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The Maximal Factorable Minorant

Do Cong Khanh and Nguyen Minh Hang
Department of Mathematics, College of General Studies, VNU-HCM
268 Ly Thuong Kiet, 10 District, Ho Chi Minh City, Vietnam

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Dedicated to Professor Hoang Tuy on the occasion of his 70th birthday

Abstract. In this paper, we study the maximal factorable minorant of the function $\theta(z) = \theta_2(z)\,\theta_1(z)$ from the maximal factorable minorants of $\theta_1(z)$ and $\theta_2(z)$, where $\theta_1(z)$, $\theta_2(z)$ and $\theta(z)$ are contractive operator functions analytic on the unit disk $D = \{z \in \mathbf{C} \,/\, |z| < 1\}$.

1. Introduction

In the linear dynamic system theory, the transfer function $\theta(z)$ is an important characteristic. For some system classes, the system corresponding to a given transfer function is unique. Let $\theta(z) : U \to V$ be a contractive operator function analytic on the unit disk $D = \{z \in \mathbf{C} \,/\, |z| < 1\}$, a theorem of Nagy and Foias [8] asserts that there exists an outer function $\varphi(z)$ on D, whose values are operators from U to an auxiliary space E such that

$$\varphi^*\varphi \le I - \theta^*\theta \text{ a.e. on } \partial D$$

and if $\phi(z)$ is an analytic contractive operator function such that

$$\phi^*\phi \le I - \theta^*\theta \text{ a.e., then } \phi^*\phi \le \varphi^*\varphi \text{ a.e.}$$

The function $\varphi(z)$ is unique up to a constant unitary factor on the left and is called the maximal factorable minorant (MFM) of $I - \theta^*\theta$.

Some important qualitative properties of unitary systems such as observability, controllability ... are characterized by the MFM $\varphi(z)$ and these properties are often not conserved through the cascade coupling of two systems. So we can use the MFM as a tool to consider the conditions for the conservation of qualitative properties for a cascade coupling.

We have a result that if the system α is a cascade coupling of two systems α_1 and α_2, then the transfer function of α is a product of the two transfer function of α_1 and α_2. Thus, building the MFM of $\theta(z) = \theta_2(z)\theta_1(z)$ from the MFM of $\theta_1(z)$

and $\theta_2(z)$, as well as searching for the conditions in which the MFM of $\theta(z)$ has simplest form, is an interesting problem. This is the main purpose of the paper.

2. The Maximal Factorable Minorant for the Product of Contractive Analytic Operator-Valued Functions in the Unit Disk of the Complex Plane

Here we denote by $B(U,V)$ the class of all analytic functions in the unit disk D having values as contractive operators from the Hilbert space U to the Hilbert space V. Let Ω be a subspace of a Hilbert space H and $\mathscr{U}$ an isometric operator in H such that $\mathscr{U}^p\Omega \perp \mathscr{U}^q\Omega$, for all nonnegative integers p, q $(p \neq q)$, we define $M_+(\Omega) = \bigoplus_0^\infty \mathscr{U}^n\Omega$.

An isometric operator $\mathscr{U}$ in the space H is called a unilateral translation if there exists a subspace Ω of H such that $H = M_+(\Omega)$.

The Fourier representation ϕ^Ω of $M_+(\Omega)$ is a unitary operator from $M_+(\Omega)$ onto $H^2(\Omega)$, defined by

$$\left(\phi^\Omega \sum_{k=0}^{\infty} \mathscr{U}^k a_k\right)(\lambda) = \sum_{k=0}^{\infty} \lambda^k a_k, \qquad \left(a_k \in \Omega, \sum_{k=0}^{\infty} \|a_k\|^2 < +\infty, |\lambda| < 1\right),$$

where $H^2(\Omega)$ denotes the Hardy vector space of Ω-valued functions on D.

Proposition 1. [8] *Let $\mathscr{U}$ and $\mathscr{U}'$ be the unilateral translations in the separable Hilbert spaces $R = \bigoplus_0^\infty \mathscr{U}^n\Omega$ and $R' = \bigoplus_0^\infty \mathscr{U}'^n\Omega'$, respectively. Let Q be a contraction from R into R' such that*

$$Q\mathscr{U} = \mathscr{U}'Q,$$

then there exists an analytic contractive operator function $\mathscr{A}(z) : \Omega \to \Omega'$ such that

$$\phi^{\mathscr{U}'}Q = \mathscr{A}\phi^{\mathscr{U}}.$$

The function $\mathscr{A}(z)$ is

(a) *outer if and only if $\overline{QR} = R'$ (by definition, the function $\mathscr{A}(z)$ is outer if $\overline{\mathscr{A}H^2(\Omega)} = H^2(\Omega')$),*

(b) *unitary constant if and only if Q is a unitary operator from R onto R', ($\mathscr{A}(z)$ is unitary constant if $\mathscr{A}(z) = \mathscr{A}_0$ where $\mathscr{A}_0$ is a unitary operator from Ω onto Ω').*

Let $\theta(z) = \theta_2(z)\theta_1(z)$ be a factorization of the contractive operator function $\theta(z) \in B(U,V)$, where $\theta_k(z) \in B(U_k, V_k)$, $k = 1, 2$, $U_1 = U$, $V_1 = U_2$, $V_2 = V$. We define $\Delta \equiv \Delta(e^{it}) = (I - \theta(e^{it})^*\theta(e^{it}))^{1/2}$.

We will build the MFM for $\theta(z)$ from the MFM of $\theta_1(z)$ and $\theta_2(z)$.

Denote by $\mathscr{U}_k (k = 1, 2)$ the multiplication by e^{it} on $L^2(U_k)$. Since Δ commutes with $\mathscr{U}_1$ and $H^2(U)$ is invariant for $\mathscr{U}_1$, the subspace $\overline{\Delta H^2(U)}$ of $L^2(U)$ is also invariant for $\mathscr{U}_1$. Thus, $\mathscr{U}_1$ induces an isometry in $\overline{\Delta H^2(U)}$.

Let

$$\overline{\Delta H^2(U)} = M_+(F) \oplus N$$

be the Wold decomposition of $\overline{\Delta H^2(U)}$ for the isometry. Then

$$F = \overline{\Delta H^2(U)} \ominus \mathscr{U}_1 \overline{\Delta H^2(U)}, \tag{1}$$

$M_+(F) = \bigoplus_{n\geq 0} \mathscr{U}_1^n F$, $N = \bigcap_{n\geq 0} \mathscr{U}_1^n \overline{\Delta H^2(U)}$, $\mathscr{U}_1|_N$ is unitary and $\mathscr{U}_1|_{M_+(F)}$ is a unilateral translation.

Let $\mathscr{P}$ be the orthoprojection from $\overline{\Delta H^2(U)}$ onto the subspace $M_+(F)$, then there exists a contractive analytic outer function $\varphi(z) : U \to F$ such that

$$\phi^F \mathscr{P} \Delta v = \varphi v$$

for all v belonging to $H^2(U)$, and $\varphi(z)$ is precisely the MFM of $I - \theta^*\theta$.

Similarly, for $k = 1, 2$, we have

$$\overline{\Delta_k H^2(U_k)} = M_+(F_k) \oplus N_k,$$

where $F_k = \overline{\Delta_k H^2(U_k)} \ominus \mathscr{U}_k \overline{\Delta_k H^2(U_k)}$, $N_k = \bigcap \mathscr{U}_k^n \overline{\Delta_k H^2(U_k)}$. $M_+(F_k) = \bigoplus_{n\geq 0} \mathscr{U}_k^n F_k$, and $\phi^{F_k} \Delta_k v = \varphi_k v$ for all $v \in H^2(U_k)$, where $\varphi_k(z) : U_k \to F_k$ is the MFM of $I - \theta_k^*\theta_k$.

Let

$$Z_+ : \Delta h \mapsto \Delta_1 h \oplus \Delta_2 \theta_1 h, h \in H^2(U) \tag{2}$$

be the operator from $\overline{\Delta H^2(U)}$ into $\overline{\Delta_1 H^2(U_1)} \oplus \overline{\Delta_2 H^2(U_2)}$, then Z_+ is unitary from $\overline{\Delta H^2(U)}$ onto $\overline{(\Delta_1 \oplus \Delta_2\theta_1) H^2(U)}$ and we have

$$\begin{aligned} Z_+\overline{\Delta H^2(U)} &= Z_+(M_+(F) \oplus N) = Z_+ M_+(F) \oplus Z_+ N \\ &= \bigoplus_{n\geq 0} Z_+ \mathscr{U}_1^n F \oplus Z_+ N. \end{aligned} \tag{3}$$

Because Δ_1, Δ_2 and θ_1 commute with the multiplication by e^{it} in the respective spaces, we have $Z_+\mathscr{U}_1 = \mathscr{U} Z_+$, where $\mathscr{U} = (\mathscr{U}_1, \mathscr{U}_2)$ is the multiplication by e^{it} on $L^2(U_1 \oplus U_2)$. From (3), it follows that

$$Z_+\overline{\Delta H^2(U)} = M_+(Z_+F) \oplus Z_+N$$

and

$$\|\mathscr{P}h\| = \|QZ_+h\|, \forall h \in \overline{\Delta H^2(U)}, \tag{4}$$

where Q is the orthoprojection from $Z_+\overline{\Delta H^2(U)}$ onto the subspace $M_+(Z_+F)$.

Note that

$$\begin{aligned} Z_+N = Z_+\left(\bigcap_{n\geq 0} \mathscr{U}_1^n \overline{\Delta H^2(U)}\right) &= \bigcap_{n\geq 0} \mathscr{U}^n Z_+ \overline{\Delta H^2(U)} \subset \\ &\subset \bigcap_{n\geq 0} \mathscr{U}^n (\overline{\Delta_1 H^2(U_1)} \oplus \overline{\Delta_2 H^2(U_2)}) \\ &= \bigcap_{n\geq 0} \mathscr{U}_1^n \overline{\Delta_1 H^2(U_1)} \oplus \bigcap_{n\geq 0} \mathscr{U}_2^n \overline{\Delta_2 H^2(U_2)} = N_1 \oplus N_2. \end{aligned}$$

Since Z_+ is isometric then Z_+N is a closed subspace of $N_1 \oplus N_2$.

Put $K = (N_1 \oplus N_2) \ominus Z_+N$, then

$$M_+(Z_+F) \subset M_+(F_1 \oplus F_2) \cap K. \tag{5}$$

Because the subspace $\overline{\Delta H^2(U)}$ is invariant for $\mathscr{U}_1$ we have $\mathscr{U}_1 N = N$, and it follows that $Z_+N = Z_+\mathscr{U}_1 N = \mathscr{U} Z_+ N$, then

$$\mathscr{U} K = K. \tag{6}$$

From (5), (6) and [8, Theorem 1.1], it follows that

$$\begin{aligned}
Z_+F &= M_+(Z_+F) \ominus \mathscr{U} M_+(Z_+F) \\
&= [M_+(Z_+F \cap (F_1 \oplus F_2)) \oplus M_+(Z_+F) \cap K] \ominus \\
&\quad [\mathscr{U} M_+(Z_+F \cap (F_1 \oplus F_2)) \oplus \mathscr{U} M_+(Z_+F) \cap K] \\
&= [M_+(Z_+F \cap (F_1 \oplus F_2)) \ominus \mathscr{U} M_+(Z_+F \cap (F_1 \oplus F_2))] \oplus \\
&\quad [(M_+(Z_+F) \cap K) \ominus (\mathscr{U} M_+(Z_+F) \cap K)] \\
&= (Z_+F \cap (F_1 \oplus F_2)) \oplus (Z_+F) \cap K).
\end{aligned}$$

Thus, $Z_+\overline{\Delta H^2(U)} = M_+(Z_+F \cap (F_1 \oplus F_2)) \oplus M_+(Z_+F \cap K) \oplus Z_+N$. Denote by Q_1 and Q_2 the orthoprojections from $Z_+\overline{\Delta H^2(U)}$ onto $M_+(Z_+F \cap (F_1 \oplus F_2))$ and $M_+(Z_+F \cap K)$, respectively. By $\delta = \Delta_1 \oplus \Delta_2\theta_1$, $\mathscr{X} = Q_2\delta$, $\mathscr{U}' = \mathscr{U}|_{M_+(Z_+F \cap K)}$, $\mathscr{U}_+ = \mathscr{U}|_{H^2(U)}$, we have that $\mathscr{X}$ is a contractive operator from $H^2(U)$ into $M_+(Z_+F \cap K)$. Moreover, note that the operator $\mathscr{U}$ commutes with δ, so we have

$$\mathscr{U}\mathscr{X} = \mathscr{U} Q_2 \delta = Q_2(\mathscr{U}|_{M_+(Z_+F \cap K)})\delta = Q_2 \mathscr{U} \delta = Q_2 \delta \mathscr{U} = \mathscr{X}\mathscr{U}.$$

This implies

$$\mathscr{U}'\mathscr{X} = \mathscr{X}\mathscr{U}_+.$$

From Proposition 1, there exists an analytic contractive operator function $\psi(z) : U \to Z_+F \cap K$ such that

$$\phi^{Z_+F \cap K} \mathscr{X} v = \psi v, \quad \forall v \in H^2(U). \tag{7}$$

Moreover, for all v belonging to $H^2(U)$, we have

$$\begin{aligned}
\phi^{Z_+F \cap (F_1 \oplus F_2)} Q_1 \delta v &= \phi^{Z_+F \cap (F_1 \oplus F_2)} Q_1(\Delta_1 \oplus \Delta_2\theta_1)v \\
&= \phi^{F_1 \oplus F_2} (\mathscr{P}_1 \oplus \mathscr{P}_2)(\Delta_1 \oplus \Delta_2\theta_1)v \\
&= \phi^{F_1} \mathscr{P}_1 \Delta_1 v \oplus f^{F_2} \mathscr{P}_2 \Delta_2 \theta_1 v \\
&= \varphi_1 v \oplus \varphi_2 \theta_1 v \\
&= \varphi_1 v \varphi_2 \theta_1 v.
\end{aligned}$$

Put $\mathscr{Y} = (Q_1 \oplus Q_2)\delta = Q\delta$, then $\mathscr{Y}$ is a contractive operator from $H^2(U)$ into $M_+(Z_+F \cap (F_1 \oplus F_2)) \oplus M_+(Z_+F \cap K) = M_+(Z_+F)$ and we have

$$\mathscr{Y}\overline{H^2(U)} = \overline{Q\delta H^2(U)} = \overline{Q\overline{\delta H^2(U)}} = \overline{QZ_+\overline{\Delta H^2(U)}} = M_+(Z_+F) \tag{8}$$

and

$$\phi^{Z_+F}\, Q\delta v = \phi^{Z_+F\cap(F_1\oplus F_2)}\, Q_1\delta v \oplus \phi^{Z_+F\cap K}\, Q_2\delta v$$
$$= \begin{pmatrix} \varphi_1 \\ \varphi_2\theta_1 \end{pmatrix} v \oplus \psi v\,. \tag{9}$$

From (8), (9) we conclude that $(\begin{pmatrix} \varphi_1 \\ \varphi_2\theta_1 \end{pmatrix} \oplus \psi)(z) : U \to Z_+F$ is outer.

Denote by $\mathscr{J} = \begin{pmatrix} \varphi_1 \\ \varphi_2\theta_1 \end{pmatrix}$, from (4) and (9), we have for all v belonging to $H^2(U)$

$$\|(\mathscr{J} \oplus \psi)v\| = \|\phi^{Z_+F}\, Q\delta v\| = \|Q\delta v\| = \|QZ_+\Delta v\|$$
$$= \|\mathscr{P}\Delta v\| = \|\phi^F \mathscr{P}\Delta v\| = \|\varphi v\|.$$

So we have

$$\frac{1}{2\pi}\int_0^{2\pi} \|(\mathscr{J} \oplus \psi)(t)v(t)\|^2 dt = \frac{1}{2\pi}\int_0^{2\pi} \|\varphi(t)v(t)\|^2 dt, \quad \forall v \in H^2(U). \tag{10}$$

Particularly, (10) holds for $v(\lambda) = p(\lambda)c$, where c is an element of U and $p(\lambda)$ is a polynomial of λ. Because every trigonometric polynomial $q(e^{it})$ derives from the form $e^{-\mathrm{int}}p(e^{it})$, where $p(\lambda)$ is an ordinary polynomial, so (10) holds for $v(\lambda) = q(\lambda)c$. Hence,

$$\frac{1}{2\pi}\int_0^{2\pi} |q(e^{it})|^2 \|(\mathscr{J} \oplus \psi)(t)c\|^2 dt = \frac{1}{2\pi}\int_0^{2\pi} |q(e^{it})|^2 \|\varphi(t)c\|^2 dt. \tag{11}$$

Similar to the proof in [8], the equality (11) holds when $q(e^{it})$ is substituted by a positive measurable function $\rho(t)$ bounded on $[0, 2\pi]$. By choosing $\rho(t)$ to be the characteristic function of the interval $(\tau, \tau + \varepsilon)$ and by giving $\varepsilon \to 0$, we get

$$\|(\mathscr{J} \oplus \psi)(t)c\|^2 = \|\varphi(t)c\|^2 \tag{12}$$

for all t outside a set E_c of null measure.

Because U is separable, there exists a set E of null measure such that (12) holds for all $t \notin E$ and all $c \in U$. Thus, we have

$$(\mathscr{J} \oplus \psi)(t)^*(\mathscr{J} \oplus \psi)(t) = \varphi(t)^*\varphi(t) \text{ a.e.}$$

Since $(\mathscr{J} \oplus \psi)$ and φ are both outer, there exists a unitary operator $E : F \to Z_+F$ such that $(\mathscr{J} \oplus \psi)$ is the MFM of $I - \theta^*\theta$. By some computations, we can prove that $E = Z_+|_F$.

We can now state our result concerning the MFM for the product of operator functions.

Theorem 1. *Let φ_k be the MFM of $I - \theta_k^*\theta_k$ $(k = 1, 2)$, then* $\begin{pmatrix} \varphi_1 \\ \varphi_2\theta_1 \end{pmatrix} \oplus \psi$ *is the MFM of $I - \theta^*\theta$.*

Moreover, we have $\begin{pmatrix} \varphi_1 \\ \varphi_2\theta_1 \end{pmatrix} \oplus \psi = E\varphi$, *where $E = Z_+|_F$ is a unitary operator from F onto Z_+F and F, Z_+, ψ are defined in* (1), (2) *and* (7).

In [6], a notion of (+) regular factorization was introduced. The factorization $\theta(z)=\theta_2(z)\theta_1(z)$ is said to be (+) regular if

$$\overline{\{\Delta_1 h\oplus\Delta_2\theta_1 h : h\in H^2(U)\}}=\overline{\Delta_1 H^2(U_1)}\oplus\overline{\Delta_1 H^2(U_2)}$$

or equivalently, the operator

$$Z_+ : \Delta h\mapsto\Delta_1 h\oplus\Delta_2\theta_1 h, \forall h\in H^2(U)$$

can be continuously extended to a unitary operator from $\overline{\Delta H^2(U)}$ onto $\overline{\Delta_1 H^2(U_1)}\oplus\overline{\Delta_2 H^2(U_2)}$.

From Theorem 1, we have the following:

Corollary 1. *If the factorization $\theta=\theta_2\theta_1$ is (+) regular, then $\mathcal{J}=\begin{pmatrix}\varphi_1\\ \varphi_2\theta_1\end{pmatrix}$ is the MFM of $I-\theta^*\theta$.*

Proof. From the proof above, we can see that if the factorization $\theta(z)=\theta_2(z)\theta_1(z)$ is (+) regular, then the space K is reduced to $\{0\}$ and this implies $\psi=0$.

From the definition for the MFM of $I-\theta^*\theta$, we introduced an analogue notion for the *-MFM of $I-\theta^*\theta$. The *-outer function $\alpha(z)\in B(E',V)$ is called the *-MFM of $I-\theta\theta^*$ if

$$\alpha\alpha^*\le I-\theta\theta^* \text{ a.e. on } \partial D$$

and if $\beta(z)$ is an analytic contractive operator function such that

$$\beta\beta^*\le I-\theta\theta^* \text{ a.e. then } \beta\beta^*\le\alpha\alpha^* \text{a.e.}$$

We recall that the function $\alpha(z)$ is *-outer if the function $\tilde\alpha(z)\in B(V,E')$ is outer. One easily sees that $\alpha(z)$ is the *-MFM of $I-\theta\theta^*$ if and only if $\tilde\alpha(z)$ is the MFM of $I-\tilde\theta^*\tilde\theta$, where $\tilde\alpha(z)=\alpha(\bar z)^*$, $\tilde\theta(z)=\theta(\bar z)^*$. ■

Similarly to Theorem 1, we have

Theorem 2. *Let α_k be the *-MFM of $I-\theta_k\theta_k^*$ $(k=1,2)$, then $\begin{pmatrix}\theta_2\alpha_1\\ \alpha_2\end{pmatrix}\oplus\beta$ is the *-MFM of $I-\theta\theta^*$, where*
$\beta(z): Z_-F'\cap K'\to V$, $F'=\overline{\Delta_* L^2_-(V)}\ominus\mathcal{U}'_2\overline{\Delta_* L^2_-(V)}$,

$$K'=\left[\bigcap_{n\ge0}\mathcal{U}_1'^{\,n}\overline{\Delta_{1*}L^2_-(V_1)}\oplus\overline{\Delta_{2*}L^2_-(V_2)}\right]\ominus Z_-\left(\bigcap_{n\ge0}\mathcal{U}_2'^{\,n}\overline{\Delta_* L^2_-(V)}\right),$$

$Z_- : \Delta_* h\mapsto\Delta_{1*}\theta_2^* h\oplus\Delta_{2*}h$, $h\in L^2_-(V)$, $\mathcal{U}'_k$ is the multiplication by e^{-it} on $L^2(V_k)$ $(k=1,2)$, $\Delta_*=(I-\theta\theta^*)^{1/2}$, $\Delta_{k*}=(I-\theta_k\theta_k^*)^{1/2}$, $k=1,2$.

Moreover, we have $\begin{pmatrix}\theta_2\alpha_1\\ \alpha_2\end{pmatrix}\oplus\beta=E_-\alpha$, where $E_-=Z_-|_{F'}$ is a unitary operator from F' onto Z_-F'.

In [6], it was also introduced a dual notion of (−) regular factorization. The factorization $\theta(z)=\theta_2(z)\theta_1(z)$ is said to be (−) regular if

$$\overline{\{\Delta_{1*}\theta_2^* h\oplus\Delta_{2*}h / h\in L^2_-(V)\}}=\overline{\Delta_{1*}L^2_-(V_1)}\oplus\overline{\Delta_{2*}L^2_-(V_2)}$$

or equivalently, the operator

$$Z_- : \Delta_* h \mapsto \Delta_{1*}\theta_2^* h \oplus \Delta_{2*}h, h \in L^2_-(V)$$

can be continuously extended to a unitary operator from $\overline{\Delta_* L^2_-(V_1)}$ onto $\overline{\Delta_{1*}L^2_-(V_1)} \oplus \overline{\Delta_{2*}L^2_-(V_2)}$.

From this notion, we have the duality of Corollary 1.

Corollary 2. *If the factorization* $\theta = \theta_2\theta_1$ *is* $(-)$ *regular, then* $\mathscr{F} = \begin{pmatrix} \theta_2\alpha_1 \\ \alpha_2 \end{pmatrix}$ *is the* $*$*-MFM of* $I - \theta\theta^*$.

3. The Necessary and Sufficient Condition for $\mathscr{J} = \begin{pmatrix} \varphi_1 \\ \varphi_2\theta_1 \end{pmatrix}$ to be the MFM of $I - \theta^*\theta$

Given a contractive operator function $\theta(z)$ analytic on the unit disk D, Branges and Rovnyak introduced the Hibert space B^θ of vector-valued analytic functions with reproducing kernel [3]

$$\mathscr{K}^\theta(w,z) = \begin{pmatrix} \dfrac{I - \theta(z)\theta(w)^*}{I - z\overline{w}} & \dfrac{\theta(z) - \theta(\overline{w})}{z - \overline{w}} \\ \dfrac{\tilde{\theta}(z) - \tilde{\theta}(\overline{w})}{z - \overline{w}} & \dfrac{I - \tilde{\theta}(z)\tilde{\theta}(w)^*}{I - z\overline{w}} \end{pmatrix},$$

where $\tilde{\theta}(z) = \theta(\bar{z})^*$.

Let us consider the following two subspaces of B^θ

$$B^\theta_+ = \{(0,g) \,|\, (0,g) \in B^\theta\},$$
$$B^\theta_- = \{f,0) \,|\, (f,0) \in B^\theta\}.$$

In the linear dynamic system theory, the subspace B^θ_+ characterizes the non-observable subspace of the unitary system having $\theta(z)$ as the transfer function, while B^θ_- is the non-controllable subspace of it.

In [4], Ball and Kriete proved the following result:

Theorem 3. *The subspace* B^θ_+ *is precisely the following subspace*

$$\{(0, \tilde{\varphi}f)/f \in H^2(U)\},$$

where $\varphi(z) \in B(U,E)$ *is the MFM of* $I - \theta^*\theta$. *Moreover,*

$$\|(0,\tilde{\varphi}f)\|_{B^\theta} = \|f\|_{H^2(U)}.$$

Similar to Theorem 3, we can state for the subspace B^θ_- the following:

Theorem 4. *The subspace* B^θ_- *can be represented as follows:*

$$B^\theta_- = \{(\alpha h, 0)/h \in H^2(E')\},$$

where $\alpha(z) \in B(E', V)$ *is the* **-MFM of* $I - \theta\theta^*$. *Moreover,*

$$\|(\alpha h, 0)\|_{B^\theta} = \|h\|_{H^2(E')}.$$

Before giving the proof of this theorem, let us consider the functional models of Nagy and Foias for a given contractive analytic function $\theta(z) \in B(U, V)$ of the forms

$$N^\theta = [L^2_+(V) \oplus \overline{\Delta L^2(U)}] \ominus \{(\theta\omega, \Delta\omega) \,|\, \omega \in L^2_+(U)\},$$
$$N^\theta_* = [L^2_-(U) \oplus \overline{\Delta_* L^2(V)}] \ominus \{(\theta^*\omega, \Delta_*\omega) \,|\, \omega \in L^2_-(V)\},$$

where $\Delta_* = (I - \theta\theta^*)^{1/2}$, $L^2_+(U) \equiv H^2(U)$, $L^2_-(U) = L^2(U) \ominus L^2_+(U)$.

We have the operator

$$\mathscr{W}^\theta : (f, g) \mapsto (\theta^* f + \Delta g,\, \Delta_* f - \theta g)$$

which acts unitarily from N^θ onto N^θ_*.

Let j_U be the operator on $L^2(U)$ defined by $(j_U f)(e^{it}) = e^{-it} f(e^{-it})$. One can easily see that j_U is a unitary involution on $L^2(U)$ which maps $L^2_+(U)$ onto $L^2_-(U)$ and $L^2_-(U)$ onto $L^2_+(U)$. The basic connection between N^θ and B^θ is that they are unitarily equivalent under the map Γ^θ defined by $\Gamma^\theta(f, g) = (f, J_U(\theta^* f + \Delta g))$ for $(f, g) \in N^\theta$.

Proof of Theorem 4. Let Γ be the unitary operator defined by

$$\Gamma = \Gamma^{\theta_1} \oplus \Gamma^{\theta_2} : N^{\theta_1} \oplus N^{\theta_2} \to B^{\theta_1} \oplus B^{\theta_2},$$

then we have

$$B^\theta_- = \Gamma^\theta N^\theta_-,$$

where

$$N^\theta_- = \{(f, g) \in N^\theta \,|\, \theta^* f + \Delta g = 0\}.$$

For each $(f, 0) \in B^\theta$, there exists an element $(f, g) \in N^\theta$ such that $\Gamma^\theta(f, g) = (f, 0)$.

Let $m = \Delta_* f - \theta g$, we have $f = \Delta_* m$ and

$$m \in \overline{\Delta_* L^2(V)} \ominus \overline{\Delta_*(L^2_-(V)}. \tag{13}$$

Note that

$$j_V(\overline{\Delta_* L^2(V)} \ominus \overline{\Delta_*(L^2_-(V)}) = \overline{\Delta_*(e^{-it}) L^2(V)} \ominus \overline{\Delta_*(e^{-it}) L^2_+(V)}, \tag{14}$$

and according to [4, Theorem 5], we have

$$\overline{\Delta_*(e^{-it}) L^2(V)} \ominus \overline{\Delta_*(e^{-it}) L^2_+(V)} = \mathscr{P}(e^{it})^* L^2_-(E'), \tag{15}$$

where $\mathscr{P}(e^{it}) : V \to E'$ is the solution of the equation

$$\tilde{\alpha}(e^{it}) = \mathscr{P}(e^{it}) \Delta_*(e^{-it}) \tag{16}$$

with $\mathscr{P}(e^{it}) = 0$ on $(\Delta_*(e^{-it})(V))^{\perp}$ a.e. and $\mathscr{P}(e^{it})^*$ is isometric a.e., $\tilde{\alpha}(z)$ is the MFM of $\Delta_{\tilde{\theta}}^2(e^{it}) = \Delta_*^2(e^{-it})$.

From (14) and (15), we have

$$\overline{\Delta_* L^2(V)} \ominus \overline{\Delta_*(L_-^2(V)} = j_V \mathscr{P}(e^{it})^* L_-^2(E') = \tilde{P}(e^{it}) L_+^2(E'). \tag{17}$$

From (13) and (17), m has the form

$$m = \tilde{\mathscr{P}} h, h \in L_+^2(E')$$

and it follows that

$$f = \Delta_* m = \Delta_* \tilde{\mathscr{P}} h = \alpha h.$$

Moreover, we have

$$\begin{aligned} \|(\alpha h, 0)\|_{B^\theta} &= \|(f, 0)\|_{B^\theta} = \|\Gamma^\theta(f, g)\|_{B^\theta} = \|(f, g)\|_{N^\theta} = \|\mathscr{W}^\theta(f, g)\|_{N_*^\theta} \\ &= \|(0, m)\|_{N_*^\theta} = \|(0, \tilde{\mathscr{P}} h)\|_{N_*^\theta} = \|\tilde{\mathscr{P}} h\|_{L_+^2(V)} = \|h\|_{L_+^2(E')}. \end{aligned}$$

This completes the proof. ■

Let Σ be the following partial isometry [3]

$$\Sigma : B^{\theta_1} \oplus B^{\theta_2} \to B^\theta, \theta = \theta_2 \theta_1;$$

$$(f_1, g_1) \oplus (f_2, g_2) \mapsto (f_2 + \theta_2 f_1, g_1 + \tilde{\theta}_1 g_2)$$

We denote by

$$\Sigma_+ = \Sigma|_{B_+^{\theta_1} \oplus B_+^{\theta_2}} : B_+^{\theta_1} \oplus B_+^{\theta_2} \to B_+^\theta$$

$$(0, g_1) \oplus (0, g_2) \mapsto (0, g_1 + \tilde{\theta}_1 g_2)$$

and by

$$\Sigma_- = \Sigma|_{B_-^{\theta_1} \oplus B_-^{\theta_2}} : B_+^{\theta_1} \oplus B_-^{\theta_2} \to B_-^\theta$$

$$(f_1, 0) \oplus (f_2, 0) \mapsto (f_2 + \theta_2 f_1, 0).$$

Note that if $\varphi_k(z)$ is the MFM of $I - \theta_k^* \theta_k$ $(k = 1, 2)$, then the function $\mathscr{J}(z) = \begin{pmatrix} \varphi_1(z) \\ \varphi_2(z)\theta_1(z) \end{pmatrix}$ which belongs to the class $B(U, E_1 \oplus E_2)$ is minorant of $I - \theta^* \theta$ $(\theta = \theta_2 \theta_1)$. When does this function $\mathscr{J}(z)$ become the MFM? The answer is given by

Theorem 5. *The function $\mathscr{J}(z)$ is the MFM of $I - \theta^* \theta$ if and only if the operator Σ is unitary from $B_+^{\theta_1} \oplus B_+^{\theta_2}$ onto B_+^θ.*

Proof. Let $\mathscr{J}(z) = \begin{pmatrix} \varphi_1(z) \\ \varphi_2(z)\theta_1(z) \end{pmatrix}$ be the MFM of $I - \theta^* \theta$.

According to Theorem 3, the operator Σ_+ has the form

$$\Sigma_+ : (0, \tilde{\varphi}_1 h_1) \oplus (0, \tilde{\varphi}_2 h_2) \mapsto (0, \tilde{\varphi}_1 h_1 + \tilde{\theta}_1 \tilde{\varphi}_2 h_2) = (0, \tilde{\mathcal{J}} h),$$

where $h = h_1 \oplus h_2$, $h_k \in H^2(E_k)$, $k = 1, 2$.

We have

$$\begin{aligned} &\|(0, \tilde{\varphi}_1 h_1) \oplus (0, \tilde{\varphi}_2 h_2)\|^2_{B^{\theta_1} \oplus B^{\theta_2}} = \|(0, \tilde{\varphi}_1 h_1)\|^2_{B^{\theta_1}} + \|(0, \tilde{\varphi}_2 h_2)\|^2_{B^{\theta_2}} \\ &= \|h_1\|^2_{H^2(E_1)} + \|h_2\|^2_{H^2(E_2)} = \|h\|^2_{H^2(E_1 \oplus E_2)} = \|(0, \tilde{\mathcal{J}} h)\|_{B^\theta}. \end{aligned}$$

Thus, Σ is an isometry. Moreover, from the assumption that $\mathcal{J}(z)$ is the MFM of $I - \theta^*\theta$, the subspace $\{(0, \tilde{\mathcal{J}} h) \mid h \in H^2(E_1 \oplus E_2)\}$ is precisely the space B^θ_+, then Σ_+ is unitary.

Conversely, let Σ_+ be unitary, then we have

$$B^\theta_+ = \{(0, \tilde{\mathcal{J}} h) \mid h \in H^2(E_1 \oplus E_2)\} = \{(0, \tilde{\varphi} f) \mid f \in H^2(E)\},$$

where $\varphi(z) \in B(U, E)$ is the MFM of $I - \theta^*\theta$.

So with each element $h \in H^2(E_1 \oplus E_2)$, there exists an element $f \in H^2(E)$ such that

$$\tilde{\mathcal{J}} h = \tilde{\varphi} f. \tag{18}$$

Since $\varphi(z)$ is outer, $\varphi(z)$ has dense range for all z in D and hence, $\ker \tilde{\varphi}(z) = \{0\}$, $z \in D$. Thus, if f is in $H^2(E)$, the element $\tilde{\varphi} f$ of $H^2(U)$ determines f. So we can define an operator χ from $H^2(E_1 \oplus E_2)$ into $H^2(E)$ by

$$\chi h = f \tag{19}$$

with h, f in the expression (18).

The operator χ is evidently linear and surjective. Moreover, from Theorem 3, we have

$$\begin{aligned} &\|h\|^2_{H^2(E_1 \oplus E_2)} = \|h_1\|^2_{H^2(E_1)} + \|h_2\|^2_{H^2(E_2)} = \|(0, \tilde{\varphi}_1 h_1)\|^2_{B^{\theta_1}} + \|(0, \tilde{\varphi}_2 h_2)\|^2_{B^{\theta_2}} \\ &= \|(0, \tilde{\varphi}_1 h_1) \oplus (0, \tilde{\varphi}_2 h_2)\|^2_{B^{\theta_1} \oplus B^{\theta_2}} = \|(0, \tilde{\mathcal{J}} h)\|^2_{B^\theta} = \|(0, \tilde{\varphi} f)\|^2_{B^\theta} = \|f\|^2_{H^2(E)}. \end{aligned}$$

Thus, χ is unitary.

From (18) and (19), we have

$$\tilde{\mathcal{G}} = \tilde{\mathcal{P}},$$

where $\tilde{\mathcal{G}}$ and $\tilde{\mathcal{P}}$ denote the operators on H^2 induced by the multiplication by $\tilde{\mathcal{J}}(z)$ and $\tilde{\varphi}(z)$, respectively. Since $\tilde{\mathcal{G}}$ and $\tilde{\mathcal{P}}$ commute with e^{it}, so does the operator χ. According to Proposition 1, the operator χ is unitary constant, then $\mathcal{J}(z)$ is the MFM of $I - \theta^*\theta$ and the proof is complete. ■

Similarly, we have the following result for the *-MFM.

Theorem 6. *Let $\alpha_k(z) \in B(E'_k, V_k)$, $k = 1, 2$, be the *-MFM of $I - \theta_k\theta_k^*$. The function* $F(z) = \begin{pmatrix} \alpha_2(z) \\ \theta_2(z)\alpha_1(z) \end{pmatrix} \in B(E'_1 \oplus E'_2, V)$ *is the *-MFM of $I - \theta\theta^*$ if and only if the operator Σ_- is unitary from $B_-^{\theta_1} \oplus B_-^{\theta_2}$ onto B_-^{θ}.*

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