Vietnam Journal of Mathematics **25**:4 (1997) 279–318

Vietnam Journal of MATHEMATICS

Survey

On the Boundary Cohomology of Locally Symmetric Varieties*

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Received April 8, 1997

Dedicated to Professor Hoang Tuy on the occasion of his 70th birthday

Abstract. In this paper, our main purpose is to give an exposition of the geometric aspects of joint work with Michael Harris [15, 16]. They contain new results on the cohomology of the boundary of a locally symmetric variety.

By definition, a locally symmetric variety, denoted throughout by X, is a complex algebraic variety given analytically as the quotient of an Hermitian symmetric space by an arithmetically defined group of isometries. These matters are recalled in (1.1). Some examples are presented in (1.2). They admit easy, overly explicit calculations. While such examples can be of great help, the reader is cautioned to beware of oversimplifying the general theory. The role of parabolic subgroups in defining compactifications of X is treated in (1.3). Indeed, the choice of compactification defines the very notion of "boundary" mentioned in the title of this paper. Two such compactifications, both quite different in character, are the Borel-Serre compactification $\overline{X}$ from [7], which is a manifold-with-corners whose definition does not require the Hermitian structure, and the toroidal compactifications $\tilde{X}_\Sigma$ from [1], some of which are smooth projective varieties, though they are not canonically defined. The two do have a well-known common quotient, namely the Baily-Borel Satake compactification X^* from [2], which is a normal projective variety with "explicit" stratification.

The results in Sec. 2 are independent of Hermitian structure and, as one might correctly infer, are about $\overline{X}$. In (2.1), we recall how differential forms on X are

*Supported in part by the National Science Foundation, through Grant DMS-9102233, and by the Japan Society for the Promotion of Science, through a Fellowship for Research, Kyoto University, Japan, Spring, 1993.

describable in terms of Lie-theoretical objects. We then present in (2.2) and (2.3) known results on the cohomology of the Borel-Serre boundary, which are already treated in [27].

In Sec. 3, we discuss the new material. The first theme is the attainment of complex-analytic versions of the results in (2.2). We discuss, in (3.1), real quotients of torus embeddings. This facilitates a nice formulation in (3.2.12) of the analogue of (2.2.3), which makes use of equivariant cohomology. After that, we focus on Hodge-theoretic questions on the boundary cohomology. It is natural to mean here the boundary of X in $\tilde{X}_\Sigma$. However, it turns out, somewhat surprisingly, that it can as well be in $\bar{X}$ (see (3.4.2)). Thus, it makes sense to talk about the mixed Hodge structures on the cohomology of (deleted neighborhoods of) the closed faces of the boundary in $\bar{X}$,[1] and these mixed Hodge structures can be identified (3.5.5). These groups comprise the E_1-term of the nerve spectral sequence, abutting to the cohomology of the Borel-Serre boundary. We see that (3.5.5) can be used in eliminating the possibility of so-called ghost classes (see (3.3.4) for the definition) in the cohomology of the boundary. This is worked out in the case of Siegel modular 3-folds (as mentioned in [16:(5.7)], though we now employ [16:(5.6)] instead of [20]) in Appendix A. We have included another appendix (B), with the aim of demystifying the rather fundamental Cayley transform, which lies behind much of the aforementioned work. Finally, we have added a third appendix concerning the structure of equal-rank groups, i.e., those admitting a compact Cartan subgroup.

1. Preliminaries

(1.1) *Locally Symmetric Varieties.* Let G be an algebraic group defined over the rational field $\mathbf{Q}$. For any extension $\mathbf{E}$ of $\mathbf{Q}$, $G(\mathbf{E})$ denotes the group of $\mathbf{E}$-valued points of G.

There is no canonical notion of $G(\mathbf{Z})$ however, but one does have the following. For any almost-faithful representation $\rho : G \to GL_n$ defined over $\mathbf{Q}$, put

$$G(\mathbf{Z})_\rho = \rho^{-1}(GL_n(\mathbf{Z})).$$

One checks that for another such representation ρ' of G, $\Gamma' = G(\mathbf{Z})_{\rho'}$ and $\Gamma = G(\mathbf{Z})_\rho$ are *commensurable*, i.e. $\Gamma' \cap \Gamma$ is of finite index in both Γ and Γ'. One says, more generally, that a group $\Gamma \subset G(\mathbf{Q})$ is an *arithmetic* subgroup of $G(\mathbf{Q})$ if Γ is commensurable with any, hence all, $G(\mathbf{Z})_\rho$.

It is well known that any arithmetic group Γ contains a normal subgroup of finite index (hence arithmetic) containing no non-trivial elements of finite order. Indeed, the following stronger assertion holds: Γ *contains neat normal subgroups* Γ' *of finite index*, i.e., ones for which

$$\Gamma'_{H_1/H_2} =: (\Gamma' \cap H_1(\mathbf{Q}))/(\Gamma' \cap H_2(\mathbf{Q})) \tag{1.1.1}$$

[1] Said mixed Hodge structures can be deduced from considerations on X^* (see (3.3.12)).

is torsion-free (and arithmetic) whenever $H_2 \subset H_1$ is a pair of algebraic $\mathbf{Q}$-subgroups of G (see [4: Sec. 17]).

Let $\mathscr{D}$ be a space of type S–$\mathbf{Q}$ for G, as defined in [7: 2.3]. For instance, if G is semi-simple, $\mathscr{D}$ must be the symmetric space of non-compact type associated to $G(\mathbf{R})$ (with the Lie group topology). If Γ is arithmetic, it is a discrete group acting (on the left) on $\mathscr{D}$, and the quotient $Y = \Gamma\backslash\mathscr{D}$ is Hausdorff. If Γ is also torsion-free, the quotient is a real-analytic manifold; since a space of type S–$\mathbf{Q}$ is homeomorphic to a Euclidean space (see [7: 2.4]), a fortiori is contractible, Y is then an Eilenberg-MacLane space $K(\Gamma, 1)$.

We suppose henceforth that G is semi-simple and let D be the associated symmetric space. Then $G(\mathbf{R})$ acts transitively on D, with maximal compact isotropy subgroups. Thus, there exist $G(\mathbf{R})$-invariant Riemannian metrics on D (ess. unique if G is irreducible over $\mathbf{R}$). If Γ is torsion-free arithmetic, then $X := \Gamma\backslash D$, with the metric induced from D, is a complete manifold of finite volume.

One says that D (likewise X and G) is *Hermitian* if D admits a $G(\mathbf{R})$-invariant complex structure. The underlying almost-complex structure, determined by the Lie algebra of the isotropy groups, is automatically integrable and Kählerian. Actually, one can say much more about X:

(1.1.2) **Theorem.** [2] *When X is Hermitian, it is a quasi-projective variety over* $\mathbf{C}$.

Indeed, X can be embedded in complex projective space by a suitable space of holomorphic automorphic forms, and its closure is a normal projective variety X^*, which one refers to as the *Baily-Borel Satake compactification* of X (the underlying topological space of X^* is a Satake compactification in the sense of [30]). In view of the above theorem, one calls X a *locally symmetric variety*. The locally symmetric varieties are of significance in number theory as the underlying complex spaces of *Shimura varieties*, which can be shown to be varieties defined over number fields (see [22]).

An important notion is the $\mathbf{E}$-*rank* of G, denoted by $r = rk_{\mathbf{E}}G$. It is the dimension of a *maximal* $\mathbf{E}$-*split torus* T of G, i.e., a subgroup T with $T(\mathbf{E}) \cong (\mathbf{E}^\times)^r$. For $\mathbf{E} = \mathbf{C}$, one recovers the notion of a Cartan subgroup of $G(\mathbf{C})$; for general $\mathbf{E}$, it displays many of the features of a Cartan subgroup. All maximal $\mathbf{E}$-split tori are conjugate under $G(\mathbf{E})$, which gives rise to the $\mathbf{E}$-root system of G [8: 4,5]. One can describe X^* as a stratified space in terms of the $\mathbf{Q}$-root system of G; but we postpone treatment of this till later.

(1.2) *Examples.* (a) SL_2. Consider

$$SL_2 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : ad - bc = 1 \right\}. \tag{1.2.1}$$

A model for the symmetric space for SL_2 is the upper half-plane $H \subset \mathbf{C}$, on which $SL_2(\mathbf{R})$ acts as Möbius transformations; the invariant metric can be taken to be the Poincaré metric $ds^2 = y^{-2}(dx^2 + dy^2)$. If $\Gamma \subset SL_2(\mathbf{Q})$ is arithmetic, one calls $X = \Gamma\backslash H$ a *modular curve.*

The method for compactifying X was understood a hundred years ago. We first describe briefly how a point is adjoined to X corresponding to ∞ on the Riemann sphere. Let P denote the subgroup of SL_2 given by the upper-triangular matrices $\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}$. This is a **Q**-*parabolic* subgroup of SL_2, with *Langlands decomposition*

$$P = U_P M_P A_P, \tag{1.2.2}$$

where U_P is the *unipotent radical* defined here by $a = 1$; A_P is the *split component* defined here by $a > 0$ and $b = 0$,[2] and M_P (the Levi factor complementary to A_P, for a basepoint on the y-axis) is just $\{\pm 1\}$ here.

One writes Γ_P for $\Gamma \cap P$, etc. (cf. (1.1.1)). Then $\Gamma_P = \Gamma_{U_P}$, a discrete group of x-translations. The main ingredient is rather easy in this example, and goes under the name *reduction theory*:

(1.2.3) **Proposition.** *Let Γ be an arithmetic subgroup of $SL_2(\mathbf{Q})$. Then there exists $L > 0$ such that if $Im\, z_1 > L$, $Im\, z_2 > L$, and $z_2 = \gamma \cdot z_1$ for some $\gamma \in \Gamma$, then $\gamma \in \Gamma_P$.*

(1.2.4) **Corollary.** *If Γ and L are as above, the natural surjection of the punctured unit disc Δ^* onto X:*

$$\Delta^* \xrightarrow{\lambda} \Gamma_P \backslash H \longrightarrow \Gamma \backslash H = X$$

is injective on $\{t \in \Delta^* : |t| < \varepsilon(iL)\}$; here $\varepsilon(z) = \lambda^{-1}(z) = exp(2\pi i z/m)$, where $b = m > 0$ gives a generator of Γ_{U_P}.

Thus, one can "fill in" the origin ($t = 0$). Every proper **Q**-parabolic subgroup of SL_2 is an $SL_2(\mathbf{Q})$ conjugate of P above. One compactifies X by adjoining one point for each Γ-conjugacy class of **Q**-parabolic subgroups, and these are canonically parametrized by the *finite* set $\Gamma \backslash SL_2(\mathbf{Q})/P(\mathbf{Q})$. One thereby obtains X^*, which is smooth in this case.

(b) *The Hilbert modular groups.* Let **E** be a totally-real number field and put $n = [\mathbf{E} : \mathbf{Q}]$. We will recall the definition of $G = R_{\mathbf{E}/\mathbf{Q}} SL_2$. A fundamental property of $G = R_{\mathbf{E}/\mathbf{Q}} SL_2$ is that $G(\mathbf{Q}) \cong SL_2(\mathbf{E})$ and $G(\mathbf{R}) \cong SL_2(\mathbf{R})^n$, with the inclusion $G(\mathbf{Q}) \hookrightarrow G(\mathbf{R})$ induced by using all n embeddings of **E** into **R**. When $n = 1$, this reverts to (a) above. It is an algebraic group over **Q**, with $rk_{\mathbf{Q}}(G) = 1$ and $rk_{\mathbf{R}}(G) = n$. The image of $A(\mathbf{Q})$ in $A(\mathbf{R})^n$ defines a maximal **Q**-split torus of G. The **C**- and **R**-root systems coincide, and are isomorphic to nA_1, while the **Q**-root system is simply A_1; restriction[3] is given naturally.

One can describe $G = R_{\mathbf{E}/\mathbf{Q}} SL_2$ explicitly. For the sake of simplicity, we restrict ourselves to the case $n = 2$, so $\mathbf{E} = \mathbf{Q}(\sqrt{N})$ for some square-free natural number N. Now, decompose $a, b, c, d \in \mathbf{E}$ into rational and irrational parts: $a = a' + a''\sqrt{N}$ ($a', a'' \in \mathbf{Q}$), etc. The equation defining SL_2 (see (1.2.1)) is equivalent to a pair of

[2] A_P is, in fact, the connected component of a maximal **Q**-split torus, thus, $r = 1$.

[3] For the general notion of restriction, see [8: 6], and also [2: 2.6].

equations in $a', a'', \ldots$, viz.

$$a'd' - b'c' + N(a''d'' - b''c'') = 1 \qquad a'd'' - b'c'' + a''d' - b''c' = 0.$$

These are the defining equations of $R_{\mathbf{E}/\mathbf{Q}}SL_2$ over $\mathbf{Q}$.

We return to the general case (i.e., n arbitrary). The symmetric space D being determined by $G(\mathbf{R})$ is clearly H^n. An example of an arithmetic group Γ coming from G is *not* $SL_2(\mathbf{Z})^n$, but rather $SL_2(\mathbf{O})$, where $\mathbf{O}$ denotes the ring of integers in $\mathbf{E}$. Thus, for any ideal $\mathscr{A} \subset \mathbf{O}$, the *congruence subgroup*

$$\Gamma(\mathscr{A}) = \{g \in SL_2(\mathbf{O}) : g \equiv I \,(\mathrm{mod}\, \mathscr{A})\}$$

is likewise arithmetic and is torsion-free if $\mathscr{A}$ is sufficiently small. One calls $X = \Gamma\backslash D$ a *Hilbert modular variety* of dimension n.

One has the $\mathbf{Q}$-parabolic subgroup $\tilde{P} = R_{\mathbf{E}/\mathbf{Q}}P$ of G. In this example, reduction theory takes the following form:

(1.2.5) **Proposition.** *Let* Γ *be an arithmetic subgroup of* $SL_2(\mathbf{E})$. *Then there exists* $L > 0$ *such that if* $\mathbf{z_j} = (z_{j1}, \ldots, z_{jn}) \in H^n$, $\prod_{1 \le k \le n} Im\, z_{jk} > L$ *for* $j = 1, 2$ *and* $\mathbf{z}_2 = \gamma \cdot \mathbf{z}_1$ *for some* $\gamma \in \Gamma$, *then* $\gamma \in \Gamma_{\tilde{P}}$.

We state now without explanation that the Baily-Borel Satake compactification X^* is obtained once again by adjoining to X a finite number of points, in one-to-one correspondence with $\Gamma\backslash SL_2(\mathbf{E})/P(\mathbf{E})$, and these are singular points of X^* whenever $n > 1$.

(c) Sp_{2r}. This is the group of $2r \times 2r$ symplectic matrices, which provides the simplest example of an irreducible group of $\mathbf{Q}$-rank r. It is a second generalization of (a), as $Sp_2 = SL_2$. The locally symmetric varieties X associated to Sp_{2r} are the moduli spaces of abelian varieties of dimension r with level structure. For a detailed treatment of this example, we refer the reader to [24].

(1.3) *The role of* $\mathbf{Q}$*-parabolic subgroups.* For any compactification $\hat{X}$ of X, one puts $\partial\hat{X} = \hat{X} - X$ and calls the latter the *boundary* of X in $\hat{X}$. The interesting compactifications of X have boundaries that can be described in terms of the $\mathbf{Q}$-parabolic subgroups (as in [8: Sec. 4]) of G. The ones we have in mind in the title of this paper are X^* (from 1.1), $\bar{X}$ (the *Borel-Serre compactification* [7], a manifold-with-corners, to be described below), and $\tilde{X}_\Sigma$ (the *toroidal compactification* [1], a non-singular projective variety, also to be described below, depending on a suitable combinatorial parameter Σ). These are related by the diagram

$$\begin{array}{ccc} \bar{X} & & \tilde{X}_\Sigma \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle g} & \\ & X^* & \end{array} \tag{1.3.1}$$

in which f is continuous (by [30]; see also [32]) and g is regular (from the construction of $\tilde{X}_\Sigma$). It is quite rare (for G of real rank 1 or when X is already compact) that one has a mapping from $\bar{X}$ to $\tilde{X}_\Sigma$ (extending the identity mapping of X); otherwise, there is no mapping of compactifications from either one to the other (see [16: (1.5)] and [34: Sec. 8]).

(a) *Rough description of* $\bar{X}$. For any space $\mathscr{D}$ of type S–**Q** for G (terminology and notation as in (1.1)), and neat arithmetic subgroup Γ, $Y = \Gamma\backslash\mathscr{D}$ admits a compactification $\bar{Y}$ that is a manifold-with-corners. Actually, one first attaches a boundary to $\mathscr{D}$, yielding a manifold-with-corners $\overline{\mathscr{D}}$, equivariantly for the action of $G(\mathbf{Q})$, and then puts $\bar{Y} = \Gamma\backslash\overline{\mathscr{D}}$; to see that $\bar{Y}$ is a manifold-with-corners, one uses the neatness of Γ and reduction theory.

The codimension-one faces of $\overline{\mathscr{D}}$ are naturally parametrized by the set $\mathscr{P}(G)_{\max}$ of maximal **Q**-parabolic subgroups of G, which is composed of $r = rk_{\mathbf{Q}}G$ G-conjugacy classes. For $P \in \mathscr{P}(G)_{\max}$, $P(\mathbf{R})$ acts transitively on $\mathscr{D}$ and the corresponding open face $e(P)$ of $\overline{\mathscr{D}}$ is canonically isomorphic to $\mathscr{D}/A_P$, where A_P is as in the Langlands decomposition (1.2.2) (actually defined for *any* **Q**-parabolic subgroup) and is one-dimensional. Then in $\bar{Y}$, the open face corresponding to P is $e'(P) \cong \Gamma_P\backslash e(P)$, and P must be taken modulo Γ-conjugacy. Since $e(P)$ is of type S–**Q** for P (which has **Q**-rank $r-1$), one can also attach a boundary with corners to it, yielding $\overline{e(P)}$, which is in fact homeomorphic, in the sense of compactifications of $e(P)$ (see [16: (1.1)]), to the closure of $e(P)$ in $\overline{\mathscr{D}}$.[4] One has similarly $\overline{e'(P)} \cong \Gamma_P\backslash\overline{e(P)}$. Thus, the boundary is easiest to describe when $r = 1$. Of course, to obtain $\bar{X}$, one starts with $\mathscr{D} = D$ in the above.

We should say something about how the space $e(P)$ is adjoined to D. It is placed as the set of limit points of $A_P(\mathbf{R})$-orbits in D for an action of $A_P(\mathbf{R})$ on D, not the standard one (given by restriction of the homogeneous action of $G(\mathbf{R})$), but rather the *geodesic action* [7: Sec. 3]. In the case of $P \subset SL_2$ (1.2(a)), it is easy to give the two actions explicitly: the standard action of $A_P(\mathbf{R})$ is given by $z = x + iy \mapsto a^2 z = a^2 x + ia^2 y$; the geodesic action is $z = x + iy \mapsto x + ia^2 y$. The latter produces a *line* at infinity, which arises by letting $a \to \infty$ (parametrized by x), and therefore, a *circle* at infinity in the arithmetic quotient. There are finitely many such circles in $\bar{X}$, and by collapsing each of them to a point, one obtains X^*. The construction of the mapping $\bar{X} \to X^*$ in general is carried out in [30] (see also [32]).

The face $e'(P)$ can be viewed as an arithmetic quotient associated to ${}^0P = U_P M_P$ (${}^0P$ from [7: 1.1]), which splits $P \to P/A_P$.[5] Because U_P is normal in P, one obtains therefrom a fibration

$$\begin{array}{ccc} \Gamma_{U_P}\backslash U_P(\mathbf{R}) & \longrightarrow & e'(P) \\ & & \Big\downarrow \kappa \\ & & \hat{e}'(P) \end{array} \tag{1.3.2}$$

[4] Another feature of $\overline{\mathscr{D}}$ is that $\overline{e(P_1)}$ and $\overline{e(P_2)}$ intersect in $\overline{\mathscr{D}}$ if and only if $P_1 \cap P_2$ is parabolic. This hints at the role of the non-maximal parabolic subgroups of G in the construction of $\overline{\mathscr{D}}$ [7: 5.3]. Note, though, that what is called μ_x in [7: 5.4 (4), (5)] should be labelled μ_x^P; this would avert the incorrect statement **5.4**(9), and the ensuing problems with Sec. 10.

[5] ${}^0P$ is an intrinsically defined algebraic **Q**-subgroup of P. After selection of a basepoint for D, M_P is defined as the intersection of the corresponding Levi subgroup with ${}^0P$.

with $\hat{e}'(P) = \Gamma_{M_P}\backslash\hat{e}(P)$, an arithmetic quotient of the symmetric space $\hat{e}(P)$ of M_P.

However, in general, it is *false* that a ${}^0P(\mathbf{R})$-orbit in D projects to a nice cross-section over $e'(P)$ to the geodesic orbits of A_P.[6] To make a cross-section that is well-defined, and extends to $\overline{e'(P)}$, one has to make a change of variables ([31: (3.19)], or see [15: 3.11.3]): the usual mapping

$$\Phi : {}^0P(\mathbf{R}) \times A_P(\mathbf{R}) \to D, \qquad \Phi(p,a) = (pa) \cdot x_0 \tag{1.3.3}$$

must be adjusted by a certain mapping g, given as the composite

$${}^0P(\mathbf{R}) \to e'(P) \xrightarrow{g_0} A_P(\mathbf{R}) \cong \mathbf{R}^+,$$

in which g_0 goes to ∞ at a prescribed rate at the boundary of $e'(P)$; one can arrange that g is constant on right U_P-cosets. This yields

$$\Psi : {}^0P(\mathbf{R}) \times A_P(\mathbf{R}) \to D, \qquad \Psi(p,a) = (pg(p)a) \cdot x_0. \tag{1.3.4}$$

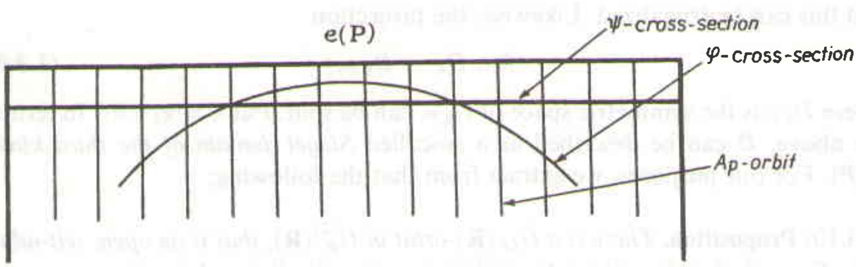
By the construction of g, the restriction of (1.3.4) to a sufficiently "elevated" ray in $A_P(\mathbf{R})$ is just a reparametrization of a deleted collar O_P of $\overline{e'(P)} \subset \partial\overline{X}$. One obtains thereby a surjection

$$\psi : {}^0P(\mathbf{R}) \times \mathbf{R}^+ \to O_P. \tag{1.3.5}$$

Then for any $r \in \mathbf{R}^+$,

$$\psi({}^0P(\mathbf{R}) \times \{r\}) \tag{1.3.6}$$

is a cross-section of the desired sort. A picture of the above is:



(b) *The construction of $\hat{X}_\Sigma$.* This is inextricably related to the structure of X^*. We must begin with the Siegel domain picture of D associated to $P \in \mathscr{P}(G)_{\max}$. We assume that G is irreducible over $\mathbf{Q}$. Then the $\mathbf{Q}$-root system of G is of type BC (or its "degenerate" form C) [2: 2.9], so in particular has a linear Dynkin diagram Δ with distinguished end.

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[6] The trouble is that the image of ${}^0P(\mathbf{R})/K_P$ in D goes outside the known domain of reduction theory for P. The statement is true, though, over relatively compact subsets of $e'(P)$. See [31: (1.3)].

The G-conjugacy type of P, by the usual system of indexing parabolic subgroups, corresponds to the omission of a single simple $\mathbf{Q}$-root β from Δ. That divides Δ into two pieces: $\Delta - \{\beta\} = \Delta_\ell \sqcup \Delta_h$, with Δ_h containing the distinguished end (unless $\Delta_h = \emptyset$). This corresponds to the decomposition of M_P as the almost-direct product of two normal subgroups:

$$M_P = G_{\ell,P} \cdot G_{h,P}.^7 \tag{1.3.7}$$

For future reference, we also set

$$\tilde{G}_{\ell,P} = G_{\ell,P} A_P \qquad \tilde{G}_{h,P} = A_P G_{h,P}. \tag{1.3.8}$$

For $\Delta \ni \alpha \neq \beta$, $\alpha|_{A_P}$ is trivial. The center of the unipotent radical U_P is the product of the root spaces for those $\mathbf{Q}$-roots involving 2β in its expansion in terms of Δ; all other roots occurring in U_P have β with coefficient 1. We thus write $U_P^{(2)}$ for the said center. One can see that under the adjoint action of M_P on U_P, $G_{h,P}$ acts trivially on $U_P^{(2)}$; in other words, $G_{h,P}$ and $U_P^{(2)}$ commute.

D sits inside the $G(\mathbf{C})$-homogeneous space $\check{D}$ (the *compact dual* of D) as a $G(\mathbf{R})$-invariant subdomain. (In the case of the example (1.2(a)), $\check{D} = \mathbf{P}^1(\mathbf{C})$.) One defines $D(P)$ to be $U_P^{(2)}(\mathbf{C}) \cdot D \subset \check{D}$. It should be kept in mind that $P(\mathbf{R})$ acts transitively on D, so $D(P)$ is a homogeneous space for $P(\mathbf{R}) \cdot U_P^{(2)}(\mathbf{C})$. Via the (non-)magic of the *Cayley transform* (see Appendix B), one can make $G_{\ell,P}$ "disappear", leaving the following favorable situation associated to the group $G_h(\mathbf{R}) \cdot U_P(\mathbf{R}) \cdot U_P^{(2)}(\mathbf{C})$.

First, one has a principal $U_P^{(2)}(\mathbf{C})$-bundle

$$D(P) \xrightarrow{\tilde{\pi}_2} D_A =: U_P^{(2)}(\mathbf{C}) \backslash D(P), \tag{1.3.9.1}$$

and this can be trivialized. Likewise, the projection

$$\tilde{\pi}_1 : D_A \to D_{h,P}, \tag{1.3.9.2}$$

where $D_{h,P}$ is the symmetric space of $G_{h,P}$, can be split. Put $\tilde{\pi} = \tilde{\pi}_1 \circ \tilde{\pi}_2$. In terms of the above, D can be described as a so-called *Siegel domain of the third kind* in $D(P)$. For our purposes, we extract from that the following:

(1.3.10) **Proposition.** *There is a $\tilde{G}_{\ell,P}(\mathbf{R})$-orbit in $U_P^{(2)}(\mathbf{R})$, that is an open, self-adjoint cone C_P such that, for all $\tilde{a} \in D_A$, $\tilde{\pi}_2^{-1}(\tilde{a})$ is a "positive" translate of*

$$\{u \in U_P^{(2)}(\mathbf{C}) : Im\, u \in C_P\},$$

and the translation depends real-analytically on $\tilde{a}$.

(1.3.11) *Remarks.* (i) In fact, C_P is a model for the symmetric space of non-compact type for the reductive group $\tilde{G}_{\ell,P}(\mathbf{R})$.

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[7] For the purposes of this exposition, we will ignore such subtleties as the distinction between M_P and the quotient ${}^0P/U_P$, or whether G_h (similarly $G_{\ell,P}$) is viewed as a subgroup of M_P or a quotient, i.e., M_P/G_ℓ. These distinctions affect the arithmetic group that we call Γ_{M_P} or $\Gamma_{G_{h,P}}$ (recall (1.1.1)), but these are, of course, well-defined up to commensurability.

(ii) We now write $\Delta = \{\beta_1, \ldots, \beta_r\}$ (as an *ordered* set) with β_r at the distinguished end. In the case of $G = Sp_{2r}$ (1.2(c)), if P corresponds to the omission of β_{s+1} $(0 \leq s < r)$, one has that $\tilde{G}_{\ell,P} \cong GL_s$, and C_P is, in a simple way, isomorphic to the cone of real positive-definite $s \times s$ matrices (see [24: p. 144]).

Taking arithmetic quotients in (1.3.9), one obtains the fundamental tower of varieties associated to P:

$$X'_P \xrightarrow{\pi_2} \mathscr{A}_P \xrightarrow{\pi_1} X_P. \tag{1.3.12}$$

In (1.3.12), X_P is the locally symmetric variety associated to $G_{h,P}$ and its arithmetic group $\Gamma_{G_{h,P}}$ (that we henceforth denote $\Gamma_{h,P}$, and similarly for $\Gamma_{\ell,P}$); $\mathscr{A}_P = (\Gamma_{h,P} \cdot \Gamma_{U_P}) \backslash D_A$; $X'_P = (\Gamma_{h,P} \cdot \Gamma_{U_P}) \backslash D(P)$; π_2 is the quotient mapping associated to the principal action of the torus $\mathscr{T}_P = \Gamma_{U_P^{(2)}} \backslash U_P^{(2)}(\mathbf{C})$; the fibers of π_1 are abelian varieties (hence compact). Put $\pi = \pi_1 \circ \pi_2$.

In actuality, $G_{\ell,P}$ has not completely disappeared for there remains an action of $\Gamma_{\ell,P}$ on (1.3.12), induced by the adjoint action of M_P. The action is free on X'_P, trivial on X_P, and nontrivial on $\mathscr{A}_P$ (hence awful) unless $\mathscr{A}_P = X_P$, i.e., $U_P = U_P^{(2)}$. One has

$$\Gamma_{\ell,P} \backslash X'_P \cong \Gamma_P \backslash D(P) \supset \Gamma_P \backslash D. \tag{1.3.13}$$

Reduction theory for P is loosely expressed in the following somewhat circular statement (cf. (1.2.4)), which says, more or less, that one can consider (1.3.13) for each P separately.

(1.3.14) **Proposition.** *In a deleted neighborhood of the part of the boundary of X that comes from P, the surjective mapping $\Gamma_P \backslash D \to \Gamma \backslash D$ is one-to-one.*

In describing X^* (or similarly, D^*), one adjoins a copy of X_P to X'_P by adding a single section of π "at infinity" (as determined by C_P), on which $\Gamma_{\ell,P}$ acts trivially. The comparability and compatibility of these, as P varies, come down to the following elementary observations. Let P be determined $\beta \in \Delta$. Then as β moves toward the distinguished end, $G_{h,P}$ gets smaller while $G_{\ell,P}$ and $U_P^{(2)}$ get larger. Of course, the Baily-Borel Satake compactification of X_P is definable in its own right; $(X_P)^*$ sits inside ∂X^* as the closure of X_P in X^*.

Since π_1 in (1.3.12) is proper, an available alternative is to add a nice boundary to the torus $\mathscr{T}_P$. This can be done by the method of torus embeddings,[8] yielding $\mathscr{T}_P \subset \mathscr{T}_{P,\Sigma_P}$, on which both $\mathscr{T}_P$ and $\Gamma_{\ell,P}$ act, with the latter acting freely (away from the vertex). Here, Σ_P denotes a $\Gamma_{\ell,P}$-invariant fan of rational *simplicial* cones in the closure of C_P. The edges τ of the cones produce smooth divisors $\mathscr{T}_\tau$ at infinity, and the union of these is, by construction, a divisor with normal crossings in $\mathscr{T}_{P,\Sigma_P}$. Then, $\mathscr{T}_{P,\Sigma_P}$ can be sewn in along π_2; viz., put

$$X'_{P,\Sigma_P} = X'_P \times^{\mathscr{T}_P} \mathscr{T}_{P,\Sigma_P}, \tag{1.3.15}$$

[8] For the general theory of torus embeddings, see [25].

on which $\Gamma_{\ell,P}$ acts. Each edge τ in Σ_P determines a smooth divisor $Z_\tau = X'_P \times^{\mathcal{T}_P} \mathcal{T}_\tau$ in X'_{P,Σ_P}. The (closed) divisor with normal crossings created by the set of edges *interior to* C_P will be denoted $\tilde{Z}_{P,\Sigma}$, and we put

$$Z_{P,\Sigma} = \Gamma_{\ell,P}\backslash\tilde{Z}_{P,\Sigma}. \tag{1.3.16}$$

If Σ denotes a compatible specification of Σ_P, as P varies (see [1: p. 252]), one sees that open subsets of the various quotients $\Gamma_{\ell,P}\backslash X'_{P,\Sigma_P}$ (cf. (1.3.13)) patch together to produce a smooth compactification $\tilde{X}_\Sigma$ of X, with $\partial\tilde{X}_\Sigma = \cup_P Z_{P,\Sigma}$ a divisor with normal crossings. One can arrange that $\tilde{X}_\Sigma$ is a projective variety.
(c) *Real quotients in torus embeddings.* Let $\mathcal{T}_P^c$ denote the maximal compact subgroup of $\mathcal{T}_P$, i.e., $\mathcal{T}_P^c \subset \mathcal{T}_P$ is isomorphic to $(S^1)^n \subset (\mathbf{C}^*)^n$ for some n. Suppose $\mathcal{U}$ is a $\mathcal{T}_P$-invariant subset of $\mathcal{T}_{P,\Sigma}$ and that $\mathcal{Y}$ is a $\mathcal{T}_P^c$-invariant subset of $\mathcal{U}$. Then the space obtained by collapsing the $\mathcal{T}_P^c$-orbits in $\mathcal{Y}$ to points is Hausdorff. We denote it by $\mathcal{U}_{\mathcal{Y}_\mathbf{R}}$ and call it the *real quotient of* $\mathcal{U}$ *along* $\mathcal{Y}$. This determines a subquotient space of X'_{P,Σ_P} from (1.3.15), namely,

$$X'_P \times^{\mathcal{T}_P} \mathcal{U}_{\mathcal{Y}_\mathbf{R}}; \tag{1.3.17}$$

it is a quotient of

$$\mathcal{U}'_P = X'_P \times^{\mathcal{T}_P} \mathcal{U} \subseteq X'_{P,\Sigma_P}.$$

This construction will play an auxiliary role in Sec. 3. It is a fundamental observation that the homotopy type of $(X'_{P,\Sigma_P})_{(\partial X'_{P,\Sigma_P})_\mathbf{R}}$ is independent of the choice of Σ_P (cf. [16: (1.4.13)]).

2. Results Independent of Hermitian Structure

(2.1) *Differential forms.* Let D be the symmetric space of non-compact type associated to the semi-simple algebraic group G, and let Γ be, say, a neat algebraic subgroup of $G(\mathbf{Q})$. By choosing a basepoint for D, we can write $D \cong G(\mathbf{R})/K$, where K is maximal compact in $G(\mathbf{R})$.

Let $\rho : G \longrightarrow GL(V)$ be a representation of G. This gives rise to a local system $\tilde{\mathbf{V}}_\Gamma$ on $X = \Gamma\backslash D$, whose associated vector bundle is $\mathcal{V}_\Gamma = \Gamma\backslash(D \times V_\mathbf{C})$.

By the de Rham theorem, the cohomology groups

$$H^\bullet(\Gamma, V_\mathbf{C}) \cong H^\bullet_\Gamma(\{pt\}, V_\mathbf{C}) \cong H^\bullet(X, \tilde{\mathbf{V}}_\Gamma) \tag{2.1.1}$$

may be computed as the cohomology of the complex of $\tilde{\mathbf{V}}_\Gamma$-valued C^∞ differential forms on X.

It is a standard device to express these differential forms as simply vector-valued *functions* on $\Gamma\backslash G(\mathbf{R})$. This is because the latter space is a principal K-bundle over X, and the vector bundles involved are isomorphic to equivariant bundles associated to representations of K. Specifically, differential forms on X come from $\wedge^\bullet(\boldsymbol{g}/\boldsymbol{k})^*$, where $\boldsymbol{g}$ and $\boldsymbol{k}$ are the Lie algebras of $G(\mathbf{R})$ and K respectively, with the adjoint action of K; $\mathcal{V}_\Gamma$ from $V_\mathbf{C}$ and $\rho|_K$. The complex of $\tilde{\mathbf{V}}_\Gamma$-valued C^∞ differential forms on $\Gamma\backslash G(\mathbf{R})$ is isomorphic to

$$C^{\infty}(\Gamma\backslash G(\mathbf{R})) \otimes V_{\mathbf{C}} \otimes \wedge^{\bullet}(\boldsymbol{g})^{*}. \tag{2.1.2}$$

One identifies the elements in (2.1.2) that come, by pullback, from X ([21: Sec. 4]; see also [28: Sec. 3]):

(2.1.3) **Proposition.** *The complex of $\tilde{\mathbf{V}}_{\Gamma}$-valued C^{∞} differential forms on X is isomorphic to*

$$[C^{\infty}(\Gamma\backslash G(\mathbf{R})) \otimes V_{\mathbf{C}} \otimes \wedge^{\bullet}(\mathbf{p})^{*}]^{K},$$

where $\boldsymbol{p}$ is the Cartan complement of $\boldsymbol{k}$ in $\boldsymbol{g}$. The differential d is the sum of two operators on the above, $d = D + d_{\rho}$, where D induced by differentiating the C^{∞} functions, as one does in taking the exterior derivative of vector-valued forms, and d_{ρ} is defined by using ρ to make operators on $V_{\mathbf{C}}$—e.g., for 0-forms,

$$[d_{\rho}(1 \otimes v)](Q) = 1 \otimes \rho(Q)v$$

whenever $Q \in \boldsymbol{p}$.

Since $X \subset \overline{X}$ is a homotopy equivalence, the local system $\tilde{\mathbf{V}}_{\Gamma}$ extends canonically to $\overline{X}$, and we therefore regard $\tilde{\mathbf{V}}_{\Gamma}$ as being defined on $\overline{X}$. Then, one has the analogue of (2.1.1):

$$H^{\bullet}(\Gamma_P, V_{\mathbf{C}}) \cong H^{\bullet}_{\Gamma_P}(\{pt\}, V_{\mathbf{C}}) \cong H^{\bullet}(e'(P), \tilde{\mathbf{V}}_{\Gamma}) \cong H^{\bullet}(\Gamma_P\backslash D, \tilde{\mathbf{V}}_{\Gamma_P}). \tag{2.1.4}$$

It should come as no surprise that one has an assertion analogous to (2.1.3) for the open faces of $\overline{X}$. Noting that

$$K_P =: K \cap P(\mathbf{R}) = K \cap {}^{0}P(\mathbf{R}) = K \cap M_P(\mathbf{R}),$$

one gets:

(2.1.5) **Proposition.** (i) *The complex of $\tilde{\mathbf{V}}_{\Gamma_P}$-valued C^{∞} differential forms on $\Gamma_P\backslash D$ is isomorphic to*

$$[C^{\infty}(\Gamma_P\backslash P(\mathbf{R})) \otimes V_{\mathbf{C}} \otimes \wedge^{\bullet}(\boldsymbol{u}_P \oplus \boldsymbol{p}_{M_P} \oplus \boldsymbol{a}_P)^{*}]^{K_P}.$$

(ii) *The complex of $\tilde{\mathbf{V}}_{\Gamma}$-valued*[9] *$C^{\infty}$ differential forms on $e'(P)$ is isomorphic to*

$$[C^{\infty}(\Gamma_P\backslash {}^{0}P(\mathbf{R})) \otimes V_{\mathbf{C}} \otimes \wedge^{\bullet}(\boldsymbol{p}_{M_P} \oplus \boldsymbol{u}_P)^{*}]^{K_P}.$$

Here $\boldsymbol{p}_{M_P}$ denotes the intersection of $\boldsymbol{p}$ and the Lie algebra of M_P, $\boldsymbol{u}_P$ is the Lie algebra of U_P, and $\boldsymbol{a}_P$ is the Lie algebra of A_P.

In view of (2.1.3) and (2.1.5), computations with differential forms on arithmetic quotients can be expressed in terms of vector-valued functions on the real

[9] equivalently, $\hat{\mathbf{V}}_{\Gamma_P}$-valued.

points of the group that are invariant under the actions of the arithmetic group and the (compact) isotropy group. Given $f \in C^\infty(\Gamma_{U_P}\backslash G(\mathbf{R}))$, one defines a C^∞ function f_P on $G(\mathbf{R})$ by

$$f_P(g) = \int_{N_P} f(ug)\, dm(u), \tag{2.1.6}$$

where m is the measure induced by the left Haar measure of U_P on the compact nilmanifold $N_P = \Gamma_{U_P}\backslash U_P$, normalized so that $m(N_P) = 1$. One calls (2.1.6) the *constant term of f with respect to P* (cf. the constant term of a Fourier series). It is clear that (2.1.6) makes sense for forms with values in any finite-dimensional vector space. One sees rather easily:

(2.1.7) **Lemma.** (i) *The function f_P is constant on right U_P-cosets.*
(ii) *If f is left-invariant under Γ_P, then so is f_P. If f is K_P-invariant, then so is f_P.*
(iii) *If $df = 0$, then $df_P = 0$.*

(2.1.8) *Remark.* One should keep in mind that if $Q \in \mathfrak{g}$ (i.e., Q is a left-invariant vector field on $G(\mathbf{R})$), the formula for $(Qf)(g)$ is $\frac{d}{dt}\big|_{t=0} f(g\, exp\,(tQ))$, and that for (2.1.7), one is taking $Q \in \mathfrak{p}$.

It is well known that the cohomology groups in (2.1.1) or (2.1.4) can, in fact, be computed by means of a subcomplex of (2.1.3) or (2.1.5) respectively, consisting of functions that satisfy a growth condition at infinity.

(2.1.9) **Definition.** *A function f on $G(\mathbf{R})$ is said to be slowly increasing, or have moderate growth, if for some, hence any, ρ as in* (1.1), *there is a natural number N so that*

$$|f(g)| \leq \|\rho(g)\|^N.$$

(2.1.10) *Remark* (see [3: 1,7]). The above is one of the conditions imposed in the definition of an automorphic form. It is equivalent to a uniform polynomial growth condition along all orbits of the geodesic action of a maximal $\mathbf{R}$-split torus.

It is not hard to see:

(2.1.11) **Lemma.** *If a function f has moderate growth, then so does its constant term f_P.*

(2.2) *The constant term and restriction to the boundary.*[10] Let f be a cocycle for (2.1.3), thus giving a cohomology class in $H^\bullet(X, \tilde{\mathbf{V}}_\Gamma)$. Consider the diagram:

[10] This section is a reworking of [27: 1.10].

$$\begin{tikzcd} H^\bullet(X,\tilde{\mathbf{V}}_\Gamma) \arrow[r, "l_P"] & H^\bullet(\Gamma_P\backslash D,\tilde{\mathbf{V}}_{\Gamma_P}) \arrow[r] & H^\bullet(\Gamma_P\backslash {}^0P(\mathbf{R})/K_P,\tilde{\mathbf{V}}_{\Gamma_P}) \\ H^\bullet(\overline{X},\tilde{\mathbf{V}}_\Gamma) \arrow[u, "\cong"] \arrow[rr, "r_P"] & & H^\bullet(e'(P),\tilde{\mathbf{V}}_\Gamma) \arrow[u, "\phi", "\cong"'] \end{tikzcd} \tag{2.2.1}$$

in which all arrows are induced by mappings of spaces.

(2.2.2) **Proposition.** *The diagram* (2.2.1) *commutes.*

Proof. The assertion would be clear if (1.3.3) gave a cross-section to the geodesic orbits, for we would then have a natural description of ϕ^{-1}. Since it usually does not, we will have to circumvent this detail. Note that since $e'(P)$ is the interior of a manifold-with-corners, it possesses compact deformation retracts E_P. Over E_P, (1.3.3) does, in fact, stay within a collar of $\overline{e'(P)}$ (as in (1.3.6); see figure at the end of (1.3(a))), for a sufficiently large. We can therefore deform E_P to the corresponding part of $\Gamma_P\backslash {}^0P(\mathbf{R})/K_P$ along geodesic orbits. Since

$$H^\bullet(e'(P),\tilde{\mathbf{V}}_\Gamma) \xrightarrow{\sim} H^\bullet(E_P,\tilde{\mathbf{V}}_\Gamma),$$

the assertion follows. ∎

Consider next the cohomology class $[r_P(f)] \in H^\bullet(e'(P),\tilde{\mathbf{V}}_\Gamma)$. Using the analogue of (2.2.1) for $\Gamma_P\backslash\overline{D}$, we see that $[r_P(f)]$ is also the restriction of $\tilde{f} = l_P(f)$, which defines a class

$$[\tilde{f}] \in H^\bullet(\Gamma_P\backslash D,\tilde{\mathbf{V}}_{\Gamma_P}).$$

On the other hand, note that (2.1.7) implies that the constant term of f (or equivalently, $\tilde{f}$) also defines a class

$$[f_P] \in H^\bullet(\Gamma_P\backslash D,\tilde{\mathbf{V}}_{\Gamma_P}).$$

(2.2.3) **Proposition.** *In* $H^\bullet(\Gamma_P\backslash D,\tilde{\mathbf{V}}_{\Gamma_P})$, $[\tilde{f}] = [f_P]$.

Proof. We use (2.1.5) to give a cohomological proof of this assertion.[11] As $\Gamma_P\backslash D$ is a compact nilmanifold fibration (cf. (1.3.2)), we consider first the $\tilde{\mathbf{V}}_{\Gamma_P}$-valued differential forms on the fiber:

$$V \otimes \wedge^\bullet \boldsymbol{u}_P^* \cong R^\bullet(\Gamma_{U_P}\backslash U_P(\mathbf{R}),\tilde{\mathbf{V}}_{\Gamma_P})^{U_P} \subset R^\bullet(\Gamma_{U_P}\backslash U_P(\mathbf{R}),\tilde{\mathbf{V}}_{\Gamma_P}), \tag{2.2.4}$$

where "$R^\bullet$" denotes C^∞ de Rham complex. It is not hard to write down a cochain homotopy from $R^\bullet(\Gamma_{U_P}\backslash U_P(\mathbf{R}),\tilde{\mathbf{V}}_{\Gamma_P})$ to $V \otimes \wedge^\bullet \boldsymbol{u}_P^*$, and likewise on the bundle, by simply viewing $\Gamma_{U_P}\backslash U_P(\mathbf{R})$ as an iterated circle bundle (see [29: Sec. 4(c)]

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[11] The proof given in [27: 1.10] is for automorphic forms f and uses growth estimates on $\tilde{f} - f_P$.

and [31: (2.5)]). The associated projection of the form given by $\tilde{f}$ onto the U_P-invariants is just f_P.

(2.3) *Calculation of $\boldsymbol{u}_P$-cohomology.* By definition, the cohomology of

$$C^\bullet(\boldsymbol{u}_P, V) =: V \otimes \wedge^\bullet \boldsymbol{u}_P^* \tag{2.3.1}$$

is the *Lie algebra cohomology* $H^\bullet(\boldsymbol{u}_P, V)$. Taking the cohomology in (2.2.4) gives

$$H^\bullet(\Gamma_{U_P}\backslash U_P(\mathbf{R}), \tilde{\mathbf{V}}_{\Gamma_P}) \cong H^\bullet(\boldsymbol{u}_P, V). \tag{2.3.2}$$

One observes that $C^\bullet(\boldsymbol{u}_P, V)$ is a complex of finite-dimensional representations of the reductive group M_PA_P. Therefore, $H^\bullet(\boldsymbol{u}_P, V)$ is likewise a representation of M_PA_P. By frequently invoked theorems of Kostant, one knows:

(2.3.3) **Theorem.** [18] (i) *There is an (M_PA_P)-equivariant embedding of complexes*

$$H^\bullet(\boldsymbol{u}_P, V) \hookrightarrow C^\bullet(\boldsymbol{u}_P, V)$$

(where the left-hand side is given the zero differential), inducing the identity mapping on cohomology.
(ii) *Assume that V is irreducible over $\mathbf{C}$, and let Λ denote the highest weight of V. Then*

$$H^i(\boldsymbol{u}_P, V) \cong \bigoplus_{w \in \mathbf{W}^P,\, l(w)=i} E_w,$$

where E_w is the irreducible representation of M_PA_P with highest weight $(w\Lambda + w\delta - \delta)$.

In the above, $\boldsymbol{W}^P$ is a specified subset of the absolute Weyl group $\boldsymbol{W}$ of G,[12] a positive Weyl chamber has been fixed, and δ is the half-sum of the positive roots in $\boldsymbol{g}$; for more details, see also [27:(2.4)], [31:(3.4)], etc.

Using (2.3.3(i)) and (2.1.5), one constructs an embedding of complexes

$$R^\bullet(\hat{e}'(P), \tilde{\mathbf{H}}^\bullet(\boldsymbol{u}_P, V)_{\Gamma_{M_P}}) \hookrightarrow R^\bullet(e'(P), \tilde{\mathbf{V}}_{\Gamma_P}), \tag{2.3.4}$$

One sees quite readily:

(2.3.5) **Proposition.** (i) *The Leray spectral sequence of κ in* (1.3.2) *degenerates at E_2;* (ii) *the inclusion* (2.3.4) *induces an isomorphism*

$$H^\bullet(e'(P), \tilde{\mathbf{V}}_{\Gamma_P}) \cong H^\bullet(\hat{e}'(P), \tilde{\mathbf{H}}^\bullet(\boldsymbol{u}_P, V)_{\Gamma_{M_P}}),$$

splitting the Leray filtration of $H^\bullet(e'(P), \tilde{\mathbf{V}}_{\Gamma_P})$ associated to κ.

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[12] It consists of the coset representatives of smallest length for $\boldsymbol{W}/\boldsymbol{W}_P$, where $\boldsymbol{W}_P$ denotes the Weyl group of P.

(2.3.6) *Remark.* Note that the isomorphism in (2.3.5(ii)) is defined over $\mathbf{Q}$. It can be written

$$H^\bullet(\Gamma_P, V) \cong H^\bullet(\Gamma_{M_P}, H^\bullet(\mathfrak{u}_P, V)).$$

3. Recent Results in the Hermitian Case

(3.1) *The Dolbeault/de Rham isomorphism for real torus embeddings.* Let T be (the $\mathbf{C}$-points of) a torus and thus isomorphic to $(\mathbf{C}^*)^n$. By a *real torus embedding,* we mean the quotient of a torus embedding T_Σ by the maximal compact subgroup T^c (i.e., real quotient, cf. (1.3(c))); we denote it $(T_\Sigma)_\mathbf{R}$ (see also [15: 2.1]). The name has been chosen because one can regard $(T_\Sigma)_\mathbf{R}$ as a *subspace* of T_Σ, namely, the closure of $T^0_\mathbf{R} \cong (\mathbf{R}^+)^n$ in T_Σ, which in turn can be constructed directly from $T^0_\mathbf{R}$ and Σ. Note also that $(T_\Sigma)_\mathbf{R}$ is always a manifold-with-corners, with interior $T^0_\mathbf{R}$. Therefore,

(3.1.1) **Proposition.** *For any torus embedding T_Σ, the associated real torus embedding $(T_\Sigma)_\mathbf{R}$ is contractible.*

(3.1.2) *Examples.* (i) $\mathbf{C}_\mathbf{R} = \mathbf{C}/S^1 \cong \mathbf{R}^{\geq 0} = [0, \infty)$.
(ii) The real quotient of $\mathbf{P}^n(\mathbf{C})$ is the n-simplex.

By construction, every point of T_Σ has a neighborhood contained in a maximal affine torus embedding isomorphic to $\mathbf{C}^n$. Playing the same role in the real quotient $(T_\Sigma)_\mathbf{R}$ is $\mathbf{C}^n/(S^1)^n \cong (\mathbf{C}_\mathbf{R})^n \cong (\mathbf{R}^{\geq 0})^n$, which is a corner. In other words:

(3.1.3) **Lemma.** *The local structure of the quotient mapping $p : T_\Sigma \to (T_\Sigma)_\mathbf{R}$ is that of $\mathbf{C}^n \to (\mathbf{R}^{\geq 0})^n$.*

We need to define the de Rham complex of a real torus embedding. Since the notion of a differential form is local in nature, it is enough, by use of (3.1.3), to consider the case of $(\mathbf{R}^{\geq 0})^n$ as the real quotient of $\mathbf{C}^n$. Then, we have the n-fold product of (3.1.2(i)), and one eventually realizes that it will suffice to consider that case.

If we regard the real quotient as a subspace or even as an abstract manifold-with-corners, a C^∞ differential form can be taken to be just a smooth form that is (locally) extendable across the boundary. As a quotient, it fits into the following general framework. Let Y be a C^∞ manifold, on which the compact Lie group H acts. If the action were free so that Y/H were smooth, one would have that the forms on Y/H pull back injectively to Y, yielding the space of those C^∞ forms on Y that are both:

(i) invariant under the action of H,
(ii) annihilated by interior multiplication with the elements of the Lie algebra of H (the latter defining vector fields on Y). (3.1.4)

One can *define* differential forms on Y/H in general by means of the above description [19].

There is a disparity between the two notions for $Y = \mathbf{C}$, $H = S^1$, $Y/H \cong \mathbf{R}^{\geq 0}$, coming from the fact that 1 is not the only element of $S^1 \cap \mathbf{R}^*$. Let (r, θ) be the usual polar coordinates in $\mathbf{C}$. For the subspace, we have $\theta = 0$, and at the origin, 0-forms $f(r)$ and 1-forms $g(r)\,dr$ for any smooth functions f and g. For the quotient, the forms cannot involve either θ or $d\theta$ and must be invariant under rotation by $\theta = \pi$, i.e., multiplication by -1; so the de Rham complex of the quotient is the subcomplex of the preceding generated by 1 and rdr over the even C^∞ functions. *We always understand the de Rham complex of a real torus embedding to be that of the quotient.*[13] The sheaf of forms thus determined on $(T_\Sigma)_{\mathbf{R}}$ will be denoted $\mathscr{R}^\bullet_{(T_\Sigma)_{\mathbf{R}}}$.

We are now in a position to state the most direct, though not final, form of the Dolbeault/de Rham isomorphism. The Dolbeault ($\bar\partial$-) complex of sheaves on T_Σ will be denoted $\mathscr{R}^{0,\bullet}_{T_\Sigma}$. Keeping in mind that

$$\mathscr{R}^\bullet_{(T_\Sigma)_{\mathbf{R}}} \subset (p_* \mathscr{R}^\bullet_{T_\Sigma})^{T^c},$$

we have:

(3.1.5) **Proposition.** *The projection of an i-form onto its $(0, i)$-component induces an isomorphism*[14] *of complexes of sheaves*

$$\mathscr{R}^\bullet_{(T_\Sigma)_{\mathbf{R}}} \xrightarrow{\sim} (p_* \mathscr{R}^{0,\bullet}_{T_\Sigma})^{T^c}.$$

Proof. This is a local assertion on $(T_\Sigma)_{\mathbf{R}}$. In view of (3.1.3), it is enough to check it for $\mathbf{C}^n \to (\mathbf{R}^{\geq 0})^n$. Using the product structure, we see that it will suffice to consider the case $n = 1$, at $0 \in \mathbf{R}^{\geq 0}$. There, the left-hand side is generated over the even C^∞ functions by 1 and $2rdr = d(z\bar z)$, whereas the right-hand side is generated by 1 and $zd\bar z$. (Here, $z = re^{i\theta}$ as usual.) One checks that dr projects onto $\frac{1}{2}e^{i\theta}d\bar z = \frac{1}{2}(dr - ird\theta)$, so $2rdr$ projects to $zd\bar z$. It remains to recognize that d projects to $\bar\partial$, but this is evident.

(3.1.6) *Remark.* By making use of simplicial constructions, one obtains the simplicial analogue of (3.1.5), which applies, for example, to the boundary divisor of a torus embedding.

Since we will need to consider fiber bundles of torus embeddings (recall (1.3.15)),[15] we need a more flexible version of (3.1.5). So let T be a torus, as above and Z a space on which T^c acts, such that $\pi^Z : Z \to B$ is a locally-trivial fiber

[13] However, both complexes are seen to give the same de Rham cohomology.

[14] More than just a quasi-isomorphism!

[15] We actually must also allow such bundles to degenerate, although we will not treat this here. See the latter half of [15: 2.8].

bundle. Let

$$Z \xrightarrow{p} Z_{\mathbf{R}} \xrightarrow{q} B \tag{3.1.7}$$

be the factorization of π^Z through the real quotient (i.e., the quotient for the action of T^c).

(3.1.8) **Proposition.** [15: 2.8.7] *Let $\mathscr{E}$ be a locally free sheaf on B and put $\mathscr{F} = (\pi^Z)^*\mathscr{E}$. Then the canonical inclusion*

$$q^{-1}\mathscr{E} \to (p_*\mathscr{F})^{T^c}$$

is an isomorphism.

Proof. Since this is a local question, we can assume B is a polydisc, so $\mathscr{E}$, hence $\mathscr{F}$, is free. Carrying out the argument with parameters, we reduce to the case where B is a point. So once again, we see that the crucial case is $Z = \mathbf{C}$. The asserted surjectivity says: an S^1-invariant holomorphic function on an S^1-invariant planar domain is constant. This follows from the Cauchy-Riemann equations in polar coordinates.

(3.1.9) *Remark.* When B is a point, we have that (3.1.5) is the version of (3.1.8) obtained from the latter after taking fine resolutions.

We state next a further assertion of the same genre. If the reader wishes, (s)he may do the proof as an exercise.

(3.1.10) **Proposition.** [15: 2.8.4] *In the situation of* (3.1.8), *$R^ip_*\mathscr{F} = 0$ for all $i > 0$.*

(3.1.11) **Corollary.** *The natural mapping $p_*\mathscr{F} \to Rp_*\mathscr{F}$ is a quasi-isomorphism.*

(3.2) *Holomorphic analogue of Proposition* (2.2.3). We take, in (3.1.7), $B = \mathscr{A}_P$. For Z, we have three subsets of $X'_{P,\Sigma}$ in mind:

$$\begin{aligned} &\text{(i) } Z_1 = Z_P,\\ &\text{(ii) } Z_2, \text{ a } T_P^c\text{- and } \Gamma_{\ell,P}\text{-invariant neighborhood of } Z_1 \text{ that is}\\ &\qquad \text{contained inside the realm of reduction theory for } P,\\ &\text{(iii) } Z_3 = Z_2 - Z_1, \text{ a deleted neighborhood of } Z_1. \end{aligned} \tag{3.2.1}$$

Here, π^Z is just $\pi_2|_Z$, where π_2 is as in (1.3.12). Of course, we have inclusions:

$$Z_1 \hookrightarrow Z_2 \hookleftarrow Z_3. \tag{3.2.2}$$

We note the following features:

(3.2.3) **Lemma.** (i) *The spaces Z_j in* (3.2.1) *all have the property that $\Gamma_{\ell,P}$ acts freely on Z_j and on Z_j/T_P^c.*
(ii) *For Z_1, on which T_P acts,* (3.1.7) *becomes*

$$Z_1 \xrightarrow{p} Z_1/T_P^c \xrightarrow{q} Z_1/T_P = \mathscr{A}_P.$$

One can identify the space Z_1/T_P^c, and see that it has fibers of the homotopy type of $\hat{C}_P$, the quotient of C_P by its cone dilations (see [16: (1.4.12)]), which is, in particular, contractible. As a result, we reach a curious conclusion (keep in mind the discussion of the action of $\Gamma_{\ell,P}$ on (1.3.12)):

(3.2.4) **Proposition.** *The mapping $q : Z_1/T^c_P \to \mathscr{A}_P$ is a homotopy equivalence, and hence is a model for the Borel construction for the action of $\Gamma_{\ell,P}$ on $\mathscr{A}_P$.*

(3.2.5) *Remark.* In fact, the same is true for Z_2/T^c_P and Z_3/T^c_P.

Next, let $\mathscr{F}$ be the so-called *canonical extension*, as defined in [23], of an automorphic vector bundle (holomorphic homogeneous bundle $\mathscr{V}_\Gamma$). From its construction, one sees that $\mathscr{F}$ is determined by the construction of an $\mathscr{F}'_{P,\Sigma}$ on each $X'_{P,\Sigma}$, and $\mathscr{F}|_Z = \mathscr{F} \otimes \mathscr{O}_Z$ is of the form $\pi_2^*\mathscr{E}$, as required for (3.1.8).

For all Z_j from (3.2.1), we have diagrams compatible with (3.2.2):

$$\begin{tikzcd}
H^\bullet(\Gamma_{\ell,P}\backslash Z_j, [\mathscr{F}|_{Z_j}]_{\Gamma_{\ell,P}}) \arrow[r, "\sim"] & H^\bullet_{\Gamma_{\ell,P}}(Z_j, \mathscr{F}|_{Z_j}) & H^\bullet_{\Gamma_{\ell,P}}(Z_j/T^c_P, Rp_*[\mathscr{F}|_{Z_j}]) \arrow[l, "\sim"'] \\
& & H^\bullet_{\Gamma_{\ell,P}}(Z_j/T^c_P, p_*[\mathscr{F}|_{Z_j}]) \arrow[u, "\simeq"'] \arrow[d, "\mathrm{inv}"] \\
H^\bullet_{\Gamma_{\ell,P}}(\mathscr{A}_P, \mathscr{E}) \arrow[r, "\sim"] & H^\bullet_{\Gamma_{\ell,P}}(Z_j/T^c_P, q^{-1}\mathscr{E}) \arrow[r, "\sim"] & H^\bullet_{\Gamma_{\ell,P}}(Z_j/T^c_P, \{p_*[\mathscr{F}|_{Z_j}]\}^{T^c_P}) \arrow[d, "\simeq"] \\
& & H^\bullet_{\Gamma_{\ell,P}}(Z_j/T^c_P, p_*[\mathscr{F}|_{Z_j}])^{T^c_P}
\end{tikzcd} \tag{3.2.6}$$

With the exception of the vertical arrow labelled "*inv*", every morphism in (3.2.6) is an isomorphism, as follows from (3.1.8), (3.1.11), (3.2.3), (3.2.4) and (3.2.5). The former is induced by taking the decomposition of $p_*[\mathscr{F}|_{Z_j}]$ as the direct sum of its weight spaces for T^c_P, and then projecting onto a direct factor. However, in the case of Z_1, one sees that *inv* is also an isomorphism for global reasons [15: 3.9].

This leaves us with the following picture:

(3.2.7) **Proposition.** *There is a commutative diagram*

$$\begin{tikzcd}
H^\bullet(Z_{P,\Sigma}, \mathscr{F} \otimes \mathscr{O}_{Z_{P,\Sigma}}) & H^\bullet_{\Gamma_{\ell,P}}(\mathscr{A}_P, \mathscr{E}) \arrow[l, "\sim"'] \arrow[d, equal] \\
H^\bullet(\Gamma_{\ell,P}\backslash Z_2, \mathscr{F}|_{Z_2}) \arrow[u] \arrow[d] \arrow[r, shift left, "\mathrm{inv}"] & H^\bullet_{\Gamma_{\ell,P}}(\mathscr{A}_P, \mathscr{E}) \arrow[l, hook, shift left] \arrow[d, equal] \\
H^\bullet(\Gamma_{\ell,P}\backslash Z_3, \mathscr{F}|_{Z_3}) \arrow[r, shift left, "\mathrm{inv}"] & H^\bullet_{\Gamma_{\ell,P}}(\mathscr{A}_P, \mathscr{E}) \arrow[l, hook, shift left]
\end{tikzcd}$$

The Dolbeault complex of an automorphic vector bundle on X can be expressed as vector-valued functions on $G(\mathbf{R})$ in a way completely analogous to what was done in (2.1) for deRham complexes. As such, we again have the notion of slowly increasing forms, as given in (2.1.9). (See [15: 3.8] for more on this.)

Let $j_\Sigma\colon X \to \tilde{X}_\Sigma$ denote the inclusion. The condition that a form be slowly increasing (together with its $\bar\partial$-derivative) is actually local on $\tilde{X}_\Sigma$, and hence defines a complex of sheaves on $\tilde{X}_\Sigma$, viz.

$$\mathscr{R}^{0,\bullet}(\mathscr{F})_{\mathrm{si}} \subset (j_\Sigma)_*(\mathscr{R}_X^{0,\bullet} \otimes j_\Sigma^*\mathscr{F}). \tag{3.2.8}$$

It is easy to see that it contains $\mathscr{R}^{0,\bullet}_{\tilde{X}_\Sigma}(\mathscr{F})$.

(3.2.9) **Proposition.** [14] *Let $\mathscr{F}$ be the canonical extension of an automorphic vector bundle on X. Then, the inclusion*

$$\mathscr{R}^{0,\bullet}_{\tilde{X}_\Sigma}(\mathscr{F}) \hookrightarrow \mathscr{R}^{0,\bullet}(\mathscr{F})_{\mathrm{si}}$$

is a quasi-isomorphism. In particular, the cohomology of the Dolbeault complex of C^∞ forms of moderate growth is the sheaf cohomology of $\mathscr{F}$ on $\tilde{X}_\Sigma$.

(3.2.10) *Remark.* A weaker version of (3.2.9) is that a $\bar\partial$-closed C^∞ $(0,i)$-form η on X with values in $(j_\Sigma)^*\mathscr{F}$, whose associated function f_η has moderate growth, defines a cohomology class in $H^i(\tilde{X}_\Sigma, \mathscr{F})$.

Now, let η be as in (3.2.10). It admits a restriction to all Z_j, so induces compatible elements of the left-hand sides of the rows in (3.2.7). In particular, from the first row, we get the restriction of η to the part of the boundary of $\tilde{X}_\Sigma$ associated to P:

$$\tilde{r}_P[\eta] \in H^\bullet(Z_{P,\Sigma}, \mathscr{F} \otimes \mathscr{O}_{Z_{P,\Sigma}}) \cong H^\bullet_{\Gamma_{\ell,P}}(\mathscr{A}_P, \mathscr{E}). \tag{3.2.11}$$

On the other hand, we may again take its constant term η_P. This defines a class in $H^\bullet(\Gamma_P\backslash D, \mathscr{F})$, which can be restricted to $\Gamma_{\ell,P}\backslash Z_3$, thereby producing an element $[\eta_P] \in H^\bullet(\Gamma_{\ell,P}\backslash Z_3, \mathscr{F}|_{Z_3})$.

(3.2.12) **Theorem.** [15: 3.10.3] *In $H^\bullet_{\Gamma_{\ell,P}}(\mathscr{A}_P, \mathscr{E})$, $inv([\eta_P]) = \tilde{r}_P[\eta]$.*

With (3.2.7) already done, the proof of this is quite easy. As both $inv([\tilde\eta])$ and $\tilde{r}_P[\eta]$ of $H^\bullet_{\Gamma_{\ell,P}}(\mathscr{A}_P, \mathscr{E})$ come from a common element of $H^\bullet(\Gamma_{\ell,P}\backslash Z_2, \mathscr{F}|_{Z_2})$, it is clear that

$$\operatorname{inv}[\tilde\eta] = \tilde{r}_P[\eta]. \tag{3.2.12.1}$$

It remains to replace $[\tilde\eta]$ by $[\eta_P]$ (cf. (2.2.3)). First, let $\eta^{(2)}$ denote the form corresponding to the average of f_η over $U_P^{(2)}(\mathbf{R})$, i.e., in (2.1.6) replace N_P by $\Gamma_{U_P^{(2)}} \backslash U_P^{(2)}(\mathbf{R})$. Then (3.2.12.1) holds, of course, with η replaced by $\eta^{(2)}$ (and likewise, for η_P). Since $\eta^{(2)} = inv(\eta)$, we get that

$$\tilde{r}_P[\eta] = \operatorname{inv}([\eta^{(2)}]). \tag{3.2.12.2}$$

To finish, one needs to check that averaging $\eta^{(2)}$ over $(U_P/U_P^{(2)})(\mathbf{R})$ does not change the cohomology class on $\Gamma_{\ell,P}\backslash Z_2$. For that, see [15: 3.5.12].

There is a direct analogue of (2.3.5) that enables one to calculate $H^\bullet_{\Gamma_{\ell,P}}(\mathscr{A}_P, \mathscr{E})$.

(3.2.13) **Proposition.** [15: 3.7.7] (i) *The Leray spectral sequence*

$$E_2^{p,q} = H^p_{\Gamma_{\ell,P}}(X_P, R^q(\pi_1)_*\mathscr{E}) \Rightarrow H^{p+q}_{\Gamma_{\ell,P}}(\mathscr{A}_P, \mathscr{E}).$$

degenerates at E_2.

(ii) $R^q(\pi_1)_*\mathscr{E}$ *is isomorphic to* $\mathscr{H}^q(\mathfrak{v}^-, E)$, *the automorphic vector bundle on* X_P *associated to the representation of* $K_{h,P}$ *on* $H^q(\mathfrak{v}^-, E)$.

(iii) *There is a canonical splitting*

$$H^i_{\Gamma_{\ell,P}}(\mathscr{A}_P, \mathscr{E}) \cong \bigoplus_{p+q=i} H^p_{\Gamma_{\ell,P}}(X_P, \mathscr{H}^q(\mathfrak{v}^-, E)).$$

(3.2.14) *Remark.* (We refer to [15] for the notation $\mathfrak{v}^-$ and $K_{h,P}$). The Lie algebra cohomology $H^\bullet(\mathfrak{v}^-, E)$ can again be computed by means of Kostant's theorem (2.3.3), whose setting, we recall, is unipotent radicals of parabolic subalgebras of reductive Lie algebras. See [15: 3.6].

(3.3) *Hodge theory at the boundary: a motivation.* We motivate the Hodge theoretic constructions to come by first considering a very general question. Let $\overline{X}$ be *any* manifold-with-corners, with interior X, $\tilde{\mathbf{V}}$ a local system on $\overline{X}$ (equivalently X), and let $\mathscr{B}$ denote the closed covering of the boundary $\partial\overline{X}$ by its closed faces of (real) codimension one. Consider the restriction mappings:

$$H^\bullet(X, \tilde{\mathbf{V}}) \xleftarrow{\sim} H^\bullet(\overline{X}, \tilde{\mathbf{V}}) \xrightarrow{\varepsilon} H^\bullet(\partial\overline{X}, \tilde{\mathbf{V}}) \xrightarrow{r} \bigoplus_{B\in\mathscr{B}} H^\bullet(B, \tilde{\mathbf{V}}). \tag{3.3.1}$$

(3.3.2) **Definition.** *The spectre of* X *in* $\partial\overline{X}$ *with coefficients in* $\tilde{\mathbf{V}}$ *is the set*

$$im\,\varepsilon \cap ker\,r \subset H^\bullet(\partial\overline{X}, \tilde{\mathbf{V}}),$$

and is denoted $Spect^\bullet(X, \tilde{\mathbf{V}})$.

(3.3.3) *Remark.* (i) It is immediate that

$$Spect^\bullet(X, \tilde{\mathbf{V}}) = ker\left\{ H^\bullet(\partial\overline{X}, \tilde{\mathbf{V}}) \to H^\bullet(\overline{X}, \partial\overline{X}; \tilde{\mathbf{V}}) \oplus \bigoplus_{B\in\mathscr{B}} H^\bullet(B, \tilde{\mathbf{V}}) \right\}.$$

(ii) Suppose $\mathscr{B}$ has only two elements, i.e., $\partial\overline{X} = B_1 \cup B_2$. Then $ker\,r = im\,\delta$, where

$$\delta : H^{\bullet-1}(B_1 \cap B_2, \tilde{\mathbf{V}}) \to H^\bullet(\partial\overline{X}, \tilde{\mathbf{V}})$$

is the connecting homomorphism in the Mayer-Vietoris sequence for $\partial\overline{X}$. This is an instance of a more general statement. Let $\{S_j : j \leq 0\}$ denote the simplicial filtration associated to the nerve of $\mathscr{B}$. Then

$$ker\,r = S_{-1}H^\bullet(\partial\overline{X}, \tilde{\mathbf{V}}).$$

(3.3.4) **Definition.** *A nonzero element of* $Spect^\bullet(X, \tilde{\mathbf{V}})$ *is called a ghost (class).*

In general, ε and r are quite unrelated, so there is little to be said. But suppose, in the above, we take $X = \Gamma\backslash D$ to be a locally symmetric variety and $\overline{X}$ its Borel-Serre compactification, as in (1.3(a)), and $\tilde{\mathbf{V}} = \tilde{\mathbf{V}}_\Gamma$, as in (2.1). Then

$$\mathscr{B} = \{\overline{e'(P)} : P \in \mathscr{P}(G)_{\max}\}. \tag{3.3.5}$$

There is some mild feeling that the following should be true:

(3.3.6) *Assertion.* If $\overline{X}$ is the Borel-Serre compactification of a locally symmetric variety, then $Spect^\bullet(\overline{X}, \tilde{\mathbf{V}}_\Gamma) = 0$.

It was once hoped that the statement in (3.3.6) would hold for *any* arithmetic quotient of a symmetric space (not necessarily Hermitian; recall (1.1)). The name "ghost" was given in [5: 4.1] in this setting only. I think it was chosen to suggest that although such things had not been seen, the possibility of their existence still haunts us.[16] It should be added, however, that a *non-Hermitian* case in which ghosts have been found is presented in [11: Sec. III] (see also [10: Sec. 7]).

We want to suggest here that Hodge theory may be useful as a technique for ruling out ghosts in the Hermitian case. It turns out, as we explain below, that the cohomology groups in (3.3.1) have natural mixed Hodge structures, derived from the functorial constructions of [26], and that ε and r are morphisms of mixed Hodge structure. It follows that $Spect^\bullet(\overline{X}, \tilde{\mathbf{V}}_\Gamma)$ gets a mixed Hodge structure; perhaps more to the point, $im\,\varepsilon$ and $ker\,r$ are both mixed Hodge substructures of $H^\bullet(\partial\overline{X}, \tilde{\mathbf{V}}_\Gamma)$, and if they are "different enough", their intersection would have to be trivial.

The main difficulty in effecting this plan is that $\overline{X}$ is far from being a variety! On the other hand, a nice toroidal compactification $\tilde{X}_\Sigma$ is a smooth projective variety. We consider the analogous set-up for $\partial\tilde{X}_\Sigma$, which has a closed covering

$$\mathscr{Z} = \{Z_{P,\Sigma} : P \in \Gamma\backslash\mathscr{P}(G)_{\max}\}. \tag{3.3.7}$$

Let $\tilde{i}_\Sigma$ denote the inclusion $\partial\tilde{X}_\Sigma \to \tilde{X}_\Sigma$. For any subset $\mathscr{S}$ of $\mathscr{P}(G)_{\max}$, we have likewise $\tilde{i}_\mathscr{S}$, the inclusion $Z_{\mathscr{S},\Sigma} = \bigcap_{P\in\mathscr{S}} Z_{P,\Sigma} \to \tilde{X}_\Sigma$. One defines the complex of sheaves

$$\mathscr{C}^\bullet_{\mathrm{dn}}(Z_{\mathscr{S},\Sigma}, \tilde{\mathbf{V}}_\Gamma) = \tilde{i}^*_\mathscr{S} Rj_{\Sigma *}\tilde{\mathbf{V}}_\Gamma. \tag{3.3.8}$$

Since our space is reasonable and $Z_{\mathscr{S},\Sigma}$ is compact, the latter has a fundamental system of open neighborhoods $N_{\mathscr{S},\Sigma}$ such that the hypercohomology of (3.3.8), $H^\bullet_{\mathrm{dn}}(Z_{\mathscr{S},\Sigma}, \tilde{\mathbf{V}}_\Gamma)$, satisfies

$$H^\bullet_{\mathrm{dn}}(Z_{\mathscr{S},\Sigma}, \tilde{\mathbf{V}}_\Gamma) \cong H^\bullet(N_{\mathscr{S},\Sigma} - \partial\tilde{X}_\Sigma, \tilde{\mathbf{V}}_\Gamma). \tag{3.3.9}$$

[16] If an automorphic form could produce such a class, it would have to be by means of some hitherto unknown construction. Another interpretation: a ghost class is there, but if you look in any particular place, you don't see it.

For this reason, (3.3.8) is called the *deleted neighborhood complex* of $Z_{\mathscr{S},\Sigma}$ with coefficients in $\tilde{\mathbb{V}}_\Gamma$, and the left-hand side of (3.3.9) is called *deleted neighborhood cohomology*, whence the subscript "*dn*". Similarly, one has $H^\bullet_{\mathrm{dn}}(\partial\tilde{X}_\Sigma, \tilde{\mathbb{V}}_\Gamma)$ as the cohomology of $\tilde{i}^*_\Sigma R j_{\Sigma *}\tilde{\mathbb{V}}_\Gamma$. The latter can be reconstituted by a simplicial construction from (3.3.8).

It is well known by now that the local system $\tilde{\mathbb{V}}_\Gamma$ on X underlies a (polarized) variation of Hodge structure [9: 1.1].[17] According to [26], the deleted neighborhood complexes (3.3.8) then underlie *mixed Hodge modules*, imparting canonical mixed Hodge structures to (3.3.9).

Of course, in actuality, we are interested in

$$H^\bullet_{\mathrm{dn}}\left(\bigcap_{P\in\mathscr{S}} \overline{e'(P)}, \tilde{\mathbb{V}}_\Gamma\right) \cong H^\bullet\left(\bigcap_{P\in\mathscr{S}} \overline{e'(P)}, \tilde{\mathbb{V}}_\Gamma\right), \tag{3.3.10}$$

where $H^\bullet_{\mathrm{dn}}$ is defined by the formulas analogous to (3.3.8) and (3.3.9); the isomorphism holds because we are on a manifold-with-corners. Concerning the intersection in (3.3.10), the situation *prior to* taking arithmetic quotients is easily described. One has, by iteration of [7: 7.4], for $\mathscr{S} \subset \mathscr{P}(G)_{\max}$,

$$\bigcap_{P\in\mathscr{S}} \overline{e(P)} = \overline{e(Q_{\mathscr{S}})} \qquad \text{if } Q_{\mathscr{S}} = \bigcap_{P\in\mathscr{S}} P \text{ is parabolic,} \tag{3.3.11}$$

(and is empty otherwise). Here, $e(Q_{\mathscr{S}})$ denotes the face of codimension $|\mathscr{S}|$ in $\bar{D}$ associated to $Q_{\mathscr{S}}$. One sees that after the quotient by Γ is taken, $\overline{e'(Q_{\mathscr{S}})}$ is one connected component of $\bigcap_{P\in\mathscr{S}} \overline{e'(P)}$ (and the others are of a similar nature), whenever $\mathscr{S}$ is as in (3.3.11) and represents $\mathscr{S}$. (See [16: App. to (3.5)].)

Although we know that it is rarely true, suppose for the moment that there was a mapping (necessarily surjective) $\tau : \bar{X} \to \tilde{X}_\Sigma$, such that $\tau|_X$ is the identity mapping of X and $\tau(\overline{e'(P)}) = Z_{P,\Sigma}$. The respective fundamental neighborhoods $\bar{N}_P$ and $N_{P,\Sigma}$ may be chosen so that $\bar{N}_P = \tau^{-1}N_{P,\Sigma}$ and $N_{P,\Sigma} = \tau(\bar{N}_P)$. Then

$$\bar{N}_P - \partial\bar{X} \cong N_{P,\Sigma} - \partial\tilde{X}_\Sigma,$$

and likewise for intersections. Thus, the deleted neighborhood cohomololgy would be the same in both $\tilde{X}_\Sigma$ and $\bar{X}$, and we could simply transport the mixed Hodge structures from one to the other. However, since τ seldom exists, one has to do something more serious (see also the recent article [33]).

(3.3.12) *Remark.* Looijenga has pointed out that the *existence* of these mixed Hodge structures follows from the interpretation of (3.3.10) as iterated deleted neighborhood cohomology on X^*, via the mapping f in (1.3.1). However, this

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[17] If V is real. In any case, it is a direct factor of one, which is sufficient for our purposes. There is no canonical determination of the *weight*, however, given that we began with G a semi-simple group in (1.1); one is chosen ad hoc in [28]. For this reason, when a Shimura variety is defined, as in [9], one needs a central torus to determine weights. However, the variation of Hodge structure is easily seen to be uniquely defined modulo Tate twists.

observation does not help in calculating them, for the latter seems to require (3.4.2) and (3.5.5).

(3.4) *Hodge theory at the boundary: a battle of nerves.* We came to suspect that the nerves of the coverings of deleted neighborhoods associated to $\mathscr{B}$ (for $\partial\overline{X}$) and $\mathscr{Z}$ (for $\partial\tilde{X}_\Sigma$) are homotopy-equivalent. More precisely, it was an issue of proving:

(3.4.1) **Proposition.** [16: (2.7.8)] *There is a system of compatible homotopy equivalences of a fundamental system of deleted neighborhoods*

$$\overline{N}_P - \partial\overline{X} \quad \textit{and} \quad N_{P,\Sigma} - \partial\tilde{X}_\Sigma,$$

as P ranges over $\Gamma\backslash\mathscr{P}(G)_{\max}$.

The proof of this turns out to imply the same for intersections. In particular, we obtain the following.

(3.4.2) **Corollary.** [16: (3.5.5)] *The spectral sequences associated to the simplicial filtration S:*

$$\overline{E}_1^{p,q} = \bigoplus_{|\mathscr{S}|=p+1} H^q\left(\bigcap_{P\in\mathscr{S}} \overline{e'(P)}, \tilde{\mathbf{V}}_\Gamma\right) \Rightarrow H^{p+q}(\partial\overline{X}, \tilde{\mathbf{V}}_\Gamma)$$

and

$$\tilde{E}_1^{p,q} = \bigoplus_{|\mathscr{S}|=p+1} H^q_{\mathrm{dn}}(Z_{\mathscr{S},\Sigma}, \tilde{\mathbf{V}}_\Gamma) \Rightarrow H^{p+q}(\partial\tilde{X}_\Sigma, \tilde{\mathbf{V}}_\Gamma)$$

are isomorphic.

In other words, we can now proceed as though a continuous mapping $\tau : \overline{X} \to \tilde{X}_\Sigma$, as at the end of (3.3), existed.

The key idea in the proof of (3.4.1) was to understand the structure of the simplicial complexes $\hat{\Sigma}_P = (\Sigma_P - \{0\})/\mathbf{R}^+$. Similarly, let $\hat{C}_P$ be the quotient of C_P by its cone dilations. First, we have an observation, whose proof is not very difficult.

(3.4.3) **Proposition.** [16: (2.1.1)] $\Gamma_{\ell,P}\backslash\hat{\Sigma}_P$ *is a triangulation of the Satake compactification* $\Gamma_{\ell,P}\backslash\hat{C}_P^*$ *of* $\Gamma_{\ell,P}\backslash\hat{C}_P$ *that is associated to* $\{\beta_1\}$.

Before proceeding, we recall a feature of the quotient mapping $f : \overline{X} \to X^*$ (from (1.3.1); see [31: (1.6)]).

(3.4.4) **Proposition.** *The fiber of f over a point of* X_P *(from* (1.3.12)*) is* $\Gamma_{\ell,P}\backslash\overline{\hat{C}_P}$.

These facts hint at a role for the quotient mapping

$$\Gamma_{\ell,P}\backslash\overline{\hat{C}_P} \xrightarrow{k} \Gamma_{\ell,P}\backslash\hat{C}_P^* \tag{3.4.5}$$

from the Borel-Serre compactification of $\Gamma_{\ell,P}\backslash\hat{C}_P$ to the Satake compactification in (3.4.3). This mapping can be regarded as a non-Hermitian analogue of f above, and it is again a case of the content of [30] (also, cf. [16: (1.3(c))]). The strata in $\partial\Gamma_{\ell,P}\backslash\hat{C}_P^*$ are parametrized in much the same way as those of X^*, viz., by $\Gamma_{\ell,P}\backslash\mathscr{P}(G_{\ell,P})_{\max}$. The stratum corresponding to $Q\in\Gamma_{\ell,P}\backslash\mathscr{P}(G_{\ell,P})_{\max}$ will be denoted $\mathscr{E}_Q$ and is of the form $\Gamma_{\ell,P'}\backslash\hat{C}_{P'}$, where $P'\in\Gamma\backslash\mathscr{P}(G)_{\max}$ corresponds to the omission of the same $\beta\in\Delta_\ell$ as Q. Similarly, if P is given by the omission of β_t, then there are t stratum types S_j $(1\le j\le t)$, whose dimensions increase with j in $\Gamma_{\ell,P}\backslash\Sigma_P$. They correspond to the chains of length at most $t-1$ containing β_1 and also the empty chain (recall (1.3.11(ii))).

A question emerges: is it possible to construct $\Gamma_{\ell,P}\backslash\overline{\hat{C}_P}$ from $\Gamma_{\ell,P}\backslash\hat{\Sigma}_P$ (given that we know from (3.4.3) that we have $\Gamma_{\ell,P}\backslash\hat{C}_P^*$)? This necessitates finding a boundary with corners. Towards an answer, let σ be any simplex of $\hat{C}_P$. We partition the vertices of σ according to the strata type that they lie in. Let σ_j denote the face (possibly empty) of σ spanned by the vertices of σ in S_j. Clearly, σ is the iterated join

$$\sigma=\sigma_1*\cdots*\sigma_t \tag{3.4.6}$$

(where one is taking the convention $\alpha*\emptyset=\alpha$).

In the case of a single join, i.e., $\sigma=\sigma_1*\sigma_2$, one has that σ is the two-way mapping cone of the two projections of $\sigma_1\times\sigma_2$. Explicitly, σ is the quotient of $\sigma_1\times[0,1]\times\sigma_2$ obtained by collapsing $\sigma_1\times\{0\}\times\sigma_2$ onto σ_1, and likewise, $\sigma_1\times\{1\}\times\sigma_2$ onto σ_2. Define $\mathscr{B}\ell_1(\sigma)$, the *blow-up of σ along* σ_1, to be the polyhedron obtained by making only the second identification above. There is the quotient mapping $\mathscr{B}\ell_1(\sigma)\to\sigma$, which can be rewritten as $Cone(\sigma_1)\times\sigma_2\to\sigma_1*\sigma_2$. This can be seen as the real quotient of blowing up of $\mathbf{P}^a\subset\mathbf{P}^{a+b+1}$, where $\mathbf{P}^{a+b+1}$ is viewed as the linear join of $\mathbf{P}^a$ and a disjoint $\mathbf{P}^b$, with morphism

$$\mathbf{P}^a\times\mathbf{P}^1\times\mathbf{P}^b\to\mathbf{P}^{a+b+1}$$

(recall (3.1.2(ii))).

By iterating the above $t-1$ times (recall (3.4.6)), in the proper order (see [16: (2.3.7)]), we obtain a polyhedron $\mathscr{B}\ell(\sigma)$ from σ. Since we have used a $\Gamma_{\ell,P}$-stable, canonically ordered partition of the vertices, the above procedure produces a polyhedral complex $\mathscr{B}\ell(\hat{\Sigma}_P)$ from $\hat{\Sigma}_P$ and then a mapping

$$\Gamma_{\ell,P}\backslash\mathscr{B}\ell(\hat{\Sigma}_P)\longrightarrow\Gamma_{\ell,P}\backslash\hat{\Sigma}_P. \tag{3.4.7}$$

Our construction gives $[0,1]$-variables coming from the join parameters and this produces a boundary-with-corners, possessing canonical collars in $\Gamma_{\ell,P}\backslash\mathscr{B}\ell(\hat{\Sigma}_P)$.[18] In this situation, given that (3.4.5) exists, it is automatic that the boundary of $\Gamma_{\ell,P}\backslash\mathscr{B}\ell(\hat{\Sigma}_P)$ has the stratified homotopy type of the Borel-Serre boundary [16: (2.6.4)]. From here, it is not hard to complete the proof of (3.4.1).

[18] For this, we must assume that $\hat{\Sigma}_P$ is not too coarse, e.g., that it be a barycentric subdivision (compare the proof of (2.2.8) in [15]).

(3.5) *Hodge theory at the boundary: determination of the mixed Hodge structure on the E_1-term.* We can now give the formula for the mixed Hodge structure on the E_1-term of the nerve spectral sequence, which is written out in its two isomorphic forms in (3.4.2).

We begin, though, with the related cohomology of X'_P (from (1.3.12)), the so-called *mixed Shimura variety* associated to P ($P \in \mathscr{P}_{\max}$). We can see easily that X'_P has the homotopy type of an N_P-fibration over X_P. From (2.3), one has formulas for this sort of thing. We get

$$H^\bullet(X'_P, \tilde{\mathbf{V}}_{\Gamma_P}) \cong H^\bullet(X_P, \tilde{\mathbf{H}}(\boldsymbol{u}_P, V)_{\Gamma_{h,P}}) \cong \bigoplus_w H^\bullet(X_P, (\tilde{\mathbf{E}}_w)_{\Gamma_{h,P}})[-\ell(w)]. \tag{3.5.1}$$

Here, "$[-\ell(w)]$" merely indicates a shift and the other notation is as in (2.3.3(ii)); we are identifying X_P with the rational cross-section of π associated to a Levi subgroup of P.[19] Recalling from (3.1.7) that the Levi subgroup decomposes as $\tilde{G}_{h,P} \cdot G_{\ell,P}$, one can write E_w as a product: $(E_w)_h \otimes (E_w)_\ell$. Inserting this into (3.5.1), we have

$$H^\bullet(X'_P, \tilde{\mathbf{V}}_{\Gamma_P}) \cong \bigoplus_w H^\bullet(X_P, ((\tilde{\mathbf{E}}_w)_h)_{\Gamma_{h,P}}) \otimes (E_w)_\ell\,[-\ell(w)]. \tag{3.5.2}$$

(3.5.3) **Theorem.** [16: (5.4.21)] *The isomorphism* (3.5.2) *is an isomorphism of mixed Hodge structures (when the two sides are given their natural mixed Hodge structures).*[20]

(3.5.4) *Remark.* Note that (3.5.3) asserts that the mixed Hodge structure on $H^\bullet(X'_P, \tilde{\mathbf{V}}_{\Gamma_P})$ decomposes into the *direct sum* of mixed Hodge structures of lower dimension, which is much stronger than the statement that $H^\bullet(X'_P, \tilde{\mathbf{V}}_{\Gamma_P})$ has a filtration for which the gradation is as given.

We now address the main task of determining the mixed Hodge structures in the E_1-term in (3.4.2). For this, we will have to use the formulation in terms of deleted neighborhood cohomology on $\tilde{X}_\Sigma$. There are two observations to be made. First, as far as the divisor $Z_{P,\Sigma}$ is concerned, we do not need to distinguish $\tilde{X}_\Sigma$ and $\Gamma_{\ell,P}\backslash X'_{P,\Sigma_P}$, as sufficiently small neighborhoods of $Z_{P,\Sigma}$ in both are analytically isomorphic. Second, the fiber of $\pi_2 : X'_P \to \mathscr{A}_P$ is a torus and the inclusion of nice deleted neighborhoods of toric boundary stratum in the torus tends to be a homotopy equivalence (e.g., $\Delta^* \hookrightarrow \mathbf{C}^*$), implying that the deleted neighborhood cohomology is isomorphic to the cohomology of the torus itself; thus is isomorphic in the sense of mixed Hodge structures, by functoriality. These considerations must be carried out in the presence of parameters (i.e., $\mathscr{A}_P$) and the $\Gamma_{\ell,P}$-action.

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[19] Here, we may have to pass to a finite covering of X_P before said cross-section exists. This is a minor detail (compare the footnote to (1.3.7)).

[20] This is discussed before (3.3.10). Of course, for (3.5.3) to be true, the weights on X'_P and X_P must be compatibly normalized. This is built into Deligne's definition of an *admissible Cayley morphism* (see [15: 1.2.2.1]).

Let R be a $\mathbf{Q}$-parabolic subgroup of G. We identify R with the set $\mathscr{S}$ of all maximal parabolic subgroups containing it. Assume R is *strictly subordinate to* $P \in \mathscr{P}(G)_{\max}$, i.e., $P \in \mathscr{S}$, and P has the largest Δ_h among all maximal parabolic subgroups in $\mathscr{S}$. Put $\Gamma_{\ell,\mathscr{S}} = \Gamma_{\ell,P} \cap R(\mathbf{Q})$. Then:

(3.5.5) **Theorem.** [16: (5.6.10)] *There is a decomposition of mixed Hodge structures*

$$H^i_{\mathrm{dn}}(Z_{\mathscr{S},\Sigma}, \tilde{\mathbf{V}}_\Gamma) \cong \bigoplus_{j,k} \bigoplus_{\ell(w)=j} H^{i-j-k}(X_P, ((\tilde{\mathbf{E}}_w)_h)_{\Gamma_{h,P}}) \otimes H^k(\Gamma_{\ell,\mathscr{S}}, (E_w)_\ell).$$

Here, $H^\bullet(\Gamma_{\ell,\mathscr{S}}, (E_w)_\ell)$ has the Hodge structure of type $(0,0)$.[21]

Appendix A: No ghosts for Sp_4

We will carry out the calculations that rule out ghosts for $G = Sp_4$. In other words, if X is the quotient of the Siegel upper half-plane of genus two by an arithmetic group $\Gamma \subset Sp_4(\mathbf{Q})$, then $Spect^\bullet(\overline{X}, \tilde{\mathbf{V}}_\Gamma) = 0$ for all local systems arising from (irreducible) finite-dimensional representations V of Sp_4. This will be achieved by topological considerations followed by a determination of *weights*.

We remind ourselves that $dim_{\mathbf{C}}\, X = 3$, so $dim_{\mathbf{R}}\, X = 6$. The root system for Sp_4 over $\mathbf{Q}$, $\mathbf{R}$ and $\mathbf{C}$ is of type C_2, which has simple roots $\{\beta_1, \beta_2\}$ (as in (1.3.11(ii))), and corresponding fundamental dominant weights $\{\lambda_1, \lambda_2\}$. There are orthogonality relations

$$(\lambda_1, \beta_2) = (\lambda_2, \beta_1) = 0; (\lambda_1, \beta_1) = 1, (\lambda_2, \beta_2) = 2. \tag{A.1.1}$$

The highest weight of V is then of the form $m_1\lambda_1 + m_2\lambda_2$, where m_1 and m_2 are nonnegative integers, and this determines the weight of the variation of Hodge structure on $\tilde{\mathbf{V}}_\Gamma$ up to a Tate twist. There are two standard maximal parabolic subgroups, P_1 and P_2 (labeled according to the *omitted* simple root), corresponding to boundary components $X_1 = X_{P_1}$ and $X_2 = X_{P_2}$ of complex dimensions 1 and 0, resp. We put $P_{12} = P_1 \cap P_2$.

In $\overline{X}$, the boundary decomposes into three strata types, coming from the conjugates of P_1, P_2 and P_{12}. The closure of each of these is the *disjoint* union of closed faces of the given type. Moreover, each $e'({}^gP_{12})$ is contained in exactly one face of type "1", viz. $\overline{e'({}^gP_1)}$,—note the canonical surjection $\Gamma\backslash G(\mathbf{Q})/P_{12}(\mathbf{Q}) \to \Gamma\backslash G(\mathbf{Q})/P_1(\mathbf{Q})$—and there is a parallel assertion for P_2 (see [16: App. to (3.5)]).

Although both $\hat{e}'(P_1) \cong X_1$ and $\hat{e}'(P_2)$ are arithmetic quotients for SL_2, the Hermitian nature of the latter is "accidental"; $e'(P_2)$ is collapsed to the point $X_2 = \hat{e}'(P_{12})$ under the quotient mapping f (in (1.3.1)). While the discussion at the end of (3.5) suggests associating $e'(P_{12})$, which lies in both $\overline{e'(P_1)}$ and $\overline{e'(P_2)}$, to the *latter* closed face, one can also view it as going with (the boundary of) X_1 (see (A.1.9) below).

[21] This is misstated in [16].

We consider the problem of determining $Spect^i(\overline{X}, \tilde{\mathbf{V}}_\Gamma)$. Let N_1 be a nice deleted neighborhood of $\bigsqcup e'({}^gP_1)$, etc. From (3.3.2), (3.3.3(ii)) and (3.4.3), our task centers around determining weights in

$$(im\,\delta)^i \cong H^{i-1}(N_{12}, \tilde{\mathbf{V}}_\Gamma)/(H^{i-1}(N_1, \tilde{\mathbf{V}}_\Gamma) \oplus H^{i-1}(N_2, \tilde{\mathbf{V}}_\Gamma)); \tag{A.1.2}$$

this is finally achieved in (A.1.18), by means of explicit computations involving the Weyl group of Sp_4.

First, we consider the quotient groups in (A.1.2) from a topological point of view, i.e., without regard to Hodge structure. Note again that, since $\overline{X}$ is a manifold-with-corners,

$$\begin{aligned} H^\bullet(N_{12}, \tilde{\mathbf{V}}_\Gamma) &\cong H^\bullet(e'(P_{12}), \tilde{\mathbf{V}}_\Gamma) \cong H^\bullet(\boldsymbol{u}_{12}, V) \\ &\cong H^\bullet(\boldsymbol{u}_{12}/\boldsymbol{u}_1, \tilde{\mathbf{H}}^\bullet(\boldsymbol{u}_1, V)) \cong H^\bullet(\boldsymbol{u}_{12}/\boldsymbol{u}_2, \tilde{\mathbf{H}}^\bullet(\boldsymbol{u}_2, V)); \end{aligned} \tag{A.1.3}$$

$$\begin{aligned} H^\bullet(N_1, \tilde{\mathbf{V}}_\Gamma) &\cong H^\bullet(e'(P_1), \tilde{\mathbf{V}}_\Gamma) \cong H^\bullet(\hat{e}'(P_1), \tilde{\mathbf{H}}^\bullet(\boldsymbol{u}_1, V)) \\ &= H^\bullet(X_1, \tilde{\mathbf{H}}^\bullet(\boldsymbol{u}_1, V)); \end{aligned} \tag{A.1.4}$$

$$H^\bullet(N_2, \tilde{\mathbf{V}}_\Gamma) \cong H^\bullet(e'(P_2), \tilde{\mathbf{V}}_\Gamma) \cong H^\bullet(\hat{e}'(P_2), \tilde{\mathbf{H}}^\bullet(\boldsymbol{u}_2, V)). \tag{A.1.5}$$

This gives rise to the diagram

$$\begin{array}{ccccc} H^\bullet(N_1, \tilde{\mathbf{V}}_\Gamma) & \cong & H^\bullet(X_1, \tilde{\mathbf{H}}^\bullet(\boldsymbol{u}_1, V)) & & \\ \downarrow & & \downarrow & & \\ H^\bullet(N_{12}, \tilde{\mathbf{V}}_\Gamma) & \cong & H^\bullet(\boldsymbol{u}_{12}/\boldsymbol{u}_1, \tilde{\mathbf{H}}^\bullet(\boldsymbol{u}_1, V)) & \cong & H^\bullet(\boldsymbol{u}_{12}/\boldsymbol{u}_2, \tilde{\mathbf{H}}^\bullet(\boldsymbol{u}_2, V)) \\ \uparrow & & & & \uparrow \\ H^\bullet(N_2, \tilde{\mathbf{V}}_\Gamma) & = & H^\bullet(N_2, \tilde{\mathbf{V}}_\Gamma) & \cong & H^\bullet(\hat{e}'(P_2), \tilde{\mathbf{H}}^\bullet(\boldsymbol{u}_2, V)) \end{array} \tag{A.1.6}$$

(A.1.7) **Lemma.** *Write $w = w_1 \cdot w^1$ with $w_1 \in W_1$ and $w^1 \in W^1$. Then, under the isomorphism in* (A.1.3), *the Kostant constituent E_w of $H^\bullet(\boldsymbol{u}_{12}, V)$ is the Kostant constituent of $H^{\ell(w_1)}(\boldsymbol{u}_{12}/\boldsymbol{u}_1, E_{w^1})$ corresponding to $w_1 \in W_1$.*

(Of course, the corresponding assertion holds for W_2.) There are two Hodge-theoretical decompositions of $H^k(N^{12}, \tilde{\mathbf{V}}_\Gamma))$. The first is the one given by (3.5.5):

$$\begin{aligned} H^k(N^{12}, \tilde{\mathbf{V}}_\Gamma) \cong & \bigoplus_{w \in W^2,\, l(w)=k} (T_w \otimes H^0(\Gamma_\ell, E_{w,\ell})) \\ & \oplus \bigoplus_{w \in W^2,\, l(w)=k-1} (T_w \otimes H^1(\Gamma_\ell, E_{w,\ell})) \end{aligned}$$

(Γ_ℓ is cyclic for Sp_4), and the other one comes from iterated deleted neighborhood cohomology:

$$H^k(N^{12}, \tilde{\mathbf{V}}_\Gamma) \cong \bigoplus_{w \in W^1,\, l(w)=k} H^0_{\mathrm{dn}}(U_2, \mathbf{E}_w) \oplus \bigoplus_{w \in W^1,\, l(w)=k-1} H^1_{\mathrm{dn}}(U_2, \mathbf{E}_w).$$

(here, T_w is a Tate Hodge structure on the point X_2, U_2 denotes a deleted neighborhood of X_2 in the closure of X_1, a punctured disc.) We write the above in terms of the Kostant decomposition

$$\begin{aligned} H^\bullet(N^{12}, \tilde{\mathbf{V}}_\Gamma) &\cong H^\bullet(\boldsymbol{u}^{12}, V) \cong \bigoplus_{w \in \boldsymbol{W}} V_w[-l(w)] \\ &\cong \bigoplus_{w^1 \in \boldsymbol{W}^1} H^\bullet(\boldsymbol{u}^{12}/\boldsymbol{u}^1, E_{w^1}) \cong \bigoplus_{w^1 \in \boldsymbol{W}^1} H^\bullet(\boldsymbol{u}^{12}/\boldsymbol{u}^1, E_{w^1}); \end{aligned} \tag{A.1.8}$$

note that this also equals the corresponding thing for $\boldsymbol{W}^2$:

$$H^\bullet(N^{12}, \tilde{\mathbf{V}}_\Gamma) \cong \bigoplus_{w^2 \in \boldsymbol{W}^2} H^\bullet(\boldsymbol{u}^{12}/\boldsymbol{u}^2, E_{w^2}).$$

We substantiate directly something that is already implied by the naturality of the construction of the mixed Hodge structures in [26]:

(A.1.9) **Proposition.** *The weights in $H^\bullet(\boldsymbol{u}_{12}, V)$ are the same, whether one views it as*

$$H^\bullet_{\mathrm{dn}}(e'(P_{12}), \tilde{\mathbf{V}}_\Gamma) \tag{a}$$

or as the iterated deleted neighborhood cohomology

$$H^\bullet_{\mathrm{dn}}(X_2, \mathscr{C}^\bullet_{\mathrm{dn}}(X_1^*, \tilde{\mathbf{V}}_\Gamma)). \tag{b}$$

Proof. In the above, it is clearly equivalent to verify the assertion for the absolute parabolic weights. Let $\boldsymbol{W}^1$ denote the Kostant subset of the Weyl group $\boldsymbol{W}$ of G over $\mathbf{C}$ (as in (2.3.3(ii))), associated to P_1, etc. Note that $\boldsymbol{W}^{12} = \boldsymbol{W}$ and $\boldsymbol{W} = \boldsymbol{W}_1 \cdot \boldsymbol{W}^1$, where $\boldsymbol{W}_1$ is the Weyl group of P_1. Note that our notation gives $\boldsymbol{W}_1$ to be generated by the reflection with respect to β_2. The Kostant constituents E_w of $H^\bullet(\boldsymbol{u}_{12}, V)$ have the highest weights of the form $w(\lambda+\delta)-\delta$. For $w \in \boldsymbol{W}$, write it as $w = w_1 \cdot w^1$, with $w_1 \in \boldsymbol{W}_1$ and $w^1 \in \boldsymbol{W}^1$. To get the absolute parabolic weights (see Question below (A.1.11)), one must respectively:
(a) *take the coefficient of β_2 in $w(\lambda+\delta)-\delta$ (written in terms of the simple roots), for the weight is determined by $(E_w)_h$.*
(b) *start with $\mu = w^1(\lambda+\delta)-\delta$ and restrict it to the line spanned by β_2; call the restriction $\bar{\mu}$. Then take the coefficient of β_2 in $w_1(\bar{\mu}+\delta_1)-\delta_1$.*

If one writes $\bar{\mu} = \mu - c\lambda_1$ as λ_1 vanishes on β_2, one sees that

$$\begin{aligned} w_1(\bar{\mu}+\delta_1)-\delta_1 &= w_1(\mu - c\lambda_1 + \delta_1) - \delta_1 \\ &= w_1(w^1(\lambda+\delta)-\delta) - cw_1(\lambda_1) + w_1(\delta_1) - \delta_1 \\ &= [w(\lambda+\delta)-\delta] + [(\delta-\delta_1) - w_1(\delta-\delta_1)] - cw_1(\lambda_1). \end{aligned}$$

The last two terms vanish for w_1 permutes the roots occurring in $\boldsymbol{u}_1$, and $w_1(\lambda_1) = \lambda_1$, which annihilates β_2. Thus, the two determinations of weight agree. ∎

The following elementary fact will get applied to X_1 and $\hat{e}'(P_2)$, and vastly simplifies the problem:

(A.1.10) **Lemma.** *Let S^* be a compact, connected orientable real surface, let S be the complement of a nonempty finite set of points J in S^*, and denote the inclusions by $S \overset{j}{\hookrightarrow} S^* \overset{i}{\hookleftarrow} J$. Let $\tilde{\mathbf{V}}$ be a non-trivial irreducible local system on S. Then:*
(i) $H^0(S, \tilde{\mathbf{V}}) \cong H^2(S^*, j_*\tilde{\mathbf{V}}) = 0$,
(ii) *The mapping $H^1(S, \tilde{\mathbf{V}}) \to H^0(S^*, R^1 j_* \tilde{\mathbf{V}}) \cong \bigoplus_{p \in J} (R^1 j_* \tilde{\mathbf{V}})_p$ is surjective.*

For $\tilde{\mathbf{V}}$ trivial, the cohomology group $H^0(S, \tilde{\mathbf{V}})$ in (i) *is one-dimensional, as is the cokernel in* (ii): $H^0(S^*, R^1_{j_*}\mathbf{C}) \to H^2(S^*, \mathbf{C}) \cong \mathbf{C}$.

Moreover, for any $\tilde{\mathbf{V}}$, $H^2(S, \tilde{\mathbf{V}}) = 0$.

The distinction in (A.1.10), raises the following general question:

(A.1.11) *Question.* Let V be the representation of G of highest weight λ and $w \in W^P$. Under what conditions is $\tilde{\mathbf{E}}_w$ trivial on $\hat{e}'(P)$?

One can see that $\tilde{\mathbf{E}}_w$ is usually nontrivial. Recall that the fundamental dominant weights are dual to the simple roots by means of a pairing that is obtained by rescaling the inner product, viz.,

$$\langle \lambda, \alpha \rangle = 2(\lambda, \alpha)/(\alpha, \alpha).$$

Let P correspond to the omission of the set of roots $\Theta \subseteq \Delta_{\mathbf{C}}$. One needs to look at the nonnegative numbers

$$(w(\lambda + \delta) - \delta, \alpha) = (\lambda + \delta, w^{-1}(\alpha)) - (\delta, \alpha)$$

for all $\alpha \notin \Theta$, and determine when all of them are actually zero. This is a system of linear equations in λ. Furthermore, if $\Theta = \{\alpha_k\}$ (P maximal), the quantity $(w(\lambda + \delta) - \delta, \lambda_k)$ gives what we will call the *absolute parabolic weight* of $\tilde{\mathbf{E}}_w$ (λ_k determines $(m_F)^{-1}$ of [15: 1.2.2.1]); it must be subtracted from the weight of $\tilde{\mathbf{V}}_\Gamma$ to give the Hodge-theoretic weight.

We address (A.1.11) in the case of $G = Sp_4$.

(A.1.12) **Lemma.** *For $G = Sp_4$, $|W| = 8$, $|W^1| = |W^2| = 4$, and*
(i) *$W^1 \cup W^2 = W - \{w_0\}$, where w_0 denotes the longest element of W;*
(ii) $W^1 \cap W^2 = \{1\}$.

Using (A.1.1), we have $(\delta, \lambda_1) = (\delta, \alpha_2) = 2$ in the C_2 root system, so we obtain:

(A.1.13) **Lemma.** *Let $w \in W^1$. Then:*
(i) *$\tilde{\mathbf{E}}_w$ has absolute parabolic weight*

$$\omega = (w(\lambda + \delta) - \delta, \lambda_1) = (m_1 + 1)(\lambda_1, w^{-1}\lambda_1) + (m_2 + 1)(\lambda_2, w^{-1}\lambda_1) - 2.$$

(ii) *$\tilde{\mathbf{E}}_w$ is trivial on X_1 if and only if*

$$(m_1 + 1)(\lambda_1, w^{-1}\alpha_2) + (m_2 + 1)(\lambda_2, w^{-1}\alpha_2) = 2.$$

Similarly, since $(\delta, \lambda_2) = 3$ and $(\delta, \alpha_1) = 1$, we have:

(A.1.14) **Lemma.** *Let $w \in W^2$. Then:*
(i) $\tilde{\mathbf{E}}_w$ *has absolute parabolic weight*

$$\omega = (m_1+1)(\lambda_1, w^{-1}\lambda_2) + (m_2+1)(\lambda_2, w^{-1}\lambda_2) - 3.$$

(ii) $\tilde{\mathbf{E}}_w$ *is trivial on* $\hat{e}'(P_2)$ *if and only if*

$$(m_1+1)(\lambda_1, w^{-1}\alpha_1) + (m_2+1)(\lambda_2, w^{-1}\alpha_1) = 1.$$

In the C_2 root system, there are four positive roots:

$$\text{short: } \alpha_1, \alpha_1+\alpha_2; \qquad \text{long: } \alpha_2, 2\alpha_1+\alpha_2.$$

One can verify the following table of facts about the Weyl group:

W^1

ℓ	$w^{-1}(\varepsilon_1)$	$w^{-1}(\varepsilon_2)$	$w^{-1}(\lambda_1)$	$w^{-1}(\alpha_2)$	parab-wt	high-wt
0	ε_1	ε_2	λ_1	α_2	m_1+m_2	m_2
1	ε_2	ε_1	ε_2	$2\varepsilon_1$	m_2-1	m_1+m_2+1
2	$-\varepsilon_2$	ε_1	$-\varepsilon_2$	$2\varepsilon_1$	$-m_2-3$	m_1+m_2+1
3	$-\varepsilon_1$	ε_2	$-\lambda_1$	α_2	$-m_1-m_2-4$	m_2

W^2

ℓ	$w^{-1}(\varepsilon_1)$	$w^{-1}(\varepsilon_2)$	$w^{-1}(\lambda_2)$	$w^{-1}(\alpha_1)$	parab-wt	high-wt
0	ε_1	ε_2	λ_2	α_1	m_1+2m_2	m_1
1	ε_1	$-\varepsilon_2$	α_1	$\varepsilon_1+\varepsilon_2$	m_1-2	m_1+2m_2+2
2	ε_2	$-\varepsilon_1$	$-\alpha_1$	$\varepsilon_1+\varepsilon_2$	$-m_1-4$	m_1+2m_2+2
3	$-\varepsilon_2$	$-\varepsilon_1$	$-\lambda_2$	α_1	$-m_1-2m_2-6$	m_1

(A.1.15)

This gives the status of (A.1.11) in our example:

(A.1.16) **Proposition.** *In the above,* $\tilde{\mathbf{E}}_w$ *is trivial if and only if:*

$$m_2 = 0, w \in W^1, \ell(w) = 0, 3; \qquad m_1 = 0, w \in W^2, \ell(w) = 0, 3.$$

The complement to (A.1.12) is

(A.1.17) **Lemma.**
(i) $w_1 W^1 \cup w_2 W^2 = W - \{1\}$,
(ii) $w_1 W^1 \cap w_2 W^2 = \{w_0\}$.

We now look again at (A.1.2), (A.1.6), and (A.1.7). Generically, $\tilde{\mathbf{E}}$ is nontrivial (when $m_1, m_2 > 0$), then, (A.1.10) gives that the only contribution to (A.1.2) is $H^0(N_{12}, \tilde{\mathbf{V}}_\Gamma)$, for $i = 1$. Otherwise, we get some H^4 as well (as in $H^1(\ldots, \tilde{\mathbf{H}}^3)$), corresponding to $w = w_0$ and $\lambda = 0$, when the local systems on both X_1 and $\hat{e}'(P_2)$ are trivial; the latter is for $i = 5$.

Behind (3.5.5) is the fact that the deleted neighborhood cohomology of N_{12}, being associable to P_2, carries trivial Hodge theory. However, there are Tate twists hidden in $H^0(X_2)$, the first factor of (3.5.5) corresponding to the weights. *Since the weight of* $\tilde{\mathbf{V}}_\Gamma$ *contributes equally to both objects in* (3.3.2), *we may subtract it from both.* The resulting numbers will be called the *adjusted weights.*

The adjusted weights in $(im\,\delta)^i$ are computed with the help of (A.1.15). Keep in mind that for a point on a curve, the deleted neighborhood cohomology H^0_{dn} is carried by the *lowest* Hodge weight, we must add the highest weight to the parabolic weight to get the negative of the adjusted weight.

$$i = 1: \quad -(m_1 + 2m_2) \le 0; \qquad i = 5\ (m_1 = m_2 = 0): \quad 6. \tag{A.1.18}$$

A necessary condition for $Spect^i(\bar{X}, \tilde{\mathbf{V}}_\Gamma) \neq 0$ is that the set of weights (occurring nontrivially) in $(im\,\delta)^i$ have nonempty intersections with that for $H^i(X, \tilde{\mathbf{V}}_\Gamma)$. For $i = 1$, there is an isomorphism

$$IH^1(X^*, \tilde{\mathbf{V}}_\Gamma) \to H^1(X, \tilde{\mathbf{V}}_\Gamma).$$

The intersection homology is pure of adjusted weight 1 (see [29: (3.20)]; it is also implied by the [proven] Zucker conjecture, for which see [31: (3.2)]). The positive adjusted weight do not match the non-positive weights in (A.1.18). The weight for $i = 5$ looks plausible, but $H^5(\partial X, \mathbf{C}) \cong H^6_c(X, \mathbf{C})$, so there are no ghost classes here either. This gives what we wanted to prove.

Appendix B: The Cayley Transform (semi-simple or non-arithmetic version)

Calculating with the inherently simple Cayley transform has caused the author enough confusion that he wishes to present here an exposition in his own image.

(B.1) $SL(2)$. We start with $\boldsymbol{su}(1,1)$ as a real form of the Lie algebra $\boldsymbol{sl}(2,\mathbf{C})$, so that $SU(1,1) \subset SL(2,\mathbf{C})$ is our corresponding initial real Lie group. The corresponding symmetric space is then seen to be, in the familiar way, the unit disc:

$$D_0 = \{z \in \mathbf{C} : |z| < 1\} \tag{B.1.1}$$

($SU(1,1)$ acts by linear-fractional transformations). As such, it is visibly a bounded symmetric domain. One chooses $0 \in D_0$ as basepoint. Its stabilizer K in $SU(1,1)$ is the subset of diagonal matrices, isomorphic to $U(1)$. For future use, denote by $\imath$ the inclusion $K \simeq U(1) \hookrightarrow SU(1,1)$. (One sees that (B.1.1) is the Harish-Chandra realization.)

(B.1.2) **Definition.** *The matrix*

$$c = 2^{-\frac{1}{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \in SL(2,\mathbf{C})$$

is called the Cayley element.

We list some of the more relevant properties of c:

(B.1.3) **Proposition.** (i) *c takes $0 \in D_0$ to i (a point on the boundary of D_0); c^{-1} takes 0 to $-i$ (also a point on the boundary of D_0).*

(ii) *$Ad(c)$ sets up an isomorphism $\mathfrak{su}(1,1) \xrightarrow{\sim} \mathfrak{sl}(2,\mathbf{R})$, thus, $Int(c)$ defines an isomorphism $SU(1,1) \simeq SL(2,\mathbf{R})$, and c maps D_0 onto the upper half-plane.*

(iii) $c^4 = -1$, $c^2 = i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, *$c^{-1} = \bar{c}$ (usual complex conjugation here) and $Int(c^2)$ is an involution of $SL(2,\mathbf{C})$ that preserves both $SU(1,1)$ and $SL(2,\mathbf{R})$, also K.*

(B.1.4) *Remark.* Of course, the factor of $2^{-\frac{1}{2}}$ in the Cayley element may be dropped when computing $Ad(c)$ or $Int(c)$.

Let $\boldsymbol{k} \oplus \boldsymbol{a} \oplus \boldsymbol{u}$ be the Iwasawa decomposition of $\mathfrak{su}(1,1)$. Then

$$i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \boldsymbol{k}, \quad i\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \boldsymbol{a}, \quad \text{and} \quad \begin{pmatrix} i & 1 \\ 1 & -i \end{pmatrix} \in \boldsymbol{u}. \tag{B.1.5}$$

The complex structure on $\boldsymbol{p}$ is given by $J = Ad\begin{pmatrix} e^{-i\pi/4} & 0 \\ 0 & e^{i\pi/4} \end{pmatrix}$ and is determined by the assertion that $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \in \boldsymbol{p}^+$. (Keep in mind that complex conjugation on $\mathfrak{sl}(2,\mathbf{C})$ with respect to $\mathfrak{su}(1,1)$ is given by $X \mapsto -H{}^t\bar{X}H$, where $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.) Of course, $Ad(c)$ takes these things to the corresponding ones for $\mathfrak{sl}(2,\mathbf{R})$. However, with the real form fixed as $\mathfrak{su}(1,1)$, one sees:

(B.1.6) **Proposition.** (i) *$Ad(c)$ maps $\boldsymbol{k}$ to $i\boldsymbol{a}$, $\boldsymbol{a}$ to $i\boldsymbol{k}$, and $\boldsymbol{u}$ into $\boldsymbol{p}^-$.*

(ii) *$Ad(c^2)$ switches $\boldsymbol{p}^+$ and $\boldsymbol{p}^-$ (and is -1 on $\boldsymbol{k}$).*

We will later see (from (B.2)) that the "nicest" formulation of (B.1.6) is:

(B.1.7) **Corollary.** $Ad(c)(\boldsymbol{p}^+ \oplus \boldsymbol{k}_{\mathbf{C}}) = \boldsymbol{u}_{\mathbf{C}} \oplus \boldsymbol{a}_{\mathbf{C}}$.

(B.1.8) *Weights.* We have that

$$\boldsymbol{k} = \left\{\begin{pmatrix} \alpha & 0 \\ 0 & -\alpha \end{pmatrix} : \operatorname{Re}\alpha = 0\right\}.$$

In the standard representation S_1 of $SL(2)$, the vectors $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ are weight vectors with respect to $\boldsymbol{k}$, with weights α, and $-\alpha$ respectively. One declares α to be positive. It follows that

$$\tilde{e}_1 = c(e_1) \quad \text{and} \quad \tilde{e}_2 = c(e_2) \tag{B.1.9}$$

are weight vectors for $\boldsymbol{a}$. We transport to $\boldsymbol{a}_{\mathbf{C}}$ (by $Ad(c) : \boldsymbol{k}_{\mathbf{C}} \to \boldsymbol{a}_{\mathbf{C}}$) the notion of positivity, and denote the respective weights by $\tilde{\alpha}$ and $-\tilde{\alpha}$. We can write:

$$e^+ = \tilde{e}_1 - i\tilde{e}_2 = e_2, \qquad e^- = \tilde{e}_1 + i\tilde{e}_2 = e_1. \tag{B.1.10}$$

The second symmetric power of S_1, denoted by S_2, is isomorphic to the adjoint representation. Under this identification, it has $\boldsymbol{k}$-weight vectors $(e^+)^2 \in \boldsymbol{p}^+$,

$e^+e^- \in \boldsymbol{k}$, and $(e^-)^2 \in \boldsymbol{p}^-$. Here, we see that $\boldsymbol{p}^+$ is the *lowest* weight space in $\boldsymbol{sl}(2, \mathbf{C})$ (convention as in [28] and its predecessors). Note also that

$$c^2 e_1 = ie_2, \quad c^2 e_2 = ie_1; \qquad c^2 \tilde{e}_1 = i\tilde{e}_2, \quad c^2 \tilde{e}_1 = i\tilde{e}_2. \tag{B.1.11}$$

Let P^+, $K_{\mathbf{C}}$ and P^- denote the subgroups of $SL(2, \mathbf{C})$ corresponding to $\boldsymbol{p}^+$, $\boldsymbol{k}_{\mathbf{C}}$ and $\boldsymbol{p}^-$, resp. Explicitly,

$$P^+ = \left\{ \begin{pmatrix} 1 & 0 \\ w & 1 \end{pmatrix} : w \in \mathbf{C} \right\}, \tag{B.1.12}$$

$$K_{\mathbf{C}} = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} : \alpha \in \mathbf{C}^* \right\}, \quad P^- = \left\{ \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} : u \in \mathbf{C} \right\}.$$

The following is well known:

(B.1.13) **Proposition.** (i) *The stabilizer of $0 \in D_0 \subset \mathbf{P}^1(\mathbf{C})$ in $SL(2, \mathbf{C})$ is $P^+K_{\mathbf{C}}$; the stabilizer of ∞ is $K_{\mathbf{C}}P^-$.*
(ii) *$P^+K_{\mathbf{C}}P^-$ contains $c^m SU(1,1)$ for $m = -1, 0, 1$.*

(B.1.14) *Remark*. One can make (B.1.13, ii) explicit:
(i) That $c^{-1} \in P^+K_{\mathbf{C}}P^-$ can be argued as follows. Observe that both $\begin{pmatrix} 1 & -i \\ 0 & 1 \end{pmatrix} \in P^-$ and c^{-1} take i to 0; one uses (B.1.13(i)) to conclude that $c^{-1}\begin{pmatrix} 1 & -i \\ 0 & 1 \end{pmatrix}^{-1} \in P^+K_{\mathbf{C}}$, etc.
(ii) From (B.1.12), one deduces that a matrix $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL(2, \mathbf{C})$ belongs to $P^+K_{\mathbf{C}}P^-$ if and only if $\alpha \neq 0$, whereas one of the defining conditions for $SU(1,1)$ includes the equation $|\alpha|^2 - |\gamma|^2 = 1$. One obtains (B.1.13(ii)) therefrom by direct verification. In fact, the solution of

$$\begin{pmatrix} 1 & 0 \\ w & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \tag{B.1.14.1}$$

is

$$a = \alpha, \quad u = \alpha^{-1}\beta, \quad w = \alpha^{-1}\gamma. \tag{B.1.14.2}$$

In particular, the canonical mapping

$$P^+ \times K_{\mathbf{C}} \times P^- \to P^+K_{\mathbf{C}}P^- \subset SL(2, \mathbf{C}) \tag{B.1.14.3}$$

is injective.

Along these lines, we note that if $U_{\mathbf{C}}$ denotes the exponential of $\mathbf{C} \cdot (\tilde{e}_1)^2$, then

$$Int(c)U_{\mathbf{C}} = P^-, \qquad \text{and} \qquad c \in U_{\mathbf{C}}P^+K_{\mathbf{C}}. \tag{B.1.15}$$

For the latter, it suffices to verify the corresponding assertion for $Int(c)SU(1,1) = SL(2, \mathbf{R})$, but then we observe that $\begin{pmatrix} 1 & -i \\ 0 & 1 \end{pmatrix} c$ fixes 0 (cf. (B.1.14.1)), and we appeal to (B.1.13(i)).

At this point, it is convenient to mention the *canonical automorphy factor*

$$\mathscr{J} : P^+K_{\mathbf{C}}P^- \times \mathbf{C} \to K_{\mathbf{C}} \simeq \mathbf{C}^*, \tag{B.1.16}$$

which is characterized by the properties:

$$\mathscr{J}(g,z) \text{ is, for fixed } g, \text{ holomorphic in } z, \tag{B.1.16.1}$$

$$\mathscr{J}(gh,z) = \mathscr{J}(g,hz)\mathscr{J}(h,z) \text{ (whenever this makes sense)}, \tag{B.1.16.2}$$

$$\begin{array}{l}\mathscr{J}|_{(P^+K_{\mathbf{C}})\times\mathbf{C}} \text{ is the pullback of the canonical projection } P^+K_{\mathbf{C}} \to K_{\mathbf{C}} \\ \text{(so is a homomorphism, independent of } z \in \mathbf{C}).\end{array} \tag{B.1.16.3}$$

From (B.14.1.2), one sees that $\mathscr{J}$ is determined by its values at $z = 0$:

$$\mathscr{J}(g, h \cdot 0) = \mathscr{J}(gh, 0)\mathscr{J}(h,0)^{-1}; \tag{B.1.16.4}$$

it then follows that

$$\mathscr{J}(g,0) \text{ is the } K_{\mathbf{C}}\text{-component of } g \text{ (recall (B.1.14.2))}. \tag{B.1.16.5}$$

(B.2) *General groups of Hermitian type*. Let D be an *irreducible* Hermitian symmetric space of non-compact type, G its connected isometry group (a real Lie group), and $\boldsymbol{g}$ the Lie algebra of G. Fix a basepoint $z_0 \in D$, whose stabilizer is a maximal compact subgroup K_G.

There is a fundamental structure homomorphism (Harish-Chandra; see [1: p. 178]):

$$\Phi : U(1) \times SU(1,1)^r \to G, \tag{B.2.1}$$

where r is the $\mathbf{R}$-rank of G. (To do this and the sequel over $\mathbf{Q}$, which is what one really wants, one must group the $SU(1,1)$ factors into bunches, and then map in $SU(1,1)$'s multidiagonally. We will not carry this out here (see [2: 2.9] or [1: p. 193]).) To describe the features of Φ, let $\pi_j : SU(1,1)^r \to SU(1,1)$ denote projection onto the jth factor for $1 \le j \le r$, and let $\Delta_s : SU(1,1) \to SU(1,1)^r$ be the sth partial diagonal $(0 \le s \le r)$, i.e., $\pi_j \circ \Delta_s$ is the identity mapping for $j \le s$ and the trivial homomorphism for $j > s$. Then $\hat{\Phi} = \Phi|_{\{1\}\times SU(1,1)^r}$ is a finite immersion and the image of A^r, where A denotes $exp\,\boldsymbol{a}$, under $\hat{\Phi}$ is a maximal $\mathbf{R}$-split torus A_G of G. Moreover, $\Phi_U = \Phi \circ (1 \times \Delta_r \circ \iota)$ maps $U(1)$ onto the center of K.

One takes the composite of the adjoint representation of G with Φ, yielding a representation of $U(1) \times SU(1,1)^r$. Its irreducible constituents (as a *real* representation) are tensor products of an irreducible real representation of $U(1)$ and r irreducible representations of $SU(1,1)$. On the other hand, under Φ_U, $\boldsymbol{g}$ decomposes as the Cartan decomposition $\boldsymbol{g} = \boldsymbol{k}_G \oplus \boldsymbol{p}_G$ as a representation of $U(1)$, in which $\boldsymbol{k}_G$ is trivial and $\boldsymbol{p}_G$ is the direct sum of copies of the two-dimensional representation R_1 of lowest possible nonzero weight. It follows that $Ad_G \circ \Phi$ decomposes into rather small pieces. Indeed, the only possible irreducible constituents are listed below:

$$\begin{aligned}&\text{(a) } 1\otimes S_2(j);\\&\text{(b) } 1\otimes (S_1(j)\otimes S_1(k)), j<k;\\&\text{(c) } R_1\otimes S_1(j);\\&\text{(d) Trivial.}\end{aligned} \tag{B.2.2}$$

Here, $S_1(j)$ denotes the representation of $SL(2)^r$ that is trivial except on the jth factor, where it is given by S_1, etc. There are precisely r summands in $\boldsymbol{g}$ of type (a), spanning the image of $\boldsymbol{su}(1,1)^r$. All factors of type (d) lie in $\boldsymbol{k}_G$. From (B.2.2), it is not hard to determine that the $\mathbf{R}$-root system of G must be of classification type BC (or its "degenerate" form C).

Next, we need to describe the standard maximal $\mathbf{R}$-parabolic subalgebras $\boldsymbol{q}_s$ $(1\le s\le r)$ of $\boldsymbol{g}$ associated to A_G. Let $A_{G,s}$ denote the one-dimensional subtorus $\hat{\Phi}(\Delta_s(A))$ and $\boldsymbol{a}_{G,s}$ its Lie algebra, with positivity induced from $\boldsymbol{a}\subset \boldsymbol{su}(1,1)$. Then

$$\boldsymbol{q}_s=\boldsymbol{u}_s\oplus \boldsymbol{v}_s\oplus \boldsymbol{g}_s, \tag{B.2.3}$$

where:

$$\boldsymbol{u}_s \text{ is the weight space for } -2\tilde{\alpha}, \tag{B.2.3.1}$$

$$\boldsymbol{v}_s \text{ is the weight space for } -\tilde{\alpha}, \tag{B.2.3.2}$$

$$\boldsymbol{g}_s \text{ is the weight space for } 0. \tag{B.2.3.3}$$

Then $\boldsymbol{w}_s=\boldsymbol{u}_s\oplus\boldsymbol{v}_s$ is the nilpotent radical of $\boldsymbol{q}_s$, $\boldsymbol{u}_s$ the center of $\boldsymbol{w}_s$; $\boldsymbol{g}_s$ is a Levi subalgebra, with $\boldsymbol{a}_{G,s}$ its $\mathbf{R}$-split component. Bases for these are easily identified in terms of (B.1.9) and (B.2.2):

(a) $1\otimes(\tilde{e}_1)^2\in\boldsymbol{u}_s$ if $s\ge j$, $\boldsymbol{g}_s$ if $s<j$; $\quad 1\otimes(\tilde{e}_1\tilde{e}_2)\in\boldsymbol{g}_s$ (all cases); (B.2.4)
$\quad 1\otimes(\tilde{e}_2)^2\in\boldsymbol{g}_s$ if $s<j$;

(b) $1\otimes(\tilde{e}_1\otimes\tilde{e}_1)\in\boldsymbol{v}_s$ if $k>s\ge j$, $\boldsymbol{g}_s$ if $s<j$, $\boldsymbol{u}_s$ if $s\ge k$,
$\quad 1\otimes(\tilde{e}_1\otimes\tilde{e}_2)\in\boldsymbol{v}_s$ if $k>s\ge j$, $\boldsymbol{g}_s$ if $s<j$, $\boldsymbol{g}_s$ if $s\ge k$,
$\quad 1\otimes(\tilde{e}_2\otimes\tilde{e}_1)\in\boldsymbol{g}_s$ if $s<j$, $\boldsymbol{g}_s$ if $s\ge k$,
$\quad 1\otimes(\tilde{e}_2\otimes\tilde{e}_2)\in\boldsymbol{g}_s$ if $s<j$;

(c) $r^+\otimes\tilde{e}_1$, $r^-\otimes\tilde{e}_1\in\boldsymbol{v}_s$ if $s\ge j$, $\boldsymbol{g}_s$ if $s<j$.

Moreover, there is a decomposition

$$\boldsymbol{g}_s=\boldsymbol{g}_{s,\ell}\oplus\boldsymbol{g}_{s,h}, \tag{B.2.5}$$

such that (B.2.5) induces a direct sum decomposition *of Lie algebras*:

$$\boldsymbol{g}_s/\boldsymbol{a}_{G,s}=(\boldsymbol{g}_{s,\ell}/\boldsymbol{a}_{G,s})\oplus\boldsymbol{g}_{s,h},$$

which can be most directly described as being determined by a partition of the basis vectors listed in (B.2.4), viz.,

(B.2.5.1) $\boldsymbol{g}_{s,\ell}$ is spanned by the vectors $1\otimes(\tilde{e}_1\otimes\tilde{e}_2)$ and $1\otimes(\tilde{e}_2\otimes\tilde{e}_1)$, from representations of type (b) with $s\ge k$, together with their Lie brackets; it contains the vectors $1\otimes(\tilde{e}_1\tilde{e}_2)$ from representations of type (a) with $s\ge j$.

(B.2.6) **Definitions.** *For* $0 \le s \le r$,

(i) *The sth Cayley element of* $G_{\mathbf{C}}$ *associated to* Φ *is* $c_s = (\hat{\Phi} \circ \Delta_s)(c)$.

(ii) $Int(c_s)$ *(resp.* $Ad(c_s)$*) is the sth (partial) Cayley transform of* G *(resp. of* $\boldsymbol{g}$*).*

Let P_G^+ and P_G^- denote the subgroups of $G_{\mathbf{C}}$ corresponding to $\boldsymbol{p}_G^+$ and $\boldsymbol{p}_G^-$, respectively. We wish to view the partial Cayley transforms in terms of their effect on $P_G^+(K_G)_{\mathbf{C}}$ (the stabilizer of z_0 in $G_{\mathbf{C}}$) or $\boldsymbol{p}_G^+ \oplus (\boldsymbol{k}_G)_{\mathbf{C}}$ (cf. (B.1.7)). To that end, recall the discussion of weights in representations of $SL(2)$ from (B.1.8). In the representation R_1 of $U(1)$, let r^+ be a positive weight vector, and $r^- = \overline{r^+}$. We have that $\boldsymbol{p}_G^+$ has negative $\boldsymbol{k}$-weight and $\boldsymbol{p}_G^-$ positive. In terms of (B.1.10) and (B.2.2):

$$\text{(a) } 1 \otimes (e^+)^2 \in \boldsymbol{p}_G^+, \quad 1 \otimes e^+e^- \in \boldsymbol{k}_G, \quad 1 \otimes (e^-)^2 \in \boldsymbol{p}_G^-; \tag{B.2.7}$$

$$\text{(b) } 1 \otimes (e^+ \otimes e^+) \in \boldsymbol{p}_G^+, \quad 1 \otimes (e^+ \otimes e^-), \quad 1 \otimes (e^- \otimes e^+) \in \boldsymbol{k}_G,$$
$$1 \otimes (e^- \otimes e^-) \in \boldsymbol{p}_G^-;$$

$$\text{(c) } r^+ \otimes e^+ \in \boldsymbol{p}_G^+, \quad r^+ \otimes e^-, r^- \otimes e^+ \in (\boldsymbol{k}_G)_{\mathbf{C}}, \quad r^- \otimes e^- \in \boldsymbol{p}_G^-.$$

It is easy now to decompose the action of c_s according to (B.2.2): on every trivial or R_1 factor, c_s acts as the identity; on $S_1(m)$ or $S_2(m)$, c_s acts in the usual way (i.e., as in (B.1.2) or (B.1.3(ii))) if $s \ge m$, and as the identity if $s < m$. Let $\boldsymbol{k}_{s,h} = \boldsymbol{k}_G \cap \boldsymbol{g}_{s,h}$, $\boldsymbol{k}_{s,\ell} = \boldsymbol{k}_G \cap \boldsymbol{g}_{s,\ell}$, and define $\boldsymbol{p}_{s,h}^-$ analogously. One can verify without difficulty:

(B.2.8) **Proposition.** (i) *The maximal* $\Phi(SU(1,1)^r)$*-invariant subspace of* $\boldsymbol{g}_{\mathbf{C}}$ *on which* $Ad(c_s)$ *is the identity is* $\boldsymbol{g}_{s,h}$.

(ii) *The differential of* $\hat{\Phi} \circ \Delta_s : SU(1,1) \to G$ *takes* $\boldsymbol{u}$ *to* $\boldsymbol{u}_s$, $\boldsymbol{p}^+$ *to* $\boldsymbol{p}_G^+$, $\boldsymbol{k}$ *to* $\boldsymbol{k}_G$, *and* $\boldsymbol{p}^-$ *to* $\boldsymbol{p}_G^-$.

(iii) $Ad(c_s)$ *takes* $\boldsymbol{u}_s$ *into* $\boldsymbol{p}_G^-$, $\boldsymbol{g}_{s,\ell}$ *into* $(\boldsymbol{k}_G)_{\mathbf{C}}$, *and* $\boldsymbol{k}_{s,\ell}$ *into itself.*

(iv) $Ad(c_s)(\boldsymbol{p}_G^+ \oplus (\boldsymbol{k}_G)_{\mathbf{C}}) \supset (\boldsymbol{w}_s)_{\mathbf{C}} \oplus (\boldsymbol{g}_{s,\ell})_{\mathbf{C}} \oplus ((\boldsymbol{p}_{s,h})_{\mathbf{C}}^+ \oplus (\boldsymbol{k}_{s,h})_{\mathbf{C}})$.

(v) $Ad(c_s^2)$ *is an involution of* $\boldsymbol{g}_{\mathbf{C}}$ *that preserves* $(\boldsymbol{k}_G)_{\mathbf{C}}$.

(B.2.9) *Remark.* The formula "opposite" to (B.2.8, iv) gives that

$$Ad(c_s^{-1})((\boldsymbol{k}_G)_{\mathbf{C}} \oplus \boldsymbol{p}_G^-) \supset (\boldsymbol{g}_{s,\ell})_{\mathbf{C}} \oplus ((\boldsymbol{k}_{s,h})_{\mathbf{C}} \oplus (\boldsymbol{p}_{s,h})_{\mathbf{C}}^-).$$

Let $\mathbf{n}_s = \boldsymbol{w}_s \oplus \boldsymbol{g}_{s,\ell}$, and $N_s = W_s G_{s,\ell}$ the corresponding Lie group.

The canonical automorphy factor for G is defined as for $G = SU(1,1)$ in (B.1.16):

$$\mathcal{J}_G : P_G^+(K_G)_{\mathbf{C}} P_G^- \times D \to (K_G)_{\mathbf{C}}, \tag{B.2.10}$$

and is similarly characterized; we write that here in the form

$$\mathcal{J}_G(g,z) \text{ is holomorphic in } z, \tag{B.2.10.1}$$

$$\mathcal{J}_G(g,z_0) \text{ is the } (K_G)_{\mathbf{C}}\text{-component of } g, \tag{B.2.10.2}$$

$$\mathcal{J}_G(gh,z) = \mathcal{J}_G(g,hz)\mathcal{J}_G(h,z). \tag{B.2.10.3}$$

It follows from (B.2.10.2) that $\mathcal{J}_G$ is hereditary for subgroups. Also, the expression

$$\mathcal{J}_G(gh,z_0)\mathcal{J}_G(h,z_0)^{-1} \tag{B.2.10.4}$$

is invariant under $h \mapsto hk$ ($k \in K_G$), which shows that *$\mathcal{J}_G$ is completely determined by its values at $z = z_0$.* We have, moreover, the following:

(B.2.10.5) **Lemma.** *If $q \in P^+(K_G)_{\mathbb{C}}$ and $g \in P^+(K_G)_{\mathbb{C}} P_G^-$, then*

$$\mathcal{J}_G(qg, z_0) = \mathcal{J}_G(q, z_0)\mathcal{J}_G(g, z_0).$$

(B.2.11) **Lemma.** (i) *$g \in P_G^+(K_G)_{\mathbf{C}} P_G^-$ is in $P_G^+(K_G)_{\mathbf{C}}$ if and only if $\mathcal{J}_G(g, z)$ is independent of z.*
(ii) *For all s, c_s^m is contained in $P_G^+(K_G)_{\mathbf{C}} P_G^-$ for $m = -1, 0, 1$ (cf.* (B.1.13, (ii))).

One writes ${}^hG$ for $Int(h)G = hGh^{-1}$. There is an easy formula:

$$\mathcal{J}_{{}^hG}(Int(h)g, hz) = Int(h)(\mathcal{J}_G(g, z)) \tag{B.2.12}$$

(with hz_0 as basepoint for the left-hand side).

(B.2.13) To conclude, we recall the canonical automorphy factor $\mathcal{J}_{G,P_s}$ of the pair (G, P_s) (where P_s is the subgroup of G corresponding to $\boldsymbol{q}_s$), with domain $G(\mathbf{R})^0 \times D$, treated in [13: (5.2)] (see also [15: (1.8.7)]). It has the basic properties (cf. (B.2.10)):

$$\mathcal{J}_{G,P_s}(g, z) \text{ takes values in } K_{\mathbf{C}}, \text{ and is holomorphic in } z, \tag{B.2.13.1}$$

$$\mathcal{J}_{G,P_s}(k, z_0) = k \text{ if } k \in K, \tag{B.2.13.2}$$

$$\mathcal{J}_{G,P_s}(gh, z) = \mathcal{J}_{G,P_s}(g, hz)\mathcal{J}_{G,P_s}(h, z). \tag{B.2.13.3}$$

There are also the further properties:

$$\mathcal{J}_{G,P_s}(g, z) \text{ is independent of } z \text{ if } g \in N_s, \tag{B.2.13.4}$$

$$\mathcal{J}_{G,P_s}(g, z_0) = \mathcal{J}_{G_{s,h}}(g, c_s z_0) \text{ if } g \in G_{s,h}, \tag{B.2.13.5}$$

plus some finer ones (see [15: (1.8.7.5),(1.8.7.6)]), which contain the specification of the homomorphism of N_s given by (B.2.13.4).

Note that [H1] gives a formula $\mathcal{J}_{G,P_s}$ in terms of $\mathcal{J}_G$ and the Cayley transform:

$$\mathcal{J}_{G,P_s}(g, z_0) = \mathcal{J}_G(c_s^{-1}, z_0)^{-1}\mathcal{J}_G(c_s^{-1}g, z_0). \tag{B.2.14}$$

Since $Int(c_s^{-1})N_s \subset P_G^+(K_G)_{\mathbb{C}}$ (cf. (B.2.8(iv))), we have from (B.2.10.5) that for $g \in N_s$,

$$\mathcal{J}_G(c_s^{-1}g, z_0) = \mathcal{J}_G(Int(c_s^{-1})g, z_0)\mathcal{J}_G(c_s^{-1}, z_0);$$

hence,

$$\mathcal{J}_{G,P_s}(g, z_0) = Int(\mathcal{J}_G(c_s^{-1}, z_0)^{-1})\mathcal{J}_G(Int(c_s)^{-1}g, z_0) \quad \text{if } g \in N_s. \tag{B.2.14.1}$$

Moreover, if $g \in U_s$ then $Int(c_s)^{-1}g \in P_G^+$, and it follows that

$$\mathcal{J}_{G,P_s}(g, z_0) = 1 \quad \text{if } g \in U_s. \tag{B.2.14.2}$$

Finally, since c_s centralizes $G_{h,s}$, we have

$$\begin{aligned}\mathscr{J}_{G,P_s}(g,z_0) &= \mathscr{J}_G(c_s^{-1},z_0)^{-1}\mathscr{J}_G(gc_s^{-1},z_0) \\ &= Int(\mathscr{J}_G(c_s^{-1},z_0)^{-1})\mathscr{J}_G(g,c_s^{-1}z_0) \quad \text{if } g \in G_{h,s}. \end{aligned} \tag{B.2.14.3}$$

We can actually rewrite (B.2.14), for arbitrary $g \in G$ as:

$$\mathscr{J}_{G,P_s}(g,z_0) = Int(\mathscr{J}_G(c_s^{-1},z_0)^{-1})\mathscr{J}_G(Int(c_s)^{-1}g,c_s^{-1}z_0).$$

This suggests the formula

$$\mathscr{J}_{G,P_s}(g,z) = Int(\mathscr{J}_G(c_s^{-1},z_0)^{-1})\mathscr{J}_G(Int(c_s)^{-1}g,c_s^{-1}z), \tag{B.2.15}$$

which is actually correct; property (B.2.13.3) follows directly from (B.2.10.3).

Appendix C. Equal-Rank Groups

The material outlined below is, or deserves to be, well known:

(1) Let g be a real Lie algebra whose complexification is simple. There are two involutions of the Dynkin diagram of the **C**-root system: ι_g, induced by multiplication by -1, and thus a feature of $g_{\mathbf{C}}$, and γ_g, induced by complex conjugation, which, of course, depends on the real form. According to [6: Sec. 1], g admits a compact Cartan subalgebra (is *equal-rank*)[22] if and only if $\iota_g = \gamma_g$. It follows that the Dynkin diagram of the **C**-root system of a Lie algebra that is *not* equal-rank has a nontrivial automorphism. It follows that its classification type must be one of the following: A_r $(r > 1)$, D, E_6.

(2) As in [17: Sec. 4], Harish-Chandra's theory of strongly orthogonal roots carries through in the equal-rank setting, and one obtains a homomorphism of real Lie algebras: $(sl_2)^{\oplus r} \to g$. While there is much less structure than in the Hermitian case, one can see at once that there is a set of orthogonal real (restricted) roots that span.[23] This rules out A_r $(r > 1)$ and E_6 as the real root system.

(3) Note that DI can go either way, depending on the ranks. Indeed, -1 is an element of the Weyl group of D_r, and therefore, $\iota_g = 1$ if and only if r is even.

(4) g is of Hermitian type if and only if its Cartan involutions come from inner derivations.

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[22] Groups of Hermitian type are automatically equal-rank. It is easy to see that the symmetric space of an equal-rank group is of even real dimension. It is expected that arithmetic quotients of the latter share many of the cohomological properties of locally symmetric varieties.

[23] Note that we did not say *simple* roots.

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