangle, \ h \in H, \ x^* \in X^*.$$

Consequently, we obtain that

$$\|T^* x^*\|^2 = \sum_n |\langle e_n, T^* x^*\rangle|^2 = \sum_n |\langle Te_n, x^*\rangle|^2 = \mathbf{E}|\langle S, x^*\rangle|^2.$$

From the Khinchin inequality, it follows that

$$\sum_{i=1}^{n} \|T^* x_i^*\| = \sum_{i=1}^{n} \sqrt{\mathbf{E}|\langle S, x_i^*\rangle|^2} \le \sqrt{2} \sum_{i=1}^{n} \mathbf{E}|\langle S, x_i^*\rangle|$$

$$= \sqrt{2} \sum_{i=1}^{n} \mathbf{E}\Big|\Big\langle \frac{S}{\|S\|}, x_i^*\Big\rangle\Big| \, \|S\| \le \sqrt{2}\mathbf{E}\|S\| \sup\Big\{\sum_{i=1}^{n} |\langle x, x_i^*\rangle| : \|x\| \le 1\Big\}.$$

By (3), we have that $T^* \in \pi_1(X^*, H)$ and $\Pi_1(T^*) \le \sqrt{2}\mathbf{E}\|S\|$. ■

We now give a sufficient condition for the a.s. convergence of the series (2).

Lemma 2. *If for some $1 \le p < \infty$, T is an absolutely p-summing operator from H into X, then the series (2) converges a.s. in X.*

Proof. Putting

$$S_n = \sum_{i=1}^{n} r_i T e_i\,,$$

$$h_n = \sum_{i=1}^{n} r_i e_i\,,$$

we have

$$\|S_m - S_n\|^p = \|T(h_m - h_n)\|^p.$$

If $T \in \Pi_p(H, X)$, by Pietsch's Theorem 1 (inequality (4)), we get:

$$\|T(h_m - h_n)\|^p \le \pi_p^p(T) \int_{B_H} |\langle h_m - h_n, h\rangle|^p \mu(dh).$$

Consequently, by the Khinchin inequality, there is a number $C_p > 0$ such that

$$\mathbf{E}\|T(h_m - h_n)\|^p \le \pi_p^p(T) \int_{B_H} \mathbf{E}|\langle h_m - h_n, h\rangle|^p \mu(dh)$$
$$\le C_p \pi_p^p(T) \int_{B_H} \Big(\sum_{k=n+1}^{m} |\langle e_k, h\rangle|^2 \Big)^{p/2} \mu(dh).$$

It shows that (S_n) is a Cauchy sequence in the Banach space $L_p(X)$, so that the series (2) (as the sum of independent X-valued random variables) converges a.s. in X.

Lemma 3. *Let X be a Hilbert space K. The series (2) converges a.s. in K if and only if T is a Hilbert–Schmidt operator.*

Proof. Since (r_n) is an orthogonal sequence, we have

$$\mathbf{E}\Big\|\sum_n r_n T e_n\Big\|^2 = \sum_n \|T e_n\|^2.$$

This completes the proof. ■

By Lemmas 1, 2 and 3 it is easy to get the following interesting result of [15].

Corollary 1. *Let H, K be Hilbert spaces. Then any class $\Pi_p(H, K)$, $1 \le p < \infty$, coincides with the class of Hilbert–Schmidt operators.*

Remark. Using Lemma 3, we can prove Lemma 2 as follows. Since any Hilbert space is of cotype 2, we have $\Pi_2(H, X) = \Pi_p(H, X)$ for all $1 \le p < \infty$ by Maurey's Theorem 3. On the other hand, it is known (see [16]) that if $T \in \Pi_2(H, X)$, then T admits a factorization $T = BA$ where A is a Hilbert–Schmidt operator from H into a Hilbert space K and B is a continuous operator from K into X. This ends the proof.

Theorem 6. *The following statements are equivalent:*
(a) *X is of cotype 2.*
(b) *The series* (2) *converges a.s. in X if and only if T is absolutely 2-summing from H into X.*

Proof. (a) ⇒ (b). By Lemma 2, it is sufficient to show that if (2) converges a.s. in X, then $T \in \Pi_2(H, X)$. Indeed, let (h_n) be any sequence in H with $\sum_n |\langle h_n, h\rangle|^2 < \infty$ for all $h \in H$. Consider the following operator:

$$B : H \to H$$

$$Bh = \sum_n \langle h_n, h\rangle e_n \, .$$

By the proof of (a) ⇒ (b) in Proposition 2, the series (6) converges a.s. This implies that

$$\sum_n \|Th_n\|^2 < \infty$$

(since X is of cotype 2), i.e., T is absolutely 2-summing.

(b) ⇒ (a). Suppose (x_n) is an arbitrary sequence in X such that $\sum_n r_n x_n$ converges a.s. in X. To prove that X is of cotype 2 we must show that $\sum_n \|x_n\|^2 < \infty$. Note that

$$\sum_n |\langle x_n, x^*\rangle|^2 < \infty$$

for all $x^* \in X^*$ because of the a.s. convergence of the series

$$\sum_n r_n \langle x_n, x^*\rangle.$$

Consequently, we can define the following operator:

$$T : X^* \to H$$

$$T^* x^* = \sum_n \langle x_n, x^*\rangle e_n \, .$$

T^* is the adjoint operator of the operator T which is defined as follows:

$$T : H \to X$$

$$Th = \sum_n \langle e_n, h\rangle x_n \, .$$

In particular, we have $Te_n = x_n$, so that

$$\sum_n r_n T e_n$$

converges a.s. By (b), T is absolutely 2-summing. This implies that

$$\sum_n \|Te_n\|^2 = \sum_n \|x_n\|^2 < \infty,$$

since

$$\sum_n |\langle e_n, h\rangle|^2 < \infty, \ \forall h \in H.$$

■

As an immediate consequence of Maurey's Theorem 3, $\Pi_2(X, H) = \Pi_p(X, H)$ for all $1 \leq p < \infty$ (since X is of cotype 2 and any Hilbert space is also of cotype 2), we have the following result.

Corollary 2. *Let X be a Banach space of cotype 2. The series (2) converges a.s. in X if and only if T is p-summing for some* $1 \leq p < \infty$.

The same proof of Theorem 6 gives

Corollary 3. (See also [4]) *The following are equivalent:*
(a) *X if of cotype q,* $2 \leq q < \infty$.
(b) *If the series (2) converges a.s. in X, then T is* $(q, 2)$*-summing.*

Remark 1. Corollary 3 was proved in [4] (Proposition 12.6, p. 235 and Proposition 12.29, p. 251) for almost summing operators. Consequently, by Proposition 2, this fact is an easy corollary of the mentioned Proposition 12.29 in [4]. Here, we give a direct proof for the sake of completeness of the paper.

Remark 2. The difference between Theorem 6 and Corollary 3 is that for $2 < q < \infty$ the reverse to the statement in (b) of Corollary 3 is not true even for Hilbert spaces. In fact, let X be a Hilbert space H. In this case, it is known that X is of cotype q for any $q, 2 \leq q < \infty$ and $\Pi_{q,2}(H, H)$ (the class of absolutely $(q, 2)$-summing operators from H into H) and $S_q(H, H)$ (the Schatten class) coincide (see [4, Theorem 10.3, p. 198]). On the other hand, it is also known that the series (2) converges a.s. in X if and only if $T \in S_2(H, H)$ (the class of Hilbert–Schmidt operators). If $\dim H = \infty$, $S_2(H, H)$ is a subclass of $S_q(H, H)$ for any $q, 2 < q < \infty$.

In the case of type 2 we have the following result:

Theorem 7. *The following statements are equivalent:*
(a) *X is of type 2.*
(b) *The series (2) converges a.s. in X if and only if* T^* *is absolutely 2-summing from* X^* *into H.*

Proof. (a) ⇒ (b). By Lemma 1, it remains to show that if $T^* \in \Pi_2(X^*, H)$, then the series (2) converges a.s. in X. In fact, it is known (see [1]) that in this case the random series

$$\sum_n \gamma_n T e_n$$

converges a.s. in X, where (γ_n) is a sequence of i.i.d. real Gaussian variables with characteristic function $e^{-t^2/2}$. On the other hand, it is well known that the convergence of the latter series always implies the convergence of the series (2).

Note that if X is a GL-space (see Definition 3 below), then (a) $\Rightarrow$ (b) can be proved in the following way. By Maurey's Theorem 3 we have $\Pi_1(X^*, H) = \Pi_2(X^*, H)$ as X is of type 2, so X^* is of cotype 2. It is known (see [18]), moreover, that T^* is 1-summing, then there is a number $1 \leq p < \infty$ such that T is p-summing. By Lemma 2, the series (2) converges a.s. (also see Theorem 8 below).

(b) $\Rightarrow$ (a). Assume that (x_n) is any sequence in X with $\sum_n \|x_n\|^2 < \infty$. We have to show that the series

$$\sum_n r_n x_n \tag{12}$$

converges a.s. in X. To this end, consider the following operator:

$$T : H \to X$$

$$Th = \sum_n \langle e_n, h\rangle x_n .$$

This operator is well defined since

$$\sum_n |\langle e_n, h| \, \|x_n\| \leq \Big(\sum_n |\langle e_n, h\rangle|^2\Big)^{1/2} \Big(\sum_n \|x_n\|^2\Big)^{1/2} .$$

The adjoint operator of T is defined as follows:

$$T^* : X^* \to H$$

$$T^* x^* = \sum_n \langle x_n, x^*\rangle e_n .$$

We next show that $T^* \in \Pi_2(X^*, H)$. In fact let (x_i^*) be any sequence in X^* with $\sum_i |\langle x, x_i^*\rangle|^2 < \infty$. By the closed graph theorem, the linear mapping

$$A : X \to H$$

$$Ax = \sum_i x, \langle x_i^*\rangle e_i$$

from X into H is bounded. It follows that

$$\sum_i |\langle x, x_i^*\rangle|^2 \leq \|A\|^2 \|x\|^2 .$$

Consequently, we have

$$\sum_i \|T^* x_i^*\|^2 = \sum_n \sum_i |\langle x_n, x_i^*\rangle|^2 \leq \|A\|^2 \sum_n \|x_n\|^2 < \infty$$

i.e., $T^* \in \Pi_2(X^*, H)$. By (b), it implies that $\sum_n r_n T e_n$ converges a.s. in X, and hence, so does the series (12) since $T e_n = x_n$. ■

To get a more general result we need the following notion.

Definition 3. (See [4, 18]) *A Banach space X is said to be a GL-space if each 1-summing operator A from X into l_2 is 1-factorable, i.e., $A = CB$ where B is a continuous operator from X into some L_1 and C is a continuous operator from L_1 into l_2.*

It is known that any Banach space with a local unconditional structure is a GL-space (see [4]). In particular, Banach spaces with Schauder unconditional basis and Banach lattices are examples of GL-spaces. It is also shown that a Banach space X is also a GL-space if and only if X^* is a Banach space.

Theorem 8. *Let X be a Banach space, H a separable Hilbert space and (c_n) an arbitrary fixed orthonormal basis of H. Consider the following statements:*

(a) *X is of a finite cotype.*

(b) *For a continuous linear operator $T : H \to X$ the random series*

$$\sum_k r_k T e_k$$

converges a.s. in X if and only if T^* is 1-summing.

Then (b) $\Rightarrow$ (a) is always true, and in addition, if X is a GL-space, (a) $\Rightarrow$ (b) is also true.

Proof. (a) $\Rightarrow$ (b). If T^* is 1-summing, then T^* is 1-factorable (since X is a GL-space). So T factorizes through L_∞. As X is of a finite cotype, by a result of Maurey (see [12]), there is a number $1 \le p < \infty$ such that T is p-summing. This ends the proof (a) $\Rightarrow$ (b) by Lemma 2.

(b) $\Rightarrow$ (a). We will make use of Proposition 1 and Lemma 1 to prove this implication. To do it, let T and U be two arbitrary operators from H into X such that the series (2) converges a.s. in X and

$$\|U^*\| \le \|T^* x^*\| \text{ for all } x^* \in X^*.$$

By Lemma 1, T^* is 1-summing. It follows immediately that U^* is also 1-summing. Therefore, (b) implies that the random series $\sum_n r_n U e_n$ also converges a.s. in X. By Proposition 1, this ends the proof. ∎

Remark. For the Gaussian case, the implication (b) $\Rightarrow$ (a) is proved more easily (see [1]).

Again by Maurey's Theorem 3 we obtain

Corollary 4. *Let X be a GL-space.*

(i) *If X is a Banach space of type 2, then the series (2) converges a.s. in X if and only if T^* is 1-summing.*

(ii) *If X is a Banach space of type p, $1 < p \le 2$, then the series (2) a.s. converges in X if T^* is r-summing for $1 \le r < p$.*

4. Examples

In this section we consider two important cases: X is an $L_p(\Omega, \mu)$ or X is a Banach space with a Schauder unconditional basis.

We begin with the following result which is an easy consequence of the Khinchin inequality (see Theorem 2).

Proposition 3. *Let T be an operator from H into $L_p(\Omega, \mu)$. Then the series (2) converges a.s. if and only if the following condition holds:*

$$\Big(\sum_n |Te_n|^2\Big)^{1/2} \in L_p(\Omega, \mu),$$

this means

$$\int_\Omega \Big(\sum_n |Te_n(\omega)|^2\Big)^{1/2} \mu(d\omega) < \infty.$$

Recall that a Banach space $X \subset L_0(\Omega, \mu)$ is a lattice if
(i) the inclusion of X into L_0 is continuous.
(ii) If $f, g \in L_0$, $|f| \leq |g|$ a.s. and $g \in X$, then $f \in X$ and $\|f\|_X \leq \|g\|_X$.

Proposition 4. *Let X be a Banach lattice of $L_0(\Omega, \mu)$ and X a finite cotype. The series (2) converges a.s. in X if and only if*

$$\Big(\sum_n |Te_n(\omega)|^2\Big)^{1/2} \in X.$$

This result is an immediate consequence of the so-called Khinchin–Maurey inequality (see [11]). There exists a number $C > 0$ such that for any finite collection $x_1, \ldots, x_n \in X$

$$\frac{1}{\sqrt{2}}\Big\|\Big(\sum_{i=1}^n |x_i|^2\Big)^{1/2}\Big\|_X \leq \Big(\int_0^1 \Big\|\sum_{i=1}^n r_i x_i\Big\|_X^2 dt\Big)^{1/2} \leq C\Big\|\Big(\sum_{i=1}^n |x_i|^2\Big)^{1/2}\Big\|_X.$$

Remark. For the Gaussian case, a statement similar to that of the above proposition was given in [5].

We now proceed to the case of the Banach space with a Schauder unconditional basis (a_n). Let (a_n^*) be the sequence of coordinate functionals associated with the above basis. The proof of the following Theorem 9 is taken from [1].

Theorem 9. *Let X be as above.*
(i) *If the series (2) converges a.s. in X, then the following series*

$$\sum_{n=1}^\infty \|T^* a_n^*\|_H a_n \tag{13}$$

converges in the norm of X.
(ii) *The following are equivalent:*
  (a) *X is of a finite cotype.*
  (b) *The convergence of the series (13) implies the a.s. convergence of the series (2).*

Proof. (i) Assume that the series (2) converges and S is its sum. Clearly, the series

$$S = \sum_{n=1}^{\infty} \langle S, a_n^* \rangle a_n$$

is unconditionally convergent. Put

$$Z = \sum_{n=1}^{\infty} |\langle S, a_n^* \rangle| a_n .$$

Noting that $\mathbf{E}\|S\| < \infty$ and $\|Z\| < K\|S\|$ for some constant $K > 0$, we obtain that the series

$$\mathbf{E}Z = \sum_{n=1}^{\infty} \mathbf{E}|\langle S, a_n^* \rangle| a_n \tag{14}$$

is unconditionally convergent in the norm of X.

On the other hand, by the Khinchin inequality, we have

$$\|T^* a_n^*\|_H = \left(\mathbf{E}|\langle S, a_n^* \rangle|^2\right)^{1/2} \leq \sqrt{2}\mathbf{E}|\langle S, a_n^* \rangle|.$$

This together with the unconditional convergence of the series (14) implies that the series (13) is also convergent in the norm of X (see [10]).

(ii) Suppose now that the series (13) is convergent. Since the basis (a_n) is unconditional, for any $x^* \in X^*$, we obtain

$$\sum_n |\langle a_n, x^* \rangle| \, \|T^* a_n^*\|_H < \infty.$$

Define the operators $A : X^* \to l_1$ and $B : l_1 \to H$ by putting

$$Ax^* = (\langle a_n, x^* \rangle \|T^* a_n^*\|_H) \ \ x^* \in X^*,$$

$$B(\lambda_n) = \sum_n \lambda_n T^* a_n^* / \|T^* a_n^*\|_H \ \ (\lambda_n) \in l_1 .$$

It is easy to verify that A and B are continuous. This implies that B is 1-summing (by the Grothendieck Theorem), so T^* is 1-summing. By Theorem 8, the series (2) is convergent a.s., so (a) $\Rightarrow$ (b).

(b) $\Rightarrow$ (a) is an immediate consequence of Proposition 1. In fact, let T, U be arbitrary operators from X into H such that the series (2) is a.s. convergent and

$$\|U^* x^*\| \leq \|T * *x^*\| \text{ for all } x^* \in X^*.$$

By (i), the series (13) converges in the norm of X. As (a_n) is an unconditional basis in X, it follows that the series

$$\sum_n \|U^* a_n^*\| a_n$$

is also convergent in the norm of X. By (b), the series

$$\sum_n r_n U e_n$$

converges a.s. and this ends the proof of the theorem. ■

Acknowledgement. The authors are very grateful to Professors M. J. Chasco, S. Chobanyan, Dang Hung Thang, V. Tarieladze, and O. Blasco for many useful discussions during the preparation of this paper. In particular, the authors are indebted to Professor O. Blasco for [4].

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