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Subdifferential Characterization of Quasiconvex and Convex Vector Functions

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Abstract. A new subdifferential of a C-lower semicontinuous vector function f from a Banach space X into R^m is defined, where $C \subseteq R^m$ is a cone generated by m linearly independent vectors. Some of its properties are shown. Especially, f is C-quasiconvex (resp. C-convex) if and only if its subdifferential is C-quasimonotone (resp. C-monotone).

1. Introduction

The problem of characterizing various classes of functions in terms of their local approximations has been studied intensively. Some new results are presented in [3–6, 8, 10] where lower semicontinuous convex, quasiconvex or pseudoconvex functions have been characterized via their Frechet derivatives [5], Clarke subdifferentials [3, 4, 6], upper and lower Dini derivatives [6, 8] or lower Dini–Hadamard derivatives [10]. Especially in [11, 12], the authors have shown necessary and sufficient conditions for a set-valued map F between Banach spaces $X$and Y to be convex and quasiconvex with respect to a convex cone $C \subseteq Y$. These conditions are written in terms of the Bouligand and Clarke derivatives of the map $\hat{F}(.) := F(.) + C$.

The aim of this paper is to characterize C-lower semicontinuous quasiconvex and convex vector functions from a Banach space X into R^m in terms of their generalized subdifferentials, where $C \subseteq R^m$ is a cone generated by m linearly independent vectors.

The paper is structured as follows. In the next section, we introduce some preliminaries. In Sec. 3, after introducing the concept of generalized subdifferentials of C-lower semicontinuous vector functions, we shall prove some of their basic properties. Section 4 is devoted to proving the equivalence between the quasiconvexity (resp. convexity) of C-lower semicontinuous vector functions and quasimonotonicity (resp. monotonicity) of their generalized subdifferentials.

2. Preliminaries

Let $C \subseteq R^m$ be a cone generated by m linearly independent vectors $c_1, c_2, ..., c_m$. Denote by $\overline{R}$ the set $R \cup \{-\infty, +\infty\}$ and by $\overline{R}^m$ the set $\{\sum_{i=1}^{m} \alpha_i c_i : \alpha_i \in \overline{R}, i = 1, 2, ..., m\}$. Define on $\overline{R}^m$ a partial order "$\preceq$" as follows. For every $x, y \in \overline{R}^m$, $x = \sum_{i=1}^{m} \alpha_i c_i$, $y = \sum_{i=1}^{m} \beta_i c_i$,

$$x \preceq y \text{ if } \alpha_i \leq \beta_i, \ i = 1, 2, ..., m.$$

It is clear that if $x, y \in R^m$, then

$$x \preceq y \text{ iff } y - x \in C.$$

Denote by pr_i the projection

$$pr_i : \sum_{i=1}^{m} \alpha_i c_i \in R^m \mapsto \alpha_i \in R.$$

Lemma 1. *Let A be a nonempty subset of $\overline{R}^m$. Then*

(a) $\inf A = \sum_{i=1}^{m} \inf(pr_i(A))c_i$.

Particularly, if $A \cap R^m \neq \emptyset$ and A is bounded below by an element of R^m then $\inf A \in R^m$.

(b) $\sup A = \sum_{i=1}^{m} \sup(pr_i(A))c_i$.

Particularly, if $A \cap R^m \neq \emptyset$ and A is bounded above by an element of R^m, then $\sup A \in R^m$.

Proof. (a) Let $x \in A$ be arbitrary. Represent x as $x = \sum_{i=1}^{m} \alpha_i c_i$, for some $\alpha_i \in \overline{R}$. It is clear that $\alpha_i \geq \inf(pr_i(A))$, $i = 1, 2, ..., m$. Then $x \succeq \sum_{i=1}^{m} \inf(pr_i(A))c_i$. Hence, $\sum_{i=1}^{m} \inf(pr_i(A))c_i$ is a lower bound of A. Now, let a be an arbitrary lower bound of A. Represent a as $a = \sum_{i=1}^{m} \alpha_i c_i$, for some $\alpha_i \in \overline{R}$. Let $\beta_i \in pr_i(A)$ be arbitrary. Then there is an element $x \in A$ such that $pr_i(x) = \beta_i$. Since $x \succeq a$, then $\beta_i \geq \alpha_i$. Hence, α_i is a lower bound of $pr_i(A)$. Then $\alpha_i \leq \inf(pr_i(A))$. Since this is true for every $i = 1, 2, \ldots, m$, then $a \preceq \sum_{i=1}^{m} \inf(pr_i(A))c_i$. Hence, $\inf A = \sum_{i=1}^{m} \inf(pr_i(A))c_i$.

Finally, assume that $A \cap R^m \neq \emptyset$ and A is bounded below by an element $b \in R^m$. Let $x \in A \cap R^m$ be arbitrary. We have $b \prec \inf A \prec x$. Hence, $\inf(pr_i(A)) \in R$, $i = 1, 2, \ldots, m$. Thus, $\inf A \in R^m$.

(b) The proof is completely similar. ∎

Denote by $\overset{+}{\infty}$ the element $(+\infty)c_1 + (+\infty)c_2 + \cdots + (+\infty)c_m$. Let $x,\ y \in \overline{R}^m$, $x = \sum_{i=1}^{m} \alpha_i c_i$, $y = \sum_{i=1}^{m} \beta_i c_i$. We shall write $x \ll y$ if $\alpha_i < \beta_i$, $i = 1, 2, ..., m$.

Now let f be a vector function from a Banach space X to $R^m \cup \{\overset{+}{\infty}\}$. The effect domain of f is defined as the set

$$\operatorname{dom} f := \{x \in X : f(x) \ll +\infty\}.$$

Represent f as

$$f(x) = \sum_{i=1}^{m} f_i(x) e_i. \tag{1}$$

It should be noted that $\operatorname{dom} f = \operatorname{dom} f_i,\ i = 1, 2, ..., m$.

A subset $W \subseteq R^m$ is said to be a neighborhood of $\overset{+}{\infty}$ if there is a point $z \in R^m$ such that $W \supseteq z + C$. Let us denote by $\overline{C}$ the set $\{\sum_{i=1}^{m} \alpha_i c_i \in \overline{R}^m : \alpha_i \geq 0,\ i = 1, 2, ..., m\}$. f is said to be C-lower semicontinuous at $x_0 \in X$ if, for every neighborhood W of $f(x_0)$, there is a neighborhood V of x_0 such that $x \in V$ implies $f(x) \in W + \overline{C}$. f is said to be C-lower semicontinuous if it is C-lower semicontinuous at every point of X. Sometimes we write "lower semicontinuous" instead of "C-lower semicontinuous" if it is clear which cone is being considered.

It is easy to see that if f is continuous at $x_0 \in \operatorname{dom} f$, then it is lower semicontinuous at x_0.

Lemma 2. *f is lower semicontinuous at $x_0 \in X$ if and only if f_i is lower semicontinuous at x_0, for every $i = 1, 2, ..., m$.*

Proof. For the "only if" part, first assume that $x_0 \in \operatorname{dom} f$. Let $\varepsilon > 0$ be arbitrary. Set

$$W := \{y \in R^m : \sum_{i=1}^{m} (f_i(x_0) - \varepsilon) c_i \preceq y \preceq \sum_{i=1}^{m} (f_i(x_0) + \varepsilon) c_i\}.$$

Then W is a neighborhood of $f(x_0)$. Hence, there is a neighborhood V of x_0 such that $x \in V$ implies $f(x) \in W + C$. We have

$$\begin{aligned} f(x) \in W + C &\Rightarrow \exists y \in W,\ \exists c \in \overline{C} : f(x) = y + c \\ &\Rightarrow \sum_{i=1}^{m} f_i(x) c_i \succeq \sum_{i=1}^{m} (f_i(x_0) - \varepsilon) c_i \\ &\Rightarrow f_i(x) \geq f_i(x_0) - \varepsilon,\ i = 1, 2, ..., m. \end{aligned}$$

Hence, f_i is lower semicontinuous at x_0, for every $i = 1, 2, ..., m$.

Now, assume that $x_0 \notin \operatorname{dom} f$. Let $\alpha > 0$ be arbitrary. Set $W := \sum_{i=1}^{m} c_i + \overline{C}$. Then W is a neighborhood of $\overset{+}{\infty}$. Hence, there exists a neighborhood V of x_0 such that $x \in V$ implies $f(x) \in W + \overline{C}$. Hence, $f_i(x) \geq \alpha$ for every $i = 1, 2, ..., m$. Then f_i is lower semicontinuous at x_0, for every $i = 1, 2, ..., m$.

For the "if" part, first we assume that $x_0 \in \operatorname{dom} f$. Let W be an arbitrary neighborhood of $f(x_0)$. Then there exists $\varepsilon > 0$ such that

$$\{y \in R^m : \sum_{i=1}^{m}(f_i(x_0) - \varepsilon)c_i \preceq y \preceq \sum_{i=1}^{m}(f_i(x_0) + \varepsilon)c_i\} \subseteq W.$$

Since f_i is lower semicontinuous at x_0, for every $i = 1, 2, ..., m$, we can find a neighborhood V of x_0 such that $f_i(x) > f_i(x_0) - \varepsilon$, for every $x \in V, i = 1, 2, ..., m$. Hence, $f(x) = \sum_{i=1}^{m} f_i(x)c_i \succeq \sum_{i=1}^{m}(f_i(x_0) - \varepsilon)c_i$. Since $\sum_{i=1}^{m}(f_i(x_0) - \varepsilon)c_i \in W$, then $f(x) \in W + \overline{C}$. This means that f is lower semicontinuous at x_0.

Now, assume that $x_0 \notin \operatorname{dom} f$. Let W be an arbitrary neighborhood of $f(x_0)$. Then there exists $z \in R^m$ such that $W \supseteq z + C$. Represent z as $z = \sum_{i=1}^{m} \alpha_i c_i$, for some $\alpha_i \in R$. Since f_i is lower semicontinuous at x_0, for every $i = 1, 2, ..., m$, there exists a neighborhood V of x_0 such that $x \in V$ implies $f_i(x) \geq \alpha_i$, for every $i = 1, 2, ..., m$. Hence, $f(x) = \sum_{i=1}^{m} f_i(x)c_i \succeq \sum_{i=1}^{m} \alpha_i c_i$, i.e. $f(x) \in W + \overline{C}$. Thus, f is lower semicontinuous at x_0.

The proof is complete. ■

We recall some definitions.

f is said to be convex (or more precisely C-convex) if for every $x, y \in X, t \in (0, 1)$ we have

$$f(tx + (1-t)y) \preceq tf(x) + (1-t)f(y).$$

f is said to be quasiconvex (or more precisely C-quasiconvex) if for every $x, y \in X$, $t \in (0, 1)$ we have

$$f(tx + (1-t)y) \preceq \sup\{f(x), f(y)\}.$$

Lemma 3.
(a) *f is convex if and only if f_i is convex, for every $i = 1, 2, ..., m$.*
(b) *f is quasiconvex if and only if f_i is quasiconvex, for every $i = 1, 2, ..., m$.*

Proof. The proof is immediate from the definitions. ■

3. Subdifferentials of Lower Semicontinuous Vector Functions

Let $C \subseteq R^m$ be a cone generated by some linearly independent vectors $c_1, c_2, ..., c_m$ and f a lower semicontinuous vector function from a Banach space X into $R^m \cup \{\overset{+}{\infty}\}$. The generalized subderivative of f at $x \in \operatorname{dom} f$ in the direction $v \in X$ is defined by

$$f^{\uparrow}(x; v) := \sup_{\varepsilon>0} \inf_{\substack{\gamma>0 \\ \delta>0 \\ \lambda>0}} \sup_{\substack{y \in B_\gamma(x) \\ f(y) \in f(x) + B_\delta(0) - C \\ t \in (0,\lambda)}} \inf_{u \in B_\varepsilon(v)} \frac{f(y+tu) - f(y)}{t}.$$

Let us represent f as (1).

Theorem 1. *For every $x \in \operatorname{dom} f$, $v \in X$, we have*

$$f^{\uparrow}(x;v) \preceq \sum_{i=1}^{m} f_i^{\uparrow}(x;v)c_i\,.$$

If, in addition, f is continuous at x, then the equality holds.

Proof. From Lemma 1, we have

$$f^{\uparrow}(x;v) = \sum_{i=1}^{m}\left(\sup_{\varepsilon>0}\ \inf_{\substack{\gamma>0\\ \delta>0\\ \lambda>0}}\ \sup_{\substack{y\in B_\gamma(x)\\ f(y)\in f(x)+B_\delta(0)-C\\ t\in(0,\lambda)}}\ \inf_{u\in B_\varepsilon(v)} \frac{f_i(y+tu)-f_i(y)}{t}\right)c_i\,.$$

Let $\varepsilon > 0$ be arbitrary. For every $i = 1, 2, ..., m$, we shall prove that

$$\begin{aligned}
&\inf_{\substack{\gamma>0\\ \delta>0\\ \lambda>0}}\ \sup_{\substack{y\in B_\gamma(x)\\ f(y)\in f(x)+B_\delta(0)-C\\ t\in(0,\lambda)}}\ \inf_{u\in B_\varepsilon(v)} \frac{f_i(y+tu)-f_i(y)}{t}\\
&\preceq \inf_{\substack{\gamma>0\\ \delta>0\\ \lambda>0}}\ \sup_{\substack{y\in B_\gamma(x)\\ f_i(y)\le f_i(x)+\delta\\ t\in(0,\lambda)}}\ \inf_{u\in B_\varepsilon(v)} \frac{f_i(y+tu)-f_i(y)}{t}\,.
\end{aligned} \tag{2}$$

Indeed, let $\gamma > 0, \delta > 0, \lambda > 0$, then the set

$$W := \left\{z = \sum_{i=1}^{m}\alpha_i c_i \in R^m \ :\ |\alpha_i| < \delta\right\}$$

is a neighborhood of 0. Then there exists $\delta_1 > 0$ such that $B_{\delta_1}(0) \subseteq W$. Hence,

$$\begin{aligned}
&\{y \in X : y \in B_\gamma(x),\ f(y) \in f(x) + B_{\delta_1}(0) - C\}\\
&\subseteq \{y \in X : y \in B_\gamma(x),\ f_i(y) \le f_i(x) + \delta\}.
\end{aligned}$$

This implies

$$\begin{aligned}
&\sup_{\substack{y\in B_\gamma(x)\\ f(y)\in f(x)+B_{\delta_1}(0)-C\\ t\in(0,\lambda)}}\ \inf_{u\in B_\varepsilon(v)} \frac{f_i(y+tu)-f_i(y)}{t}\\
&\preceq \sup_{\substack{y\in B_\gamma(x)\\ f_i(y)\le f_i(x)+\delta\\ t\in(0,\lambda)}}\ \inf_{u\in B_\varepsilon(v)} \frac{f_i(y+tu)-f_i(y)}{t}\,.
\end{aligned}$$

Then we obtain (2). From (2) one has

$$f^{\uparrow}(x; v) \preceq \sum_{i=1}^{m} f_i^{\uparrow}(x; v)c_i .$$

Now, assume that f is continuous at x. For every $\varepsilon > 0,\ i = 1, 2, ..., m$, we shall prove that

$$\inf_{\substack{\gamma>0 \\ \delta>0 \\ \lambda>0}} \sup_{\substack{y\in B_\gamma(x) \\ f(y)\in f(x)+B_\delta(0)-C \\ t\in(0,\lambda)}} \inf_{u\in B_\varepsilon(v)} \frac{f_i(y+tu)-f_i(y)}{t} \succeq \inf_{\substack{\gamma>0 \\ \delta>0 \\ \lambda>0}} \sup_{\substack{y\in B_\gamma(x) \\ f_i(y)\le f_i(x)+\delta \\ t\in(0,\lambda)}} \inf_{u\in B_\varepsilon(v)} \frac{f_i(y+tu)-f_i(y)}{t}. \tag{3}$$

Let $\gamma > 0,\ \delta > 0,\ \lambda > 0$. Since $f(x) + B_\delta(0)$ is a neighborhood of $f(x)$, then we can find a positive number γ_1 with $\gamma_1 < \gamma$ such that $y \in B_{\gamma_1}(x) \cap \operatorname{dom} f$ implies $f(y) \in f(x) + B_\delta(0)$. Hence,

$$\begin{aligned}&\{y \in X : y \in B_\gamma(x),\ f(y) \in f(x) + B_\delta(0) - C\} \supseteq B_{\gamma_1}(x) \cap \operatorname{dom} f \\ &\supseteq \{y \in X : y \in B_{\gamma_1}(x),\ f_i(y) \le f_i(x) + \delta\}.\end{aligned}$$

This implies

$$\sup_{\substack{y\in B_\gamma(x) \\ f(y)\in f(x)+B_\delta(0)-C \\ t\in(0,\lambda)}} \inf_{u\in B_\varepsilon(v)} \frac{f_i(y+tu)-f_i(y)}{t} \succeq \sup_{\substack{y\in B_\gamma(x) \\ f_i(y)\le f_i(x)+\delta \\ t\in(0,\lambda)}} \inf_{u\in B_\varepsilon(v)} \frac{f_i(y+tu)-f_i(y)}{t}.$$

Then we obtain (3). From (3) one has

$$f^{\uparrow}(x; v) \succeq \sum_{i=1}^{m} f_i^{\uparrow}(x; v)c_i .$$

The proof is complete. ■

Let $f : X \to R^m \cup \{\overset{+}{\infty}\}$ be a C-lower semicontinuous function. The generalized subdifferential of f at $x \in X$ is defined by

$$\partial^{\uparrow} f(x) := \begin{cases} \{A \in L(X, R^m) : A(v) \preceq f^{\uparrow}(x; v),\ \forall x \in X\}, & x \in \operatorname{dom} f \\ \emptyset, & x \notin \operatorname{dom} f, \end{cases}$$

where $L(X, R^m)$ denotes the space of continuous linear maps from X into R^m.

Let $A_1, A_2, ..., A_m \in L(X, R)$. Denote by $\sum_{i=1}^{m} A_i c_i$ the linear map from X into R^m defined by the rule

$$(\sum_{i=1}^{m} A_i c_i)(x) := \sum_{i=1}^{m} A_i(x) c_i .$$

Let $\mathcal{A}_1, \mathcal{A}_2, ..., \mathcal{A}_m \subseteq L(X, R)$.

Denote by $\sum_{i=1}^{m} \mathcal{A}_i c_i$ the subset of $L(X, R^m)$ defined by the rule

$$\sum_{i=1}^{m} \mathcal{A}_i c_i := \{\sum_{i=1}^{m} A_i c_i \ : A_i \in \mathcal{A}_i \ (i = 1, 2, ..., m)\}.$$

Theorem 2. *For every $x \in \operatorname{dom} f$, we have*

$$\partial^{\uparrow} f(x) \subseteq \sum_{i=1}^{m} \partial^{\uparrow} f_i(x) c_i .$$

In addition, if f is continuous at x, then the equality holds.

Proof. Let $A \in \partial^{\uparrow} f(x)$ be arbitrary. Represent A as $A = \sum_{i=1}^{m} A_i c_i$, for some $A_i \in L(X, R)$. From definitions and by Theorem 1, we have

$$\sum_{i=1}^{m} A_i(v) c_i \preceq f^{\uparrow}(x; v) \preceq \sum_{i=1}^{m} f_i^{\uparrow}(x; v) c_i ,$$

for every $v \in X$. Then $A_i(v) \leq f_i^{\uparrow}(x; v)$. Hence, $A_i \in \partial^{\uparrow} f_i(x)$. Thus, $A \in \sum_{i=1}^{m} \partial^{\uparrow} f_i(x) c_i$.

Now, assume that f is continuous at x. Let $A_i \in \partial^{\uparrow} f_i(x)$ be arbitrary. From definitions and by Theorem 1, we have

$$\sum_{i=1}^{m} A_i(v) c_i \preceq \sum_{i=1}^{m} f_i^{\uparrow}(x; v) c_i = f^{\uparrow}(x; v),$$

for every $v \in X$. Hence, $\sum_{i=1}^{m} A_i c_i \in \partial^{\uparrow} f(x)$. The theorem is proved. ■

Now, we shall consider the relation between the generalized Jacobian and the generalized subdifferential of a lower semicontinuous vector function f from R^n to R^m.

Let $x_0 \in int(\operatorname{dom} f)$. Assume that f is Lipschitz near x_0. By Radermacher's theorem, f is differentiable almost everywhere. The generalized Jacobian $Jf(x_0)$ of f at x_0 in the Clarke's sense [2] is defined as the convex hull of all $(m \times n)$ matrices obtained as the limit of a sequence of the form $(Df(x_i))_i$, where $(x_i)_i$ converges to x_0 and the classical Jacobian matrix $Df(x_i)$ of f at x_i exists.

Lemma 4. *For every $x \in \operatorname{dom} f$, $\partial^{\uparrow} f(x)$ is convex.*

Proof. This is immediate by definition. ∎

Lemma 5. *Let $g : R^n \to R \cup \{+\infty\}$ be a lower semicontinuous function. If g is Lipschitz near $x_0 \in int(\operatorname{dom} g)$, then*

$$Jg(x_0) \subseteq \partial^{\uparrow} g(x_0).$$

Proof. Let A be the limit of a sequence of the form $(Dg(x_i))_i$, where $(x_i)_i$ converges to x_0 and the classical Jacobian matrix $Dg(x_i)$ of g at x_i exists. Since g is Lipschitz near x_0, there exists $\varepsilon' > 0$, $k > 0$ such that for every $x,\ y \in B_{\varepsilon'}(x_0)$, one has

$$|g(x) - g(y)| \leq k\|x - y\|. \tag{4}$$

Let $v \in R^n$ and $\alpha > 0$ be arbitrary. Set $\varepsilon_0 := \frac{\alpha}{3k}$. From the definition we have

$$\inf_{\substack{\gamma>0\\ \delta>0\\ \lambda>0}} \sup_{\substack{y\in B_\gamma(x_0)\\ g(y)\le g(x_0)+\delta\\ t\in(0,\lambda)}} \inf_{u\in B_{\varepsilon_0}(v)} \frac{g(y+tu)-g(y)}{t} \leq g^{\uparrow}(x_0; v). \tag{5}$$

By the definition of "inf", there exist $\gamma_1 > 0, \delta_1 > 0, \lambda_1 > 0$ such that

$$\sup_{\substack{y\in B_{\gamma_1}(x_0)\\ g(y)\le g(x_0)+\delta_1\\ t\in(0,\lambda_1)}} \inf_{u\in B_{\varepsilon_0}(v)} \frac{g(y+tu)-g(y)}{t}$$

$$\leq \inf_{\substack{\gamma>0\\ \delta>0\\ \lambda>0}} \sup_{\substack{y\in B_\gamma(x_0)\\ g(y)\le g(x_0)+\delta\\ t\in(0,\lambda)}} \inf_{u\in B_{\varepsilon_0}(v)} \frac{g(y+tu)-g(y)}{t} + \frac{\alpha}{6}. \tag{6}$$

Since $x_i \to x_0$ and g is continuous at x_0, there exists $N > 0$ such that

$$i > N \Rightarrow x_i \in B_{\gamma_1}(x_0) \cap B_{\frac{\varepsilon'}{2}}(x_0),\ g(x_i) \leq g(x_0) + \delta_1\,. \tag{7}$$

For every $i > N$, since $Dg(x_i)(v) = \lim\limits_{t\downarrow 0} \dfrac{g(x_i+tv)-g(x_i)}{t}$, there exists $t_i \in (0, \lambda_1)$ such that

$$t_i < \frac{\varepsilon'}{2(\|v\|+\varepsilon_0)},\ Dg(x_i)(v) < \frac{g(x_i+t_i v)-g(x_i)}{t_i} + \frac{\alpha}{3}. \tag{8}$$

It is clear that

$$\inf_{u\in B_{\varepsilon_0}(v)} \frac{g(x_i+t_i u)-g(x_i)}{t} \leq \sup_{\substack{y\in B_{\gamma_1}(x_0)\\ g(y)\le g(x_0)+\delta_1\\ t\in(0,\lambda_1)}} \inf_{u\in B_{\varepsilon_0}(v)} \frac{g(y+tu)-g(y)}{t}, \tag{9}$$

for every $i > N$. From the definition of "inf", there exists

$$u_i \in B_{\varepsilon_0}(v) \tag{10}$$

such that

$$\frac{g(x_i + t_i u_i) - g(x_i)}{t_i} \leq \inf_{u \in B_{\varepsilon_0}(v)} \frac{g(x_i + t_i u) - g(x_i)}{t_i} + \frac{\alpha}{6}. \tag{11}$$

From (5), (6), (9) and (11), for every $i > N$, we have

$$\frac{g(x_i + t_i u_i) - g(x_i)}{t_i} \leq g^{\uparrow}(x_0; v) + \frac{\alpha}{3}. \tag{12}$$

From (4), (7), (8) and (10), we have

$$\frac{g(x_i + t_i v) - g(x_i)}{t_i} - \frac{g(x_i + t_i u_i) - g(x_i)}{t_i} \leq k\|v - u_i\| < k\varepsilon_0 = \frac{\alpha}{3}. \tag{13}$$

From (8), (12) and (13), we have

$$Dg(x_i)(v) \leq g^{\uparrow}(x_0; v) + \alpha.$$

As $i \to \infty$, one has

$$A(v) \leq g^{\uparrow}(x_0; v) + \alpha.$$

Since $\alpha > 0$ is arbitrary,

$$A(v) \leq g^{\uparrow}(x_0; v).$$

Hence $A \in \partial^{\uparrow} g(x_0)$. By Lemma 4, we obtain

$$Jg(x_0) \subseteq \partial^{\uparrow} g(x_0).$$

The proof is complete. ■

Theorem 3. *If a lower semicontinuous vector function $f : R^n \to R^m \cup \{\overset{+}{\infty}\}$ is Lipschitz near $x_0 \in int(\text{dom} f)$, then*

$$Jf(x_0) \subseteq \partial^{\uparrow} f(x_0).$$

Proof. Since f is Lipschitz near x_0, by Theorem 2, we have

$$\partial^{\uparrow} f(x_0) = \sum_{i=1}^{m} \partial^{\uparrow} f_i(x_0) c_i.$$

By Lemma 5 above and by [2, Proposition 2.6.2], one has

$$Jf(x_0) \subseteq \sum_{i=1}^{m} Jf_i(x_0)c_i \subseteq \sum_{i=1}^{m} \partial^{\uparrow} f_i(x_0)c_i = \partial^{\uparrow} f(x_0).$$

The proof is complete. ■

It should be noted that the inclusion of Theorem 3 is strict in general. For instance, consider the function $f : x \in R \to (|x|, |x|) \in R^2$, where R^2 is ordered by the nonnegative orthant. Then $Jf(0) = [(-1, -1), (1, 1)]$ and $\partial^{\uparrow} f(0) = [-1, 1] \times [-1, 1]$.

Lemma 6. *Let g be a lower semicontinuous function from a Banach space X into $R \cup \{+\infty\}$. If g is differentiable at $x_0 \in int(\mathrm{dom}\, g)$, then $Dg(x_0) \in \partial^{\uparrow} g(x_0)$.*

Proof. Let $v \in X$. For every $\varepsilon > 0$, $\gamma > 0$, $\delta > 0$, $\lambda > 0$, we have

$$\sup_{t\in(0,\lambda)} \inf_{u\in B_\varepsilon(v)} \frac{g(x_0+tu)-g(x_0)}{t} \le \sup_{\substack{y\in B_\gamma(x_0)\\ g(y)\le g(x_0)+\delta\\ t\in(0,\lambda)}} \inf_{u\in B_\varepsilon(v)} \frac{g(y+tu)-g(y)}{t}.$$

Hence,

$$\sup_{\varepsilon>0} \inf_{\lambda>0} \sup_{t\in(0,\lambda)} \inf_{u\in B_\varepsilon(v)} \frac{g(x_0+tu)-g(x_0)}{t} \le g^{\uparrow}(x_0; v). \tag{14}$$

Let $\alpha > 0$ be arbitrary. Since $Dg(x_0)$ is continuous at 0, for $r := \frac{\alpha}{6} > 0$, there exists $s > 0$ such that for every $w \in X$, one has

$$\|w\| < s \Rightarrow |Dg(x_0)(w)| < r. \tag{15}$$

Since $\lim_{w\to 0} \frac{g(x_0+w)-g(x_0)-Dg(x_0)(w)}{\|w\|} = 0$, for $r' := \frac{\alpha}{6(\|v\|+s)}$, there exists $s' > 0$ such that

$$\|w\| < s' \Rightarrow \frac{|g(x_0+w)-g(x_0)-Dg(x_0)(w)|}{\|w\|} < r'. \tag{16}$$

It is clear that

$$\inf_{\lambda>0} \sup_{t\in(0,\lambda)} \inf_{u\in B_\varepsilon(v)} \frac{g(x_0+tu)-g(x_0)}{t} \le \sup_{\varepsilon>0} \inf_{\lambda>0} \sup_{t\in(0,\lambda)} \inf_{u\in B_\varepsilon(v)} \frac{g(x_0+tu)-g(x_0)}{t}. \tag{17}$$

From the definition of "inf", there exists $\lambda_1 > 0$ such that

$$\sup_{t\in(0,\lambda_1)} \inf_{u\in B_\varepsilon(v)} \frac{g(x_0+tu)-g(x_0)}{t} \le \inf_{\lambda>0} \sup_{t\in(0,\lambda)} \inf_{u\in B_\varepsilon(v)} \frac{g(x_0+tu)-g(x_0)}{t} + \frac{\alpha}{4}. \tag{18}$$

Let $t_0 \in (0, \lambda_1)$ such that $t_0 < \frac{s'}{\|v\|+s'}$. Then

$$\inf_{u\in B_\varepsilon(v)} \frac{g(x_0+t_0u)-g(x_0)}{t_0} \le \sup_{t\in(0,\lambda_1)} \inf_{u\in B_\varepsilon(v)} \frac{g(x_0+tu)-g(x_0)}{t}. \tag{19}$$

From the definition of "inf", there exists $u_0 \in B_s(v)$ such that

$$\frac{g(x_0+t_0u_0)-g(x_0)}{t_0} \le \inf_{u\in B_\varepsilon(v)} \frac{g(x_0+t_0u)-g(x_0)}{t_0} + \frac{\alpha}{4}. \tag{20}$$

From (14), (17)–(20), we have

$$\frac{g(x_0+t_0u_0)-g(x_0)}{t_0} \leq g^{\uparrow}(x_0; v) + \frac{\alpha}{2}. \tag{21}$$

Put $\beta := \frac{s'}{\|v\|}$. For every $t \in (0, \beta)$, since $\|tv\| < s'$, then by (16), one has

$$\frac{|g(x_0+tv)-g(x_0)-Dg(x_0)(tv)|}{\|tv\|} < r'. \tag{22}$$

Since $t_0\|u_0\| < s'$,

$$\frac{|g(x_0+t_0u_0)-g(x_0)-Dg(x_0)(t_0u_0)|}{\|t_0u_0\|} < r'. \tag{23}$$

Since $u_0 \in B_s(v)$, by (15), one has

$$|Dg(x_0)(v-u_0)| < r. \tag{24}$$

From (22)–(24), for every $t \in (0, \beta)$, we have

$$\begin{aligned} &\left|\frac{g(x_0+tv)-g(x_0)}{t} - \frac{g(x_0+t_0u_0)-g(x_0)}{t_0}\right| \\ &\leq \|v\| \left|\frac{g(x_0+tv)-g(x_0)-Dg(x_0)(tv)}{\|tv\|}\right| \\ &+ |Dg(x_0)(v-u_0)| + \|u_0\| \left|\frac{g(x_0+t_0u_0)-g(x_0)-Dg(x_0)(t_0u_0)}{\|t_0u_0\|}\right| \\ &\leq \|v\|r' + r + \|u_0\|r' = \|v\|\frac{\alpha}{6(\|v\|+s)} + \frac{\alpha}{6} + \|u_0\|\frac{\alpha}{6(\|v\|+s)} < \frac{\alpha}{2}. \end{aligned} \tag{25}$$

From (21) and (25), we have

$$\frac{g(x_0+tv)-g(x_0)}{t} \leq g^{\uparrow}(x_0; v) + \alpha.$$

Taking $t \downarrow 0$, we obtain

$$Dg(x_0)(v) \leq g^{\uparrow}(x_0; v) + \alpha.$$

Since $\alpha > 0$ is arbitrary,

$$Dg(x_0)(v) \leq g^{\uparrow}(x_0; v).$$

Hence, $Dg(x_0) \in \partial^{\uparrow} g(x_0)$. The proof is complete. ■

Theorem 4. *If a lower semicontinuous vector function f from a Banach space X into $R^m \cup \{\overset{+}{\infty}\}$ is differentiable at $x_0 \in int(\operatorname{dom} f)$, then*

$$Df(x_0) \in \partial^{\uparrow} f(x_0).$$

Proof. By Theorem 2, we have

$$\partial^{\uparrow} f(x_0) = \sum_{i=1}^{m} \partial^{\uparrow} f_i(x_0) c_i .$$

By Lemma 6, one has $D_f i(x_0) \in \partial^{\uparrow} f_i(x_0)$, $i = 1, 2, ..., m$. Hence, $Df(x_0) = \sum_{i=1}^{m} f_i(x_0) c_i \in \partial^{\uparrow} f(x_0)$. The proof is complete. ■

Now, let $f : R^n \to R^m \cup \{\overset{+}{\infty}\}$ be convex. The subdifferential of f at $x \in \text{dom } f$ (see [9]) is defined as the set

$$\partial f(x) := \{A \in L(R^n, R^m) \,:\, f(y) - f(x) \succeq A(y - x),\ (\forall y \in \text{dom } f)\}.$$

We shall consider the relation between the subdifferential and the generalized subdifferential of a convex vector function.

Lemma 7. *Let $g : R^n \to R \cup \{+\infty\}$ be a lower semicontinuous convex function. Then for every $x_0 \in int(\text{dom} g)$, we have*

$$\partial g(x_0) = \partial^{\uparrow} g(x_0).$$

Proof. Since g is a scalar convex function, then $Jg(x_0) = \partial g(x_0)$. By Lemma 5, one has $\partial g(x_0) \subseteq \partial^{\uparrow} g(x_0)$.

Conversely, let $A \in \partial^{\uparrow} g(x_0)$ be arbitrary. Since g is Lipschitz near x_0, there exist $\varepsilon' > 0$, $k > 0$ such that

$$|g(x) - g(y)| \leq k\|x - y\|, \tag{26}$$

for every $x,\ y \in B_{\varepsilon'}(x_0)$. Let $v \in R^n$ such that $x_0 + v \in \text{dom} g$ and let $\alpha > 0$ be arbitrary. There exists $\varepsilon_0 \in (0, \frac{\alpha}{4k})$ such that

$$\inf_{\substack{\gamma>0 \\ \delta>0 \\ \lambda>0}} \ \sup_{\substack{y\in B_\gamma(x_0) \\ g(y)\leq g(x_0)+\delta \\ t\in(0,\lambda)}} \ \inf_{u\in B_{\varepsilon_0}(v)} \frac{g(y+tu)-g(y)}{t} \geq g^{\uparrow}(x_0; v) - \frac{\alpha}{4} . \tag{27}$$

Let $\delta_0 > 0$ be arbitrary. Then there exist $\lambda_0 > 0$, $\gamma_0 > 0$ such that

$$\lambda_0 < \min\{\frac{\varepsilon'}{2(\|v\| + \varepsilon_0)}, 1\},\ \gamma_0 < \min\{\frac{\varepsilon'}{2}, \frac{\lambda_0 \alpha}{8k}\},$$

$$g(y) \leq g(x_0) + \delta_0\ (\forall y \in B_{\gamma_0}(x_0)). \tag{28}$$

Obviously,

$$\sup_{\substack{y\in B_{\gamma_0}(x_0) \\ g(y)\leq g(x_0)+\delta_0 \\ t\in(0,\lambda_0)}} \ \inf_{u\in B_{\varepsilon_0}(v)} \frac{g(y+tu)-g(y)}{t}$$

$$\geq \inf_{\substack{\gamma>0 \\ \delta>0 \\ \lambda>0}} \ \sup_{\substack{y\in B_\gamma(x_0) \\ g(y)\leq g(x_0)+\delta \\ t\in(0,\lambda)}} \ \inf_{u\in B_{\varepsilon_0}(v)} \frac{g(y+tu)-g(y)}{t} . \tag{29}$$

From the definition of "sup", there exist $y_0 \in B_{\gamma_0}(x_0)$, $t_0 \in (0, \lambda_0)$ such that

$$\inf_{u \in B_{\varepsilon_0}(v)} \frac{g(y_0 + t_0 u) - g(y_0)}{t_0} \geq \sup_{\substack{y \in B_{\gamma_0}(x_0) \\ g(y) \leq g(x_0) + \delta_0 \\ t \in (0, \lambda_0)}} \inf_{u \in B_{\varepsilon_0}(v)} \frac{g(y + tu) - g(y)}{t} - \frac{\alpha}{4}. \tag{30}$$

Since g is convex, for every $u \in B_{\varepsilon_0}(v)$, we have

$$\frac{g(y_0 + t_0 u) - g(y_0)}{t_0} \leq \frac{g(y_0 + \lambda_0 u) - g(y_0)}{\lambda_0}.$$

Hence,

$$\inf_{u \in B_{\varepsilon_0}(v)} \frac{g(y_0 + \lambda_0 u) - g(y_0)}{\lambda_0} \geq \inf_{u \in B_{\varepsilon_0}(v)} \frac{g(y_0 + t_0 u) - g(y_0)}{t_0}. \tag{31}$$

Let $u_0 \in B_{\varepsilon_0}(v)$ be arbitrary. One has

$$\frac{g(y_0 + \lambda_0 u_0) - g(y_0)}{\lambda_0} \geq \inf_{u \in B_{\varepsilon_0}(v)} \frac{g(y_0 + \lambda_0 u) - g(y_0)}{\lambda_0}. \tag{32}$$

From (27) and (29)–(32), we have

$$\frac{g(y_0 + \lambda_0 u_0) - g(y_0)}{\lambda_0} \geq g^{\uparrow}(x_0; v) - \frac{\alpha}{2}.$$

On the other hand, we have

$$\begin{aligned}
&\left| \frac{g(x_0 + \lambda_0 v) - g(x_0)}{\lambda_0} - \frac{g(y_0 + \lambda_0 u_0) - g(y_0)}{\lambda_0} \right| \\
&\leq \left| \frac{[g(x_0 + \lambda_0 v) - g(y_0 + \lambda_0 u_0)] - [g(x_0) - g(y_0)]}{\lambda_0} \right| \\
&\leq \frac{|g(x_0 + \lambda_0 v) - g(y_0 + \lambda_0 u_0)|}{\lambda_0} + \frac{|g(x_0) - g(y_0)|}{\lambda_0} \\
&\leq \frac{k}{\lambda_0} \|x_0 - y_0 + \lambda_0 (v - u_0)\| + \frac{k}{\lambda_0} \|x_0 - y_0\| \\
&\leq k\|v - u_0\| + \frac{2k}{\lambda_0} \|x_0 - y_0\| < k x_0 + \frac{2k}{\lambda_0} \gamma_0 < \frac{\alpha}{4} + \frac{\alpha}{4} = \frac{\alpha}{2}.
\end{aligned}$$

By this and the convexity of g, one has

$$\begin{aligned}
g(x_0 + v) - g(x_0) &\geq \frac{g(x_0 + \lambda_0 v) - g(x_0)}{\lambda_0} \geq \frac{g(y_0 + \lambda_0 u_0) - g(y_0)}{\lambda_0} - \frac{\alpha}{2} \\
&\geq g^{\uparrow}(x_0; v) - \alpha \geq A(v) - \alpha.
\end{aligned}$$

Since $\alpha > 0$ is arbitrary, $A(v) \leq g(x_0 + v) - g(x_0)$. Thus, $A \in \partial g(x_0)$. The proof is complete. ∎

Theorem 5. *If a lower semicontinuous function f from R^n into $R^m \cup \{\overset{+}{\infty}\}$ is convex, then for every $x_0 \in \operatorname{int}(\operatorname{dom} f)$, we have*

$$\partial f(x_0) = \partial^{\uparrow} f(x_0).$$

Proof. Represent f as $f(x) = \sum_{i=1}^{m} f_i(x)c_i$. It is not difficult to see that

$$\partial f(x_0) = \sum_{i=1}^{m} \partial f_i(x_0)c_i\,.$$

By Lemmas 3 and 7, we have

$$\partial f_i(x_0) = \partial^{\uparrow} f_i(x_0).$$

Then by [9, Theorem 3.1] and Theorem 2 above, we have

$$\partial f(x_0) = \partial^{\uparrow} f_i(x_0).$$

The proof is complete. ■

4. Subdifferential Characterization of Convex and Quasiconvex Vector Functions

Let F be a set-valued map from a Banach space X into $L(X, R^m)$. Denote by dom$F$ the set $\{x \in X : F(x) \neq 0\}$.

F is said to be monotone if $x, y \in \operatorname{dom}F$, $A \in F(x)$, $B \in F(y)$ imply $(B - A)(y - x) \in C$.

F is said to be quasimonotone if $x, y \in \operatorname{dom}F$, $A \in F(x)$, $B \in F(y)$ and $A(y - x) \in \operatorname{int} C$ imply $B(y - x) \in C$.

Let $F_1, F_2, \ldots, F_m$ be set-valued maps from X to $L(X, R)$. Denote by $\sum_{i=1}^{m} F_i c_i$ the set-valued map from X into $L(X, R^m)$ defined by the rule

$$\Big(\sum_{i=1}^{m} F_i c_i\Big)(x) := \sum_{i=1}^{m} F_i(x)c_i\,.$$

It is easy to see that $\operatorname{dom}\Big(\sum_{i=1}^{m} F_i c_i\Big) = \bigcap_{i=1}^{m} \operatorname{dom}F_i$.

Lemma 8. *Assume that* dom F_1 = domF_2 $= \cdots =$ domF_m. *Then*

(a) $\sum_{i=1}^{m} F_i c_i$ *is quasimonotone if and only if F_i is quasimonotone for every i = 1, 2, ..., m.*

(b) $\sum_{i=1}^{m} F_i c_i$ *is monotone if and only if F_i is monotone for every $i = 1, 2, \ldots, m$.*

Proof. (a) For the "only if" part, let $x, y \in \operatorname{dom} F_i$, $A_i \in F_i(x)$, $B_i \in F_i(y)$, and $A_i(y-x) > 0$. Then $\sum_{i=1}^{m} A_i c_i \in (\sum_{i=1}^{m} F_i c_i)(x)$, $\sum_{i=1}^{m} B_i c_i \in (\sum_{i=1}^{m} F_i c_i)(y)$ and $(\sum_{i=1}^{m} B_i c_i)(y-x) = \sum_{i=1}^{m} A_i(y-x)c_i \in \operatorname{int} C$. Since $\sum_{i=1}^{m} F_i c_i$ is quasimonotone, $(\sum_{i=1}^{m} B_i c_i)(y-x) \in C$. Hence, $B_i(y-x) \geq 0, i = 1, 2, \ldots, m$. Thus, F_i is quasimonotone for every $i = 1, 2, \ldots, m$.

For the "if" part, let $x, y \in \operatorname{dom} \sum_{i=1}^{m} F_i c_i$, $A \in (\sum_{i=1}^{m} F_i c_i)(x)$, $B \in (\sum_{i=1}^{m} F_i c_i)(y)$ and $A(y-x) \in \operatorname{int} C$. Represent A, B as

$$A = \sum_{i=1}^{m} A_i c_i, \quad \text{for some } A_i \in F_i(x).$$

$$B = \sum_{i=1}^{m} B_i c_i, \quad \text{for some } B_i \in F_i(y).$$

Then $\sum_{i=1}^{m} A_i(y-x)c_i = (\sum_{i=1}^{m} A_i c_i)(y-x) \in \operatorname{int} C$. Hence, $A_i(y-x) > 0, i = 1, 2, \ldots, m$. Since F_i is quasimonotone, $B_i(y-x) \geq 0$, $i = 1, 2, \ldots, m$. This implies $B(y-x) = \sum_{i=1}^{m} B_i(y-x)c_i \in C$. Thus, $\sum_{i=1}^{m} F_i c_i$ is quasimonotone.

(b) The proof is completely similar. ■

Now, let f be a lower semicontinuous vector function from a Banach space X to $R^m \cup \{\overset{+}{\infty}\}$. Represent f as

$$f(x) = \sum_{i=1}^{m} f_i(x) c_i,$$

for some $f_i \in R \cup \{+\infty\}$.

Theorem 6. *Assume that the lower semicontinuous vector function f is continuous on* $\operatorname{dom} f$ *and* $\operatorname{dom}\partial^{\uparrow} f_1 = \operatorname{dom}\partial^{\uparrow} f_2 = \cdots = \operatorname{dom}\partial^{\uparrow} f_m$. *Then*

(a) *f is quasiconvex if and only if $\partial^{\uparrow} f$ is quasimonotone.*

(b) *f is convex if and only if $\partial^{\uparrow} f$ is monotone.*

Proof. Since f is continuous on $\operatorname{dom} f$, by Theorem 2, we have

$$\partial^{\uparrow} f = \sum_{i=1}^{m} \partial^{\uparrow} f_i c_i.$$

(a) One has

f is quasiconvex $\Leftrightarrow$ f_i is quasiconvex, $i = 1, 2, \ldots, m$, by Lemma 3 above.
$\Leftrightarrow$ $\partial^{\uparrow} f_i$ is quasimonotone, $i = 1, 2, \ldots, m$, by [6, Theorem 3.2].
$\Leftrightarrow$ $\partial^{\uparrow} f$ is quasimonotone by Lemma 8 above.

(b) One has

$$\begin{aligned} f \text{ is convex } &\Leftrightarrow f_i \text{ is convex, } i = 1, 2, \ldots, m, \text{ by Lemma 3 above.} \\ &\Leftrightarrow \partial^{\uparrow} f_i \text{ is monotone, } i = 1, 2, \ldots, m, \text{ by [6, Theorem 3.2].} \\ &\Leftrightarrow \partial^{\uparrow} f_i \text{ is monotone, by Lemma 8 above.} \end{aligned}$$

The theorem is proved. ■

Remark. We note that in [11, Theorem 4.2] some sufficient conditions for quasiconvex set-valued maps between Banach spaces were given. However, in some cases [11, Theorem 4.2] is not valid while Theorem 6 above is still applied. For instance, put $X = R, m = 2$ and $C = R^2_+$. Define a function $f : R \to R^2 \cup \{\overset{+}{\infty}\}$ as follows:

$$f(x) = \begin{cases} (x, -x) & x \in [-1, 0] \\ (x, -2x) & x \in [0, 1] \\ \overset{+}{\infty} & \text{otherwise.} \end{cases}$$

Denote by f_1, f_2 the component function of f. Obviously, f is continuous on $[-1, 1]$. By computing, we obtain

$$\partial^{\uparrow} f_1(x) = \begin{cases} (-\infty, 1] & x = -1 \\ \{1\} & -1 < x < 1 \\ [1, +\infty) & x = 1 \\ \emptyset & \text{otherwise.} \end{cases}$$

$$\partial^{\uparrow} f_2(x) = \begin{cases} (-\infty, 1] & x = -1 \\ \{-1\} & -1 < x < 0 \\ [-2, -1] & x = 0 \\ \{-2\} & 0 < x < 1 \\ [-2, +\infty) & x = 1 \\ \emptyset & \text{otherwise.} \end{cases}$$

It is not difficult to see that $\partial^{\uparrow} f_1$ and $\partial^{\uparrow} f_2$ are quasimonotone and so is $\partial^{\uparrow} f$. Hence, by Theorem 6 above, f is quasimonotone.

However, in this case, the sufficient conditions in [11, Theorem 4.2] do not hold.

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