Vietnam Journal of Mathematics **25**:4 (1997) 371–377

Vietnam Journal
of
MATHEMATICS

Short Communication

An Application of the Spectral Dichotomy Theory for Difference Equations

Phan Le Na

Department of Mathematics, Vinh Teachers Training College, Vinh, Vietnam

Received June 15, 1996
Revised December 10, 1996

Dedicated to Professor Hoang Tuy on the occasion of his 70th birthday

1. Introduction

This paper is devoted to study the existence of bounded solution for a difference equation of the form

$$x_{n+1} = A_n x_n + f_n(x_n) \tag{1}$$

under the assumption that the linear part of Eq. (1), i.e., the equation

$$x_{n+1} = A_n x_n \tag{2}$$

is exponentially dichotomic.

Suppose B is a complex Banach space and A_n belongs to the space $L(B)$ of all bounded linear operators acting in B.

To Eq. (2) we associate an operator $S : H \to H$, so-called characteristic operator of Eq. (2) acting on the Banach space H defined by

$$(Sx)_{n+1} = A_n x_n, \tag{3}$$

where $H = \{x : Z \to B \,|\, \sup_n \|x_n\| < \infty\}$ endowed with the sup-norm given as follows: if $x \in H$ then, $\|x\| = \sup_n \|x_n\|$. In [2], Aulbach and Minh have shown that the spectrum $\sigma(S)$ of the operator S does not intersect the unit circle $S_1 = \{z \in C : |z| = 1\}$ if and only if Eq. (2) has an exponential dichotomy on the condition

$$\sup_n \|A_n\| < \infty \tag{4}$$

even when A_n is not invertible. Basing on this property, the authors have given a new criterion for the exponential stability of difference equations.

It is easy to show that the spectrum $\sigma(S)$ of the operators S is invariant under the rotation around the origin. Therefore, saying that the spectrum $\sigma(S)$ does not intersect with S_1 means that the equation

$$x_{n+1} = A_n x_n + f_n \tag{5}$$

has a unique bounded solution (see Lemma 1 in Sec. 2). This result can be extended to the space L_p under the assumption that A_n is invertible and

$$\sup_n \|A_n^{-1}\| < \infty . \tag{6}$$

In this case, the equivalence between spectral dichotomy and exponential dichotomy is still valid (see Lemma 3 in Sec. 2).

The aim of this paper is to generalise their results to the nonlinear difference equation (1). Our main result (Sec. 3, Theorem 2) claims that, in the general case, Eq. (1) has a unique bounded solution if its linear part has spectral dichotomy and $\{f_n(x)\}$ satisfies the Lipschitz condition with a small Lipschitz constant. Theorem 3 follows from this result for the nonlinear difference equation (1) and the result in [2].

The paper is organized as follows. In the next section, we will give necessary definitions and state the results mentioned above. In Sec. 3, we show that the operator T of Eq. (1) is continuous.

Theorem 2 is our main result. To prove Theorem 2, we shall use the evolution operator (see [4]) and the inverse function theorem satisfying the Lipschitz condition as in [4].

In this paper, we will suppose $f_n(x_n)$ satisfies the Lipschitz condition with a Lipschitz coefficient sufficiently small.

2. Statement of Results

In this section, we state the main results of spectral dichotomy for linear spectrum of the characteristic operator, associated with the underlying linear difference equation

$$x_{n+1} = A_n x_n ,$$

where x_n belongs to a given Banach complex space B and A_n belongs to the space $L(B)$ consisting of all bounded linear operators acting in B.

Now let us introduce necessary definitions to state the results. First, we refer the concept of exponential dichotomy in the sense of Henry (see [3]) and it is defined as follows:

Definition 1. *Eq. (2) is said to have an exponential dichotomy (with M, θ) if there are positive constants M, $\theta < 1$ and a sequence of projections $\{P_n, n \in Z\}$ on B such that*

(i) $A_n P_n = P_{n+1} A_n$.
(ii) $A_n|_{R(P_n)}$ *is an isomorphism from* $R(P_n)$ *onto* $R(P_{n+1})$, *where* $R(P_n)$ *denotes the range of* P_n.
(iii) *If* $X(n, m) = A_{n-1} \cdots A_{m+1} A_m$ *for* $n > m$, $X(m, m) = Id$,

$$\|X(n, m)(I - P_m)x\| \leq M\theta^{n-m}\|x\|, \quad (n \geq m).$$

(iv) $\|X(n, m)P_m x\| \leq M\theta^{m-n}\|x\|, (n < m)$.

Further, we can define the operator $X(n, m)P_m$ when $n < m$ as the following: $X(n, m)P_m x = y \in R(P_n)$ if and only if $P_m(x) = X(m, n)y$, $(n < m)$.

To Eq. (2), we associate an operator $S : H \to H$, defined by

$$(Sx)_{(n+1)} = A_n x_n \tag{7}$$

for all $n \in Z$, where $H = \left\{ x : Z \to B \mid \sup_n \|x_n\| < \infty \right\}$.

The following definition deals with the spectral dichotomy and can be seen in [1].

Definition 2. *Eq. (2) is said to have a spectral dichotomy if the spectrum* $\sigma(S)$ *of the operator S does not intersect the unit circle.*

Under the assumptions that A_n is not invertible and suppose the condition (4) holds in [2], Aulbach and Minh have proved the following results.

Lemma 1. *Eq. (2) has a spectral dichotomy if and only if* $1 \notin \sigma(S)$ *or, in other words, for every bounded sequence* $\{f_n, n \in Z\}$, *the following difference equation*

$$x_{n+1} = A_n x_n + f_n \tag{8}$$

has a unique bounded solution $\{x_n(f), n \in Z\}$.

This means that the spectrum $\sigma(S)$ *is invariant with respect to rotation.*

Lemma 2. *Eq. (2) has an exponential dichotomy if and only if it has a spectral dichotomy.*

Under the assumptions that A_n is invertible and the condition (6) holds, they have following results in the space $L_p = L_p(Z, B)$ as in [2].

Lemma 3. *Assume that in addition to Condition (4), Eq. (2) satisfies the following conditions*
(i) A_n *is invertible for* $n \in Z$,
(ii) $\sup_n \|A_n^{-1}\| < \infty$.

Then, Eq. (2) has an exponential dichotomy if and only if it has a spectral dichotomy.

Lemma 4. *Under Lemma 3's assumptions, where Eq. (2) has an exponential dichotomy if and only if for every sequence* $\{f_n, n \in Z\} \subset L_p$, *there exists uniquely a* *solution in L_p to the following equation*

$$x_{n+1} = A_n x_n + f_n, \quad n \in Z.$$

Now we recall the so-called Inverse Function Theorem satisfying the Lipschitz condition by Nitecki (see [4] Lemma 2.1, p. 78) to estimate Eq. (1).

Lemma 5. *Suppose X is a Banach space, L is an invertible mapping from X into itself, ϕ is a mapping satisfying the Lipschitz condition such that $\mathscr{L}ip(\phi) < \|L^{-1}\|^{-1}$. Then $(L+\phi)$ is invertible and $(L+\phi)^{-1}$ satisfies the Lipschitz condition. Moreover,*

$$\mathscr{L}ip((L+\phi)^{-1}) \leq \frac{1}{\|L^{-1}\|^{-1} - \mathscr{L}ip(\phi)}. \tag{9}$$

By using an evolution operator as in [1, 2] and Lemma 5, we shall show that Eq. (1) has a unique bounded solution.

3. Main Results

Let us return to Eq. (1)

$$x_{n+1} = A_n x_n + f_n(x_n).$$

We suppose A_n is not invertible and the condition (4) holds.

We consider Eq. (1), when $x_n \in B$, $A_n \in L(B)$.

Let $H = \{x : Z \to B \mid \sup_n \|x_n\| < \infty\}$.

We define an operator $T : H \to H$ given by

$$(Tx)_{(n+1)} = A_n x_n + f_n(x_n), \quad n \in Z. \tag{10}$$

Put

$$F_n(x_n) = A_n x_n + f_n(x_n), \quad n \in Z. \tag{11}$$

In this section we only consider $f_n(x_n)$ satisfying the Lipschitz condition with a sufficiently small Lipschitz coefficient.

Definition 3. *Eq. (1) is said to satisfy the condition (H) if it satisfies the following conditions:*

(i) *$f_n(x)$ satisfies the Lipschitz condition with respect to x, where the Lipschitz coefficient is independent of n and x, i.e., there is a positive constant l, such that*

$$\|f_n(x) - f_n(y)\| \leq l\|x - y\|, \tag{12}$$

for all $n \in Z$ and $x, y \in B$.

(ii) *The linear part of Eq. (1) is exponentially dichotomic.*

Under the assumption that $f_n(x)$ satisfies the Lipschitz condition with respect to x, we shall show in the following proposition that the nonlinear operator T associated with Eq. (1) is continuous.

Proposition 1. *If $f_n(v)$ satisfies the Lipschitz condition with respect to v, then the operator T acting in H is continuous.*

Proof. Let the sequence $\{v^k, k \in N\} \subset H$ tend to v^0 as $k \to \infty$, where $\{v^k\}, v^0 \in H$. Then the sequence $\{v_n^k\}_{k \in N}$ tends to $v_n^0 \in B$ for any $n \in Z$.

From the definition of the operator T and the Lipschitz condition of $(f_n(x))$, we get the following estimate

$$\begin{aligned} \|Tv^k - Tv^0\| &= \sup_n \|F_n(v_n^k) - F_n(v_n^0)\| \\ &= \sup_n \|A_n v_n^k + f_n(v_n^k) - A_n v_n^0 - f_n(v_n^0)\| \\ &\leq \sup_n ((\|A_n\| + l)\|v_n^k - v_n^0\|). \end{aligned} \tag{13}$$

The proof is complete. ■

Remark 1. The operator T is nonlinear and in general it is not invertible.

Remark 2. Every sequence $x = \{x_n\}$ is a bounded solution of Eq. (1) if and only if it is a fixed point of the operator T.

We shall show that the existence of fixed points of the operator T is equivalent to exponential dichotomy for the linear part of Eq. (1).

Proposition 2. *The linear part of Eq. (1) has a spectral dichotomy if and only if the operator T has a fixed point.*

Proof. By virtue of Lemmas 1 and 2, since the linear part of Eq. (1) has a spectral dichotomy, the operator $(I - S)$ is invertible.

Put $v_n^{(k)} = (I - S)^{-1} f_n(v_n^{(k-1)})$. We have

$$\begin{aligned} |v_n^{(k)} - v_n^{(k-1)}| &= \|(I - S)^{-1}\| \, |f_n(v_n^{(k-1)}) - f_n(v_n^{(k-2)})| \\ &\leq \|(I - S)^{-1}\| \, l |v_n^{(k-1)} - v_n^{(k-2)}| \,. \end{aligned}$$

If $\rho = \|(I - S)^{-1}\| \, l < 1$, then $\{v^{(k)}\} \to v^0$.

It implies the desired result; the proof is complete. ■

When Eq. (1) satisfies condition (H), then we will get the following results.

Theorem 1. *Assume Eq. (1) satisfies Condition (H), and the Lipschitz coefficient l is small enough, then $(I - T)$ is invertible.*

Proof. Let S be the operator of the linear part of Eq. (1) defined by (7). By virtue of Lemma 1, since $1 \notin \sigma(S)$, then $(I - S)$ is an invertible operator. Put $L = I - S$, $\phi = S - T$.

From Condition (H) it follows that ϕ satisfies the Lipschitz condition, and $\mathscr{L}ip(\phi) = l$. Applying the inverse function theorem satisfying the Lipschitz condition (see in [4]), if $l < \|(I - S)^{-1}\|^{-1}$, then we get $(L + \phi)$ to be invertible. This means that $(I - T)$ is an invertible operator. This completes the proof of the theorem. ■

The main result of this paper is the following.

Theorem 2. *Under the assumptions of Theorem 1, Eq. (1) has a unique bounded solution.*

Proof. This is a direct conclusion from Theorem 1 and Remark 2. ■

In [2], it is proved that Eq. (2) has an exponential dichotomy if and only if, for every bounded sequence $\{f_n, n \in Z\}$, Eq. (8) has a unique bounded solution $\{f_n(f), n \in Z\}$. We shall prove below that this is still true for Eq. (1) if it satisfies Condition (H).

Theorem 3. *Under the assumptions of Theorem 1, for every bounded sequence* $\{q_n, n \in Z\}$, *the following difference equation*

$$x_{n+1} = F_n(x_n) + q_n \tag{14}$$

has a unique bounded solution $\{x_n(q_n), n \in Z\}$.

Proof. From (14), we have

$$x_{n+1} = Tx_{n+1} + q_n. \tag{15}$$

It implies that $(I - T)x_{n+1} = q_n$.

From Theorem 1, it follows that $(I - T)$ is invertible. That means $x_{n+1} = (I - T)^{-1} q_n$ is a unique bounded solution of Eq. (14).

We now consider the action of the operator T in the space $L_p = L_p(Z, B)$ consisting of all sequences in B such that

$$\sum_{n=1}^{\infty} \|x_n\|^p < \infty, \tag{16}$$

where $1 \leq p < \infty$. L_p is a Banach space on the complex field.

Under the assumptions that A_n is invertible and

$$\sup_n \|A_n^{-1}\| < \infty, \tag{17}$$

the equivalence between spectral dichotomy and exponential dichotomy is still true.

It is proved in [2] that the linear part of Eq. (2) has a spectral dichotomy if, for every sequence $\{q_n, n \in Z\}$, there exists a unique solution in L_p of Eq. (14) for all $n \in Z$.

In general, we prove that the above result is true for Eq. (14) in the same way as in Theorem 2 and Proposition 2. ■

Theorem 4. *Under the assumptions of Theorem 1 and assume Condition (17) holds, then for every sequence $\{q_n, n \in Z\}$, there exists uniquely a solution in L_p of the following equation*

$$x_{n+1} = F_n(x_n) + q_n, \quad n \in Z.$$

Acknowledgement. The author wishes to thank Dr. Nguyen Van Minh and Ph.D Nguyen Huu Du for helping her to carry out this article.

References

1. B. Aulbach, N. V. Minh, and P. P. Zabreiko, The concept of spectral dichotomy for linear difference equation, *J. Math. Anal. Appl.* **185** (1994) 275–287.
2. B. Aulbach and N. V. Minh, The concept of spectral dichotomy for linear difference equation II, *J. Math. Anal. Appl.* (to appear).
3. D. Henry, *Geometric Theory of Semilinear Parabolic Equation*, Lecture Notes in Mathematics Vol. 840, Springer-Verlag, Berlin-Heidengberg-New York, 1981.
4. Z. Nitecki, *Difference Dynamics*, Mir., Moscow, 1975 (Russian).