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Meromorphic Functions with Values in a Frechet Space and Linear Topological Invariant (DN)

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Abstract. The main aim of this paper is to prove that a Frechet space E has a continuous norm (resp., E has the property (DN)) if and only if $M(X,E) = M_w(X,E)$ holds for every open subset (resp., $\tilde{L}$-regular compact set) X of $\mathbf{C}^n$.

1. Introduction

Let X be a subset of $\mathbf{C}^n$ and E a sequentially complete locally convex space. A function f defined and holomorphic on a dense open subset X_0 of X with values in E is called meromorphic on X if it can be extended to a meromorphic function on a neighborhood of X in $\mathbf{C}^n$. In the case where this holds for x^*f with every $x^* \in E^*$, the dual space of E, we say that f is weakly meromorphic on X. Write $M(X,E)$ and $M_w(X,E)$ for vector spaces of meromorphic and weakly meromorphic functions on X with values in E, respectively. The main aim of the present paper is to find necessary and sufficient conditions for which

$$M(X,E) = M_w(X,E). \qquad (*)$$

The case where E is a Banach space and X is either open or compact, the equality has been proved in [3].

By applying this results in Sec. 2, we show that $(*)$ holds for every open set X in $\mathbf{C}^n$ if and only if E has a continuous norm. The case where X is compact in $\mathbf{C}^n$ will be investigated in Sec. 3. We will prove that $(*)$ holds for every $\tilde{L}$-regular compact set $X \subseteq \mathbf{C}^n$ if and only if E has the property (DN).

Finally, in Sec. 4, we prove that every analytic function on an open set $X \subset \mathbf{C}^n$ with values in a Frechet space E having the property (DN), can be weakly analytically extended to D. An open set in $\mathbf{C}^n$ containing X is also analytically extended to D. The case where E^* is a Baire space, the result has been established by Ligocka and Siciak [8].

2. Existence of a Continuous Norm on a Frechet Space

In this section, we give a necessary and sufficient condition for the existence of a continuous norm on a Frechet space.

Theorem 1. *Let E be a Frechet space. Then E has a continuous norm if and only if $M(X,E) = M_w(X,E)$ for every open subset X of $\mathbf{C}^n$.*

Proof. Necessity:

(i) For $n = 1$, the theorem has been proved in [4].

(ii) General case $n > 1$. Assume E has a continuous norm. Choose an increasing fundamental system $\{\|\cdot\|_k\}_{k=1}^{\infty}$ of continuous semi-norms on E. Without loss of generality, we may assume $\|\cdot\|_1$ is a norm. For each $k \geq 1$, by E_k we denote the canonical Banach space associated to $\|\cdot\|_k$ and $\omega_k : E \to E_k$ the canonical map. Let $f \in M_w(X,E)$. By [3], we have

$$f_k = \omega_k f \in M(X, E_k) \quad \text{for } k \geq 1.$$

As in the case $n = 1$, first we check that

$$P(f_k) = P(f_1) \quad \text{for } k \geq 1.$$

For each $k \geq 1$, put

$$Z_k = \{L \in CP^{n-1} : \pi^{-1}(L) \cap P(f_k) \neq \pi^{-1}(L) \cap X\},$$

where $\pi: \mathbf{C}^n \backslash \{0\} \to \mathbf{CP}^{n-1}$ is the canonical map.

It is easy to see that Z_k is dense and open in CP^{n-1} for $k \geq 1$ and by (i)

$$\pi^{-1}(L) \cap P(f_k) = \pi^{-1}(L) \cap P(f_1) \quad \text{for every } L \in Z_k \text{ and every } k \geq 1.$$

Hence, $\pi^{-1}(Z_k) \cap P(f_j)$ is dense and open in $P(f_j)$ for $k, j \geq 1$.

By the Baire theorem, this yields that

$$\pi^{-1}(Z) \cap P(f_j) = \pi^{-1}\left(\bigcap_{k\geq 1}(Z_k)\right) \cap P(f_j) = \bigcap_{k\geq 1}(\pi^{-1}(Z_k) \cap P(f_j))$$

is dense in $P(f_j)$ for $j \geq 1$, where $Z = \bigcap_{k\geq 1} Z_k$.

Since

$$\pi^{-1}(Z) \cap P(f_j) = \pi^{-1}(Z) \cap P(f_1) \quad \text{for } j \geq 1,$$

we have

$$P(f_j) = \overline{(\pi^{-1}(Z) \cap P(f_j))} = \overline{(\pi^{-1}(Z) \cap P(f_1))} = P(f_1) \quad \text{for } j \geq 1.$$

It remains to prove the meromorphicity of f at every $z_0 \in P(f_1)$.

First, consider the case where $z_0 \in RP(f_1)$, the regular locus of $P(f_1)$. We may assume $z_0 = 0$. Choose a neighborhood U of z_0 of the form $U = \Delta^n$, such that $U \cap P(f_1) = \Delta^{n-1} \times 0$, where $\Delta = \{z \in \mathbf{C} : |z| < 1\}$. Since f is holomorphic on $U \backslash P(f_1) = \Delta^{n-1} \times \Delta^*$, $\Delta^* = \Delta \backslash \{0\}$ we can write the Laurent expansion

$$f(z', z_n) = \sum_{j=-\infty}^{+\infty} a_j(z') z_n^j \quad \text{for } z = (z', z_n) \in \Delta^{n-1} \times \Delta^*,$$

where $a_j(z')$ are holomorphic functions on Δ^{n-1}.

Since $f_1(z', z_n) = \omega_1 f(z', z_n) = \sum_{j=-\infty}^{+\infty} \omega_1(a_j(z')) z_n^j$, hence,

$$\omega_1(a_j(z')) = 0 \quad \text{for } j < n_1.$$

By the injectivity of ω_1 we have

$$a_j(z') = 0 \quad \text{for } j < n_1.$$

This means that f is meromorphic at z_0. Since $\operatorname{codim} S(P(f)) \geq 2$, where $S(P(f))$ is the singular locus of f, and by the Remmert–Stein theorem [10], f can be meromorphically extended to X.

Sufficiency. See [4].

3. Existence of a (DN)-Norm on a Frechet Space

To give a characterization of Frechet spaces having the property (DN), we recall the following.

Let $\{\|\cdot\|_k\}_{k=1}^{\infty}$ be a fundamental system of continuous semi-norms of a Frechet space E. For each subset B of E, consider the general semi-norm

$$\|\cdot\|_B^* : E^* \to [0, +\infty]$$

given by

$$\|u\|_B^* = \{\sup |u(x)| : x \in B\}.$$

Write $\|\cdot\|_k^*$ for $B = U_k = \{x \in E : \|x\|_k \leq 1\}$.

We say that E has the property (DN) if and only if

$$\exists p \geq 1 \; \forall q \geq 1 \; \forall d > 0 \; \exists k \geq 1, \; C > 0 : \|x\|_q^{1+d} \leq C \|x\|_k \|x\|_p^d \quad \text{for } x \in E \quad \text{(DN)}$$

Obviously, $\|\cdot\|_p$ is a norm and we call it a (DN)-norm.

We say that E has the property $(\tilde{\Omega})$ if and only if

$$\forall p \geq 1 \; \exists q \geq 1 \; \exists d > 0 \; \forall k \geq 1, \; \exists C > 0 : \|y\|_q^{*1+d} \leq C \|y\|_k^* \|y\|_p^{*d} \text{ for } y \in E^*. \quad (\tilde{\Omega})$$

The properties (DN), $(\tilde{\Omega})$ and others were introduced and investigated by Vogt (see [13, 14, 15, etc.]). In [15], Vogt has proved that a Frechet space E has the property (DN) if and only if every continuous linear map $T : \lambda_1(\alpha) \to E$ is bounded on a neighborhood of $0 \in \lambda_1(\alpha)$ for some exponent sequence $\alpha = (\alpha_n)$, where

$$\lambda_1(\alpha) = \left\{ (\xi_j) \in \mathbf{C}^\infty : \sum_{j \geq 1} |\xi_j| r^{\alpha_j} < \infty \quad \forall r, 0 < r < 1 \right\}.$$

Let V be an open subset of $\mathbf{C}^n$. We let

$$H^\infty(V) = \{ f \in H(V) : \|f\|_V = \sup\{ |f(x)| : x \in V \} < \infty \},$$

where $H(V)$ is the space of holomorphic functions on V. $H^\infty(V)$ is a Banach space with the norm $\|\cdot\|_V$.

Let X be a compact subset of $\mathbf{C}^n$. On $\bigcup_{\substack{V \supset X \\ V \text{ open}}} H^\infty(V)$, we define the equivalence relation $\sim$ as follows: $f \sim g$ if there exists a neighborhood W of X on which $f|_W = g|_W$.

We denote by $H(X)$ the vector space of equivalence classes and the elements of $H(X)$ are called germs of holomorphic functions on X. $H(X)$ is equipped with the inductive limit topology

$$H(X) = \lim \operatorname{ind} H^\infty(V).$$

Now we say that a compact subset X in $\mathbf{C}^n$ is $\tilde{L}$-regular if $[H(X)]^*$ has the property $(\tilde{\Omega})$.

Through the forthcoming, unless otherwise specified, we shall write $Z(h)$ and $Z(g, \sigma)$ for $h^{-1}(0)$ and $g^{-1}(0) \cap \sigma^{-1}(0)$, respectively.

The main result of the section is the following:

Theorem 2. *Let E be a Frechet space. Then E has the property* (DN) *if and only if $M(X, E) = M_w(X, E)$ for every $\tilde{L}$-regular compact set X in $\mathbf{C}^n$.*

To prove Theorem 2, we first prove the following result.

Lemma 1. *Let D be a pseudoconvex domain in $\mathbf{C}^n$ and f a meromorphic function on D with values in a sequentially complete locally convex space E. Then for every relatively compact domain $\tilde{D}$ in D, there exist holomorphic functions $h : D \to E$ and $\sigma : D \to \mathbf{C}$ such that*

$$f = h/\sigma \quad \text{and } \operatorname{codim}_y Z(h, \sigma) \geq 2 \quad \text{for } y \in \tilde{D}.$$

Proof. From the hypothesis and by [7], we can write $f = h_1/\sigma_1$, where $h_1 : D \to E$ and $\sigma_1 : D \to \mathbf{C}$ are holomorphic functions with $\sigma_1 \neq 0$. By the compactness of $\bar{\tilde{D}}$, there exists a neighborhood W of $\bar{\tilde{D}}$ in D such that

$$Z(h_1) \cap W \subseteq \bigcup_{i=1}^{p} A_i \quad \text{and} \quad Z(\sigma_1) \cap W \subseteq \bigcup_{j=1}^{q} B_j$$

with

$$A_i \cap W \neq \emptyset \quad \text{and} \quad B_j \cap W \neq \emptyset \quad \text{for } i = 1, \ldots, p \text{ and } j = 1, \ldots, q.$$

Here, $Z(h_1) = \bigcup_{i \geq 1} A_i$ and $Z(\sigma_1) = \bigcup_{j \geq 1} B_j$ are irreducible branches of $Z(h_1)$ and $Z(\sigma_1)$, respectively.

Let $A_{i_0} = B_{j_0} = A$ for some $1 \le i_0 \le p$ and $1 \le j_0 \le q$.

Now, by using Cartan's theorem A, we can locally factorize h_1 and σ_1 through common factors and finally, we can find holomorphic functions $h : D \to E$ and $\sigma : D \to \mathbf{C}$ such that $f = h/\sigma$ and $Z(h,\sigma)$ does not contain an irreducible branch A of codimension 1 in W. This yields $\operatorname{codim}_y Z(h,\sigma) \ge 2$ for $y \in \tilde{D}$.

Lemma 2. *Let E be a locally convex space and $\sigma, \beta : D \to \mathbf{C}$, $g : D \to E$ holomorphic functions on an open subset $D \subset \mathbf{C}^n$. Assume $\frac{\beta g}{\sigma}$ is holomorphic on D and $\operatorname{codim} Z(g,\sigma) \ge 2$. Then $\frac{\beta}{\sigma}$ is holomorphic on D.*

Proof. Given $z_0 \in D$. Since the local ring $\mathcal{O}_{z_0}$ of germs of holomorphic functions at z_0 is factorial [6], we can write

$$\sigma = \sigma_1^{m_1} \cdots \sigma_p^{m_p}$$

in a neighborhood U of z_0 such that $\sigma_{1_{z_0}}, \ldots, \sigma_{p_{z_0}}$ are irreducible.

By the hypothesis and the equality

$$\frac{\beta g}{\sigma_1} = \frac{\beta g}{\sigma} \sigma_1^{m_1 - 1} \cdots \sigma_p^{m_p},$$

it follows that $\frac{\beta g}{\sigma_1}$ is holomorphic at z_0. On the other hand, from the hypothesis $\operatorname{codim} Z(g,\sigma) \ge 2$ and $Z(\sigma) = \bigcup_{i=1}^{p} Z(\sigma_1)$, we have $\operatorname{codim} Z(g,\sigma_i) \ge 2$, for $i = 1,\ldots,p$. Hence, from the irreducibility of $\sigma_{1_{z_0}}$, we infer that $Z(\sigma_1)_{z_0} \subseteq Z(\beta)_{z_0}$. This again implies $\beta = \beta_1\sigma_1$ at z_0. Hence, $\frac{\beta}{\sigma_1}$ is holomorphic at z_0. Continuing this process, we infer that $\frac{\beta}{\sigma}$ is holomorphic at z_0.

Lemma 3. *Let X be a $\tilde{L}$-regular compact set in $\mathbf{C}^n$. Then X is a unique set, i.e., if $f \in H(X)$, $f|_X = 0$, then $f = 0$ on some neighborhood of X.*

Proof. Let (V_p) be a decreasing neighborhood basis of X in $\mathbf{C}^n$. By the hypothesis, we have

$$\forall p \ge 1\ \exists q \ge p,\ d > 0\ \forall k \ge q\ \exists C > 0 : \|f\|_q^{1+d} \le C\|f\|_k\|f\|_p^d \quad \forall f \in H^\infty(V_p).$$

Using the above inequality for $f^n, f \in H^\infty(V_p)$, it follows that

$$\begin{aligned}\|f\|_q^{1+d} &= \lim_{t\to\infty}(\|f\|_q^{n(1+d)})^{1/n}\\ &= \lim_{n\to\infty}(\|f^n\|_q^{1+d})^{1/n}\\ &\le \lim_{n\to\infty} C^{1/n}(\|f^n\|_k\|f^n\|_p^d)^{1/n} = \|f\|_k\|f\|_p^d.\end{aligned}$$

Hence,

$$\forall p \ge 1\ \exists q \ge p,\ d > 0\ \forall k \ge q : \|f\|_q^{1+d} \le \|f\|_k\|f\|_p^d \quad \forall f \in H^\infty(V_p),$$

which implies as $k \to \infty$:

$$\forall p \geq 1\ \exists q \geq p\ \exists d > 0 : \|f\|_q^{1+d} \leq \|f\|_x \|f\|_p^d \quad \forall f \in H^\infty(V_p).$$

This means that X is a unique set.

Let E be a Frechet space with strong dual E^*. The space E', the topological dual space of E, equipped with the strongest locally convex topology having the same bounded sets as E^* is called the bornological space associated to E^* and is denoted by E^*_{bor}.

We have the following lemma.

Lemma 4. *Let E be a Frechet space and have the property* (DN). *Then $[E^*_{bor}]^*$ has the property* (DN).

Proof. It is known that E has the property (DN) if and only if

$$\exists p\ \forall q\ \exists k,\ C > 0 : \|\cdot\|_q \leq Cr\|\cdot\|_k + \frac{1}{r}\|\cdot\|_p \quad \forall r > 0$$

or, as was shown in [14], this condition is equivalent to

$$\exists p\ \forall q\ \exists k,\ C > 0 : U_q^0 \subseteq CrU_k^0 + \frac{1}{r}U_p^0 \quad \forall r > 0\,,$$

where U_q^0 is the polar of U_q.

Thus,

$$\|u\|_q^{**} = \sup_{x^* \in U_q^0} |u(x^*)| \leq \sup_{x^* \in CrU_k^0 + 1/rU_p^0} |u(x^*)|$$

$$\leq Cr \sup_{x^* \in U_k^0} |u(x^*)| + \frac{1}{r} \sup_{x^* \in U_p^0} |u(x^*)| = Cr\|u\|_k^{**} + \frac{1}{r}\|u\|_p^{**}$$

for all $r > 0$ and $u \in [E^*_{bor}]^*$.

This means that $[E^*_{bor}]^*$ has the property (DN).

Lemma 5. *Let E and F be Frechet spaces and let F have the property* (DN) *and E have the property* $(\tilde{\Omega})$. *Then every continuous linear map from F^*_{bor} into E^* is factorized through a Banach space.*

Proof. Given $f : F^*_{\text{bor}} \to E^*$ a continuous linear map. Since every continuous linear map which is bounded on some neighborhood of zero is factorized through a Banach space, it suffices to find a neighborhood V of $0 \in E$ such that

$$\sup\{\|f(u)\|_V^* : u \in U_k^0\} < \infty \quad \text{for } k \geq 1, \tag{1}$$

where $\{U_k\}$ is a neighborhood basis of $0 \in F$.

By [16], F is isomorphic to a subspace of the space $B \hat{\otimes}_\pi s$ for some Banach space B, where s is the space of rapidly decreasing sequences.

Since the restriction map R from $[B \hat{\otimes}_\pi s]^* \cong B^* \hat{\otimes}_\pi s^*$ onto F^*_{bor} is open, it remains to prove that (1) holds for $g = fR$.

Consider the continuous linear map $\tilde{g} : s^* \to L(B^*, E^*)$, the space of continuous linear maps from B^* to E^*, induced by g. Here, $L(B^*, E^*)$ is equipped with the strong topology.

Let $\{\|\cdot\|_\gamma\}_{\gamma=1}^{\infty}$ be a fundamental system of semi-norms of E.

Since E has the property $(\tilde{\Omega})$, it follows that

$$\begin{gathered}\forall \alpha \geq 1\ \exists \beta \geq \alpha,\ d > 0\ \forall \gamma \geq \beta\ \exists C_1(\gamma) > 0,\\ \|\sigma\|_\beta^{*1+d} \leq C_1(\gamma)\|\sigma\|_\gamma^*\|\sigma\|_\alpha^* \quad \text{for every } \sigma \in L(B^*, E^*),\end{gathered} \tag{2}$$

where

$$\|\sigma\|_\beta^* = \sup\{\|\sigma(v)\|_\beta^* : v \in B^*, \|v\| \leq 1\}.$$

Now for each $k \geq 1$, put

$$s^*(k) = \left\{ u = (\eta_j) \in \mathbf{C}^\infty : |||u|||_k = \sum_{j\geq 1} |\eta_j| j^{-k} < \infty \right\}.$$

Since s^* is bornological, we have

$$s^* \cong \lim_k \operatorname{ind} s^*(k)$$

and the topology of s^* can be defined by the semi-norms

$$\big|\|u\|\big|_k = \sum_{j\geq 1} |\eta_j| j^{-k}.$$

On the other hand, since s has the property (DN), it implies that

$$\begin{gathered}\exists p \geq 1\ \forall q \geq p\ \forall d > 0\ \exists k \geq q, \quad C_2(q,d) > 0,\\ \big|\|e_j^*\|\big|_q^{1+d} \geq C_2(q,d)|\|e_j^*\||_k|\|e_j^*\||_p^d \quad \text{for every } j \geq 1,\end{gathered} \tag{3}$$

where $\{e_j^*\}$ is the canonical basis of s^*.

For each $k \geq 1$, choose $\gamma = \gamma(k)$ such that

$$M(k, \gamma(k)) = \sup\{\|\tilde{g}(u)\|_{\gamma(k)}^* : |||u|||_k \leq 1\} < \infty.$$

For p in (3), put $\alpha = \gamma(p)$ and take $\beta \geq \alpha$, $d > 0$ such that (2) holds. Using d in (2) for (3), we now check that $M(q, \beta) < \infty$ for $q \geq p$. Indeed, let $q \geq p$. Choose $k \geq q$ and $C_2(q,d) > 0$ for which (3) is satisfied. For k, choose $\gamma = \gamma(k)$ such that $M(k, \gamma(k)) < \infty$.

Then for every $u = (\eta_j) \in U_q^0$, $u = \sum_{j\geq 1} \eta_j e_j^*$ with $\big|\|u\|\big|_q = \sum_{j\geq 1} |\eta_j|\, \big|\|e_j^*\|\big|_q \leq 1$, we have

$$\begin{aligned}\|\tilde{g}(u)\|_\beta^* &\leq \sum_{j\geq 1} |\eta_j||\|e_j^*\||_q \frac{\|\tilde{g}(e_j^*)\|_\beta^*}{|\|e_j^*\||_q}\\ &\leq \sum_{j\geq 1} |\eta_j||\|e_j^*\||_q \left(\frac{C_1(\gamma(k))}{C_2(q,d)}\right)^{\frac{1}{1+d}} \left[\frac{\|\tilde{g}(e_j^*)\|_{\gamma(k)}^*}{|\|e_j^*\||_k}\right]^{\frac{1}{1+d}} \left[\frac{\|\tilde{g}(e_j^*)\|_{\gamma(p)}^*}{|\|e_j^*\||_p}\right]^{\frac{1}{1+d}}\\ &\leq \left(\frac{C_1(\gamma(k))}{C_2(q,d)}\right)^{\frac{1}{1+d}} M(k,\gamma(k))^{\frac{1}{1+d}}(M(p,\gamma(p)))^{\frac{1}{1+d}} < \infty.\end{aligned}$$

This inequality implies that $\tilde{g}$ and hence, g satisfies (1).

Proof of Theorem 2. Let E have the property (DN) and $f \in M_w(X,E)$, where X is $\tilde{L}$-regular compact set in $\mathbf{C}^n$.

By [3], for each $p \geq 1$, there exists a Stein neighborhood U_p of X in $\mathbf{C}^n$ and a meromorphic function $f_p : U_p \to E_p$ such that $f_p|_X = \omega_p f$. We can suppose that $U_1 \supset U_2 \supset \cdots \supset U_p \supset \cdots$. By Lemma 2.1, we can write $f_p = h_p/\sigma_p$ where $h_p : U_p \to E_p$, $\sigma_p : U_p \to \mathbf{C}$ are holomorphic functions and $\sigma_p \neq 0$ such that

$$\operatorname{codim} Z(h_p, \sigma_p) \geq 2\,.$$

Since $\omega_1 = \omega_1^p.\omega_p$, where $\omega_1^p : E_p \to E_1$ is the canonical map, and by Lemma 3, we have

$$\frac{h_1}{\sigma_1}|_{U_p} = \frac{\omega_1^p h_p}{\sigma_p} \quad \text{and} \quad \operatorname{codim} Z(\omega_1^p h_p, \sigma_p) \geq 2\,.$$

By Lemma 2, it follows that

$$\frac{\sigma_1}{\sigma_p}|_{U_p} \text{ is holomorphic for } p \geq 1\,.$$

We can define a linear map

$$\tilde{h} : E^*_{bor} \to H(X)$$

by

$$\tilde{h}|_{E_p^*} = \left(\frac{\sigma_1}{\sigma_p}\right)\tilde{h}_p \quad \text{for } p \geq 1\,,$$

where

$$\tilde{h}_p(x^*)(z) = x^*(h_p(z)) \quad \text{for } x^* \in E_p^* \text{ and } z \in U_p \text{ and } E_p^* = E^*(U_p^0)\,.$$

Obviously, $\tilde{h}$ is continuous. Since $[E^*_{bor}]^*$ has the property (DN) (Lemma 4) and $[H(X)]^*$ has the property $(\tilde{\Omega})$, by Lemma 5, we can find a neighborhood W of $0 \in E^*_{bor}$ such that $\tilde{h}(W)$ is bounded in $H(X)$. Hence, there exists p such that $\tilde{h}(W)$ is contained and bounded in $H^\infty(U_p)$, the Banach space of bounded holomorphic functions on U_p. Thus, the form

$$\hat{h}(z)(x^*) = \tilde{h}(x^*)(z) \quad \text{for } z \in U_p,\ x^* \in E^*$$

defines a holomorphic function $\hat{h} : U_p \to E$. Since $E = \lim\operatorname{proj} E_p$ and $f_p|_X = \omega_p f$ for every $p \geq 1$, it implies that $\left.\frac{\hat{h}}{\sigma_1}\right|_X = f$ and hence, $f \in M(X,E)$.

Conversely, by [15], it suffices to show that every continuous linear map T from $H(\Delta)$ to E is bounded on a neighborhood of $0 \in H(\Delta)$. Consider $T^* : E^* \to [H(\Delta)]^* \cong H(\bar{\Delta})$. Since $T^*(x^*) \in H(\bar{\Delta})$ for every $x^* \in E^*$ and, hence, we can define a map $f : \bar{\Delta} \to E^{**}$ by

$$f(z)(x^*) = \delta_z(T^*(x^*))$$

for $x^* \in E^*$, $z \in \bar{\Delta}$ and δ_z is the Dirac functional defined by z,

$$\delta_z(\sigma) = \sigma(z) \quad \text{for } \sigma \in H(\bar{\Delta}).$$

It is easy to see that $f(z) \in E$ because of the $\sigma(E^*, E)$-continuity of $f(z)$. Moreover, $f \in M_w(\bar{\Delta}, E)$. By the hypothesis, we can find a neighborhood U of $\bar{\Delta}$ in $\mathbf{C}$ and an E-valued meromorphic function g on U such that

$$g|_{\bar{\Delta}} = f.$$

Since f is continuous on $\bar{\Delta}$, without loss of generality, we may assume g is holomorphic on U and $B = g(U)$ is bounded in E. It follows that T^* is bounded on B^0. Put $T^*(B^0) = C \subset [H(\Delta)]^*$. Thus, $V = C^0$ is a neighborhood of $0 \in H(\Delta)$ and $T(V) \subset B^{00}$ is bounded in E. The theorem is proved.

4. Weak Extension of Analytic Functions

Theorem 3. *Let X be an open subset of an open connected set D in $\mathbf{R}^n$ and E a Frechet space having the property* (DN). *Assume $f : X \to E$ is an analytic function such that uf is extended to an analytic function $\widehat{uf}$ on D for all $u \in E^*$. Then f is analytically extended to D.*

Proof. It suffices to show that f is analytically extended to every $x^0 \in \partial X$. Take a neighborhood $G = I_1 \times \cdots \times I_n$ of x^0 in D, where $I_i = [a_i, b_i]$, $a_i < b_i$, $i = 1, \ldots, n$.

For each $0 < \varepsilon < 1$, consider the linear map

$$S_\varepsilon : E^*_{bor} \to A(\varepsilon G)$$

given by

$$S_\varepsilon(u)(x) = \widehat{uf}(x) \quad \text{for } u \in E^*_{bor}, \quad x \in \varepsilon G,$$

where $A(\varepsilon G)$ is the space of analytic functions on εG.

By the uniqueness, S_ε has the closed graph. On the other hand, since

$$A(\varepsilon G) \equiv \lim_{\widetilde{W} \downarrow \varepsilon G} \operatorname{ind} H^\infty(\widetilde{W}) \equiv H(\varepsilon G)$$

where for each neighborhood $\widetilde{W}$ of εG in $\mathbf{C}^n$, by $H^\infty(\widetilde{W})$, we denote the Banach space of bounded holomorphic functions on W, it follows that $S_\varepsilon : E^*_{bor} \to A(\varepsilon G)$ is continuous.

Since

$$\begin{aligned}
[H(\varepsilon G)]^* &\cong [H(\varepsilon I_1) \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi H(\varepsilon I_n)]^* \\
&\cong H(\mathbf{C} \backslash \varepsilon I_1) \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi H(\mathbf{C} \backslash \varepsilon I_n) \\
&\cong H(\Delta) \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi H(\Delta) \cong H(\Delta^n) \text{ have the property } (\tilde{\Omega})
\end{aligned}$$

and $[E^*_{bor}]^*$ has the property (DN), we can find a neighborhood W_ε of εG in $\mathbf{C}^n$

such that $S_\varepsilon : E^*_{bor} \to H^\infty(W_\varepsilon)$ is continuous. Define a holomorphic extension

$$\hat{f}_\varepsilon : W_\varepsilon \to [E^*_{bor}]^* .$$

by

$$\hat{f}_\varepsilon(z)(u) = S_\varepsilon(u)(z) \quad \text{for } z \in W_\varepsilon ,\ u \in E^*_{bor} .$$

By the uniqueness, the family $\{\hat{f}_\varepsilon\}$ defines a holomorphic extension $\hat{f}$ of f to a connected neighborhood W of G in $\mathbf{C}^n$. Since $\hat{f}(G \cap X) \subset E$ and E is a closed subspace of $[E^*_{bor}]^*$, it follows that $\hat{f}(W) \subset E$.

This means that f can be analytically extended to x^0. The theorem is proved.

Remark. Consider the function $f : \mathbf{R} \to \mathbf{R}^{\mathbf{N}}$ which was given by Ligocka and Siciak [8],

$$f(t) = \left(\frac{1}{1+t^2}, \dots, \frac{1}{1+(nt)^2}, \dots \right), \quad t \in \mathbf{R} .$$

This function is analytic on $R \backslash 0$ and uf is analytic on $\mathbf{R}$ for all $u \in [\mathbf{R}^N]^*$. However, f is not analytic at $0 \in \mathbf{R}$.

Now let X be an arbitrary Stein manifold. In [5], we have proved that if every weakly holomorphic function with values in $H(X)$ is holomorphic, then $H(X)$ has the property (DN). Hence, in this case every pluri-subharmonic function on X, which is bounded from above, is constant (cf. [17]). However, for analytic functions, we only prove the following.

Proposition 1. *Let X be a connected complex space such that every weakly analytic function on an open set in $\mathbf{R}^n$ with values in $H(X)$ is analytic. Then every bounded holomorphic function on X is constant.*

Proof. Otherwise, let $\varphi \in H(X)$ such that $\varphi \neq$ const and

$$\sup_X |\varphi| = 1 .$$

Consider the function $f : (-1, 1) \times X \to \mathbf{C}$ given by

$$f(t, z) = \frac{1}{1 + \dfrac{t^2}{1 - \varphi(z)}} .$$

It follows that f is analytic.

First we check that $\hat{f} : (-1, 1) \to H(X)$ is weakly analytic.

Indeed, given $\mu \in [H(X)]^*$ and $t_0 \in (-1, 1)$. Choose a compact set K in X such that $\operatorname{supp} \mu \subset K$. By the compactness of K, we can find a neighborhood $U \times V$ of $\{t_0\} \times K$ in $\mathbf{C}^n \times X$ and a holomorphic function $g : U \times V \to \mathbf{C}$ for which

$$g|_{(U \times V) \cap ((-1,1) \times X)} = f|_{(U \times V) \cap ((-1,1) \times X)} .$$

Since $\hat{g} : U \to H(V)$ is holomorphic and μ can be considered as an element of $[H(V)]^*$, it follows that $\mu\hat{f}$ is extended holomorphically to $\mu\hat{g}$ on U.

By the hypothesis, $\hat{f}$ is analytic. However, this is impossible since the radius of the convergence $r(z)$ of the series

$$1 - \frac{t^2}{1-\varphi(z)} + \frac{t^4}{(1-\varphi(z))^2} - \frac{t^6}{(1-\varphi(z))^2} + \cdots$$

is $\sqrt{|1-\varphi(z)|} \to 0$ as $z \to \partial X$.

However, for the case $\dim X = 1$, we have

Proposition 2. *Let Z be a connected open set in* $\mathbf{C}$. *Then $H(Z)$ has the property* (DN) *if and only if every $H(Z)$-valued weakly analytic function is analytic.*

Proof. Necessity follows from Theorem 3. Conversely, by Proposition 1, every bounded holomorphic function on Z is constant. Hence, $\gamma(\overline{\mathbf{C}}\backslash Z) = 0$ where $\gamma(\overline{\mathbf{C}}\backslash Z)$ is the analytic capacity of $\overline{\mathbf{C}}\backslash Z$ [2]. Hence, $H(\overline{\mathbf{C}}\backslash Z) \cong H(\{0\})$. Then $H(Z) \cong [H(\overline{\mathbf{C}}\backslash Z)]^* \cong [H(\{0\})]^* \cong H(\overline{\mathbf{C}}\backslash\{0\}) \cong H(\mathbf{C})$ has the property (DN).

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