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Two Species Competition in Almost Periodic Environment*

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Abstract. This paper considers the non-autonomous competitive Lotka–Volterra system of two equations. Conditions for the existence and uniqueness of a globally attractive, almost periodic solution defined on $(-\infty, +\infty)$ whose components are bounded above and below by positive constants are given.

1. Introduction

Consider the non-autonomous system of differential equations

$$\begin{aligned} u_1' &= u_1(A_1(t) - a_{11}(t)u_1 - a_{12}(t)u_2), \\ u_2' &= u_2(A_2(t) - a_{21}(t)u_1 - a_{22}(t)u_2), \end{aligned} \tag{1.1}$$

where $A_i(t)$, $a_{ij}(t)$ $(i, j = 1, 2)$ are assumed to be continuous and bounded above and below by positive constants. Given a function $g(t)$ on $R := (-\infty, +\infty)$, we let g_L, g_M denote $\inf_{t\in R} g(t)$ and $\sup_{t\in R} g(t)$, respectively.

In [1], it was shown that if the two inequalities

$$\begin{aligned} A_{1L}a_{22L} &> a_{12M}A_{2M}, \\ A_{2L}a_{11L} &> a_{21M}A_{1M} \end{aligned} \tag{1.2}$$

hold, and if $A_i(t)$, $a_{ij}(t)$ $(i, j = 1, 2)$ are almost periodic, then (1.1) has a unique almost periodic solution whose components are bounded below and above by positive constants, which is globally asymptotically stable in $\{u = (u_1, u_2)\colon u_i > 0;\ i = 1, 2\}$. This is a generalization of a result by Gopalsamy [4] for the case of two dimensions.

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For each $i = 1, 2$ let us denote by U_i^0 the unique solution of the logistic equation

$$U_i' = U_i[A_i(t) - a_{ii}(t)U_i], \tag{1.3}$$

which is bounded above and below by positive constants. The existence and uniqueness of this solution were given by Ahmad [2]. Our main result is the following:

Suppose

$$\begin{aligned} A_1(t) - a_{12}(t)U_2^0(t) &\geq \varepsilon_1; \quad t \in R, \\ A_2(t) - a_{21}(t)U_1^0(t) &\geq \varepsilon_1; \quad t \in R \end{aligned} \tag{1.4}$$

hold for some $\varepsilon_1 > 0$. If there are positive constants ε_2, α_1, α_2 such that

$$\alpha_i a_{ii}(t) - \alpha_{3-i} a_{3-ii}(t) \geq \varepsilon_2; \quad t \in R; \quad i = 1, 2, \tag{1.5}$$

then the system (1.1) has a unique solution u^0 defined on $(-\infty, +\infty)$ whose components are bounded above and below by positive constants, and $u(t) - u^0(t) \to 0$ as $t \to +\infty$ for any positive solution $u(t)$ of (1.1).

If, in addition, A_i, a_{ij} $(i, j = 1, 2)$ are almost periodic, then the solution u^0 is also almost periodic.

It is not hard to see that $A_{iL}/a_{iiM} \leq U_i^0(t) \leq A_{iM}/a_{iiL}$ $(i = 1, 2;\ t \in R)$. Therefore, (1.2) implies (1.4). Furthermore, from (1.2) it follows that $a_{22L}a_{11L} > a_{21M}a_{12M}$. We can choose $\alpha_1, \alpha_2 > 0$ such that $a_{21M}/a_{11L} < \alpha_1/\alpha_2 < a_{22L}/a_{12M}$, then (1.5) holds for some $\varepsilon_2 > 0$. With $A_1 = 1$, $a_{11} = 1$, $a_{12} = \dfrac{1}{2}$, $A_2 = a_{22} = \dfrac{3}{2} + \dfrac{1}{2}\sin t$ and $a_{21} = \dfrac{1}{4}A_2$, we can check that the system (1.1) satisfies (1.4) and (1.5) (for $\alpha_1 = \alpha_2 = 1$) but not (1.2).

Thus, our result is stronger than that in [1]. The ecological significance of such a system was discussed in [4].

2. Existence

In this section, we shall prove that the system (1.1) has at least one solution $u^0(t)$ on $(-\infty, +\infty)$ as mentioned above. To do this we need the following lemma.

Lemma 1. *Let $u = (u_1, u_2)$ be a solution of (1.1) with $u_i > 0$; $i = 1, 2$. For each $i = 1, 2$, let U_i be a solution of (1.3) such that $U_i(t_0) \geq u_i(t_0)$ (or $U_i(t_0) \leq u_i(t_0)$) for some $t_0 \in R$, then $U_i(t) > u_i(t)$ for $t > t_0$ ($U_i(t) < u_i(t)$ for $t < t_0$, respectively).*

Proof. Let us fix $i = 1, 2$. If $U_i(t_0) = u_i(t_0)$, then $U_i'(t_0) > u_i'(t_0)$. Therefore, if $U_i(t_0) \geq u_i(t_0)$, then there exists $t_1 > t_0$ such that $U_i > u_i$ on (t_0, t_1). We claim that $\{t > t_1 : U_i(t) = u_i(t)\} = \emptyset$ which will prove that $U_i(t) > u_i(t)$ for $t > t_0$. If it is false, then it is not hard to see that $U_i(t_2) = u_i(t_2)$, where $t_2 = \inf\{t > t_1 : U_i(t) = u_i(t)\}$. Let $g(t) = U_i(t) - u_i(t)$, then $g'(t_2) = U_i'(t_2) - u_i'(t_2) > 0$. Consequently, $g'(t) > 0$ for $t \in [t_2 - \eta, t_2 + \eta]$ for some small $\eta > 0$ such that $t_2 - \eta > t_0$. By the definition of t_2, we have $g(t_2 - \eta) > 0$. Consequently, $g(t_2) = U_i(t_2) - u_i(t_2) > 0$, which is a contradiction. This proves the claim. By a similar argument we can prove that if $U_i(t_0) \leq u_i(t_0)$, then $U_i(t) < u_i(t)$ for $t < t_0$. The lemma is proved. ∎

We now recall the topological principle of Wazewski (see, for example, [5]). Let $f(t, y)$ be a continuous function defined on an open (t, y)-set $\Omega \subset \mathbf{R} \times \mathbf{R}^n$. Let Ω^0 be an open subset of Ω, $\partial\Omega^0$ the boundary and $\bar{\Omega}^0$ the closure of Ω^0. Recall that a point $(t_0, y_0) \in \Omega \cap \partial\Omega^0$ is called an egress point of Ω^0 with respect to the system

$$y' = f(t, y), \tag{2.1}$$

if for every solution $y = y(t)$ of (2.1) satisfying the initial condition

$$y(t_0) = y_0, \tag{2.2}$$

there is an $\varepsilon > 0$ such that $(t, y(t)) \in \Omega^0$ for $t_0 - \varepsilon \leq t < t_0$. An egress point (t_0, y_0) of Ω^0 is called a strict egress point if $(t, y(t)) \notin \bar{\Omega}^0$ for $t_0 < t \leq t_0 + \varepsilon$ for a small $\varepsilon > 0$. The set of egress points of Ω^0 will be denoted by Ω^0_e and the set of strict egress points by Ω^0_{se}.

If X is a topological space, V a subset of X, a continuous mapping $\pi: X \to V$ defined on all of X is called a retraction of X onto V if the restriction π/V of π to V is the identity. When there exists a retraction of X onto V, V is called a retract of X.

Remark 1. For $a_i < b_i$ $(i = 1, 2)$, let X be a 2-dimensional rectangle $\{(x_1, x_2): a_i \leq x_i \leq b_i; i = 1, 2\}$ in the Euclidean space $\mathbf{R}^2$, and V its boundary. Then V is not a retract of X. For if there exists a retraction $\pi: X \to V$, then there exists a map of X into itself,

$$(x_1, x_2) \mapsto \left(\frac{a_1 + b_1}{2}, \frac{a_2 + b_2}{2}\right) - \pi(x_1, x_2),$$

without fixed points, which is impossible by the fixed point theorem of Schauder.

Theorem 1. (Topological Principle, see [5]) *Let $f(t, y)$ be continuous on an open (t, y)-set Ω with the property that initial values determine unique solutions of (2.1). Let Ω^0 be an open subset of Ω satisfying $\Omega^0_e = \Omega^0_{se}$. Let S be a non-empty subset of $\Omega^0 \cup \Omega^0_e$ such that $S \cap \Omega^0_e$ is not a retract of S but is a retract of Ω^0_e. Then there exists at least one point (t_0, y_0) in $S \cap \Omega^0$ such that the solution $(t, y(t))$ of (2.1), (2.2) is contained in Ω^0 on its right maximal interval of existence.*

Theorem 2. *Suppose A_i, a_{ij} $(i, j = 1, 2)$ are continuous and bounded above and below by positive constants. If conditions (1.4) hold, then (1.1) has at least one solution $u^0(t) = (u^0_1(t), u^0_2(t))$ defined on $(-\infty, +\infty)$ satisfying*

$$\eta_i \leq {u_i}^0(t) \leq {U_i}^0(t); \quad i = 1, 2,$$

where η_i is a positive number such that

$$\eta_i < \min\left\{\varepsilon_1/a_{iiM}, \inf_{t\in R} U_i^0(t)\right\}.$$

Proof. First, it is easy to see that the Cauchy problem for (1.1) with the initial condition $u(t_0) = u_0 \in \{(u_1, u_2) \in R^2 : u_1 > 0, u_2 > 0\}$, $(t_0 \in R)$ has a unique solution defined on $(-\infty, +\infty)$ whose components are strictly positive for all $t \in (-\infty, +\infty)$.

Consider the system

$$\begin{aligned} v_1' &= v_1(-A_1(-t) + a_{11}(-t)v_1 + a_{12}(-t)v_2),\\ v_2' &= v_2(-A_2(-t) + a_{21}(-t)v_1 + a_{22}(-t)v_2). \end{aligned} \tag{2.3}$$

Set $\Omega^0 = \{(t, v_1, v_2) : -\infty < t < +\infty; \eta_i < v_i < U_i^0(-t);\ i = 1, 2\}$, and $\Omega = \{(t, v_1, v_2) \in R^3\}$.

Since (2.3) is the inverse time system of (1.1), Lemma 1 implies that any point (t, v_1, v_2) in

$$\begin{aligned} A = &\{(t, v_1, v_2) \in \overline{\Omega^0} : v_1 = U_1^0(-t); t \in R\}\\ &\cup \{(t, v_1, v_2) \in \overline{\Omega^0} : v_2 = U_2^0(-t); t \in R\} \end{aligned}$$

is a strict egress point of Ω^0. It is not hard to see from the definition of η_i $(i = 1, 2)$ that any point (t, v_1, v_2) in

$$B = \{(t, \eta_1, v_2) \in \overline{\Omega^0}\} \cup \{(t, v_1, \eta_2) \in \overline{\Omega^0}\}$$

is a strict egress point of Ω^0.

Therefore, $\Omega_e^0 = \Omega_{se}^0 = A \cup B$. Let us take

$$S = \{(0, v_1, v_2) : \eta_i \le v_i \le U_i{}^0(0);\quad i = 1, 2\}.$$

Then S is a rectangle. By Remark 1, $S \cap \Omega_e^0$ is not a retract of S. Define

$$\pi : \Omega_e^0 \to S \cap \Omega_e^0,$$

$$(t, v_1, v_2) \mapsto \left(0, \eta_1 + \frac{v_1 - \eta_1}{U_1^0(t) - \eta_1}(U_1^0(0) - \eta_1), \eta_2 + \frac{v_2 - \eta_2}{U_2^0(t) - \eta_2}(U_2^0(0) - \eta_2)\right).$$

Clearly the map π is continuous relative to subtopologies on Ω_e^0 and $S \cap \Omega_e^0$ of Euclidean space $\mathbf{R}^3$, and its restriction to $S \cap \Omega_e^0$ is the identity. Therefore, $S \cap \Omega_e^0$ is a retract of Ω_e^0. By Theorem 1, (2.3) has at least a solution $v^0(t)$ satisfying $\eta_i < v_i^0(t) < U_i^0(-t)$ for $t \ge 0$. In fact $u_i^*(t) = v^0(-t)$ is a solution of (1.1) for $t \le 0$. By Lemma 1 and the definition of η_i $(i = 1, 2)$, it follows that the solution $\bar{u}(t)$ of (2.1) with $\bar{u}(0) = v^0(0)$ satisfies

$$\eta_i \le \bar{u}_i(t) \le U_i{}^0(t);\quad \text{for } t \ge 0,\ i = 1, 2.$$

Let

$$u^0(t) = \begin{cases} u^*(t), & t \le 0, \\ \bar{u}(t), & t \ge 0, \end{cases}$$

then $u^0(t)$ is a solution of (1.1) satisfying

$$\eta_i \le u_i^0(t) \le U_i^0(t); \quad t \in \mathbf{R}; \quad i = 1, 2.$$

The theorem is proved. ■

3. Uniqueness and Asymptoticity

In this section we show that the solution u^0 in Theorem 2 is unique and asymptotically stable if Conditions (1.4) and (1.5) hold. The following lemma is sharper than Lemmas 3.2, 3.2′, 3.3 and 3.3′ in [6].

Lemma 2. *Suppose u^1, u^2 are two different solutions of (1.1) defined on $(-\infty, +\infty)$ and such that $u_i^j(t) > 0$ for any $t \in (-\infty, +\infty)$; $i, j = 1, 2$. Then only one of the following alternatives is met:*

(i) *$u_i^1(t) \ne u_i^2(t)$ for any $t \in (-\infty, +\infty)$; $i = 1, 2$.*

(ii) *There exist $t_0 \in \mathbf{R}$ and $i, j \in \{1, 2\}$ such that $u_i^j(t_0) = u_i^{3-j}(t_0)$, $u_i^j(t) > u_i^{3-j}(t)$ for $t < t_0$, $u_i^j(t) < u_i^{3-j}(t)$ for $t > t_0$, and $u_{3-i}^j(t) > u_{3-i}^{3-j}(t)$ for $t \in \mathbf{R}$.*

Proof. Suppose (i) does not happen. Then there are $i, j \in \{1, 2\}$ and $t_0 \in \mathbf{R}$ such that $u_i^j(t_0) = u_i^{3-j}(t_0)$, $u_{3-i}^j(t_0) > u_{3-i}^{3-j}(t_0)$. Without loss of generality, we can assume $j = 1$, $i = 1$, i.e., we have $u_1^1(t_0) = u_1^2(t_0)$, $u_2^1(t_0) > u_2^2(t_0)$. Therefore, we have to prove that $u_1^1(t) < u_1^2(t)$ for $t > t_0$, $u_1^1(t) > u_1^2(t)$ for $t < t_0$ and $u_2^1(t) > u_2^2(t)$ for $t \in R$.

Let $v_1^j = \dfrac{1}{u_1^j}$ $(j = 1, 2)$. Then

$$\begin{aligned} v_1^{j\prime} &= -A_1(t)v_1^j + a_{11}(t) + a_{12}(t)u_2^j v_1^j, \\ u_2^{j\prime} &= u_2^j\left(A_2(t) - a_{21}(t)\frac{1}{v_1^j} - a_{22}(t)u_2^j\right); \quad j = 1, 2. \end{aligned} \tag{3.1}$$

Since $v_1^{1\prime}(t_0) > v_1^{2\prime}(t_0)$, there exists $t_1 > t_0$ such that $v_1^1(t) > v_1^2(t)$ and $u_2^1(t) > u_2^2(t)$ for $t \in (t_0, t_1)$. Define

$$t_2 = \inf\Big(\{t > t_1 : v_1^1(t) = v_1^2(t)\} \cup \{+\infty\}\Big)$$

and

$$t_3 = \inf\Big(\{t > t_1 : u_2^1(t) = u_2^2(t)\} \cup \{+\infty\}\Big).$$

We claim that $t_2 = t_3 = +\infty$. If it is false, without loss of generality, we can assume $t_2 \le t_3$. Therefore, $t_2 < +\infty$. It is not hard to see that $v_1^1(t_2) = v_1^2(t_2)$. By the uniqueness, it follows that $u_2^1(t_2) > u_2^2(t_2)$. Therefore, $v_1^{1\prime}(t_2) > v_1^{2\prime}(t_2)$. By the same argument as in the proof of Lemma 1, we get $v_1^1(t_2) > v_1^2(t_2)$, a contradiction. This proves the claim. Therefore, $u_1^1(t) < u_1^2(t)$ and $u_2^1(t) > u_2^2(t)$ for $t > t_0$. We now consider the case of $t < t_0$. Let $v_i^j(t) = u_i^j(-t)$; $i, j = 1, 2$. We get

$$v_i^{j'}(t) = v_i^j(t)\big(-A_i(-t) + a_{ii}(-t)v_i^j(t) + a_{i3-i}(-t)v_{3-i}^j(t)\big); \quad i, j = 1, 2. \quad (3.2)$$

By the similar argument and using (3.2), we get

$$v_1^1(t) > v_1^2(t), \quad v_2^1(t) > v_2^2(t) \quad \text{for } t > -t_0.$$

This implies $u_1^1(t) > u_1^2(t)$, $u_2^1(t) > u_2^2(t)$ for $t < t_0$.

The lemma is proved. ■

Theorem 3. *Suppose A_i, a_{ij} $(i, j = 1, 2)$ are as in Theorem 2. If, in addition, (1.5) holds, then the system (1.1) has a unique solution u^0 defined on $(-\infty, +\infty)$, whose components are bounded above and below by positive constants.*

Proof. The existence follows from Theorem 2. We now prove the uniqueness. Suppose by contradiction that u^1, u^2 are two different solutions of (1.1) defined on $(-\infty, +\infty)$, whose components are bounded above and below by positive constants. By Lemma 2, only one of the following alternatives is met:

(i) There exists $j \in \{1, 2\}$ such that $u_i^j(t) > u_i^{3-j}(t)$ $(i = 1, 2;\ t \in \mathbf{R})$.

(ii) There exist $i, j \in \{1, 2\}$, such that

$$u_i^j(t) > u_i^{3-j}(t), \quad u_{3-i}^j(t) < u_{3-i}^{3-j}(t), \quad \text{for } t \in \mathbf{R}.$$

(iii) There exist $t_0 \in \mathbf{R}$, $i, j \in \{1, 2\}$ such that $u_i^j(t_0) = u_i^{3-j}(t_0)$, $u_i^j(t) < u_i^{3-j}(t)$ for $t > t_0$, $u_i^j(t) > u_i^{3-j}(t)$ for $t < t_0$, and $u_{3-i}^j(t) > u_{3-i}^{3-j}(t)$ for $t \in \mathbf{R}$.

Suppose (i) happens, without loss of generality, we can assume $j = 1$, i.e., $u_i^1(t) > u_i^2(t)$ $(i = 1, 2;\ t \in \mathbf{R})$. It is not hard to get from (1.1) that

$$\begin{aligned} \frac{d}{dt}\ln\frac{u_1^2(t)}{u_1^1(t)} &= -a_{11}(t)\big(u_1^2(t) - u_1^1(t)\big) - a_{12}(t)\big(u_2^2(t) - u_2^1(t)\big), \\ \frac{d}{dt}\ln\frac{u_2^2(t)}{u_2^1(t)} &= -a_{21}(t)\big(u_1^2(t) - u_1^1(t)\big) - a_{22}(t)\big(u_2^2(t) - u_2^1(t)\big). \end{aligned} \quad (3.3)$$

Since $u_i^j(t)$ is bounded above and below by positive constants for $i, j = 1, 2$, it follows that there exists a positive number M such that $\int_{-T}^{T}\left(\frac{d}{dt}\ln\frac{u_1^2(t)}{u_1^1(t)}\right)dt \le M$, for any $T > 0$. Therefore, from (3.3), it follows that

$$\int_{-\infty}^{\infty} a_{11}(t)\big(u_1^1(t) - u_1^2(t)\big) + a_{12}(t)\big(u_2^1(t) - u_2^2(t)\big)dt \le M.$$

Since $a_{11}(t)\big(u_1^1(t) - u_1^2(t)\big) > 0$ and $a_{12}(t)\big(u_2^1(t) - u_2^2(t)\big) > 0$ for any $t \in \mathbf{R}$, it follows that $u_1^1(t) - u_1^2(t) \to 0$ and $u_2^1(t) - u_2^2(t) \to 0$ as $t \to \pm\infty$. Consequently, $\frac{u_i^1(t)}{u_i^2(t)} \to 1$ $(i = 1, 2)$, as $t \to \pm\infty$, since $0 < u_{iL}^j \le u_{iM}^j < +\infty$. Hence,

$$0 = \int_{-\infty}^{\infty}\left(\frac{d}{dt}\ln\frac{u_1^2(t)}{u_1^1(t)}\right)dt = \int_{-\infty}^{\infty} a_{11}(t)\big(u_1^1(t) - u_1^2(t)\big) + a_{12}(t)\big(u_2^1(t) - u_2^2(t)\big)dt.$$

Consequently, $a_{11}(t)\big(u_1^1(t)-u_1^2(t)\big)+a_{12}(t)\big(u_2^1(t)-u_2^2(t)\big)\equiv 0$. It follows that $u^1(t)=u^2(t)$ for any $t\in\mathbf{R}$, a contradiction. Hence, (i) does not happen.

Suppose (ii) happens. Without loss of generality, we can assume that $i=1$, $j=1$, i.e., we have $u_1^1(t)>u_1^2(t)$, $u_2^1(t)<u_2^2(t)$, for $t\in\mathbf{R}$. From (3.3), we get

$$\begin{aligned}\alpha_1\frac{d}{dt}\left(\ln\frac{u_1^2(t)}{u_1^1(t)}\right)-\alpha_2\frac{d}{dt}\left(\ln\frac{u_2^2(t)}{u_2^1(t)}\right)&=\big(\alpha_1a_{11}(t)-\alpha_2a_{21}(t)\big)\big(u_1^1(t)-u_1^2(t)\big)\\&\quad+\big(-\alpha_1a_{12}(t)+\alpha_2a_{22}(t)\big)\big(u_2^2(t)-u_2^1(t)\big).\end{aligned}\tag{3.4}$$

Since $0<u_{iL}^j\le u_{iM}^j<+\infty$ $(i,j=1,2)$, it follows that there exists a positive number M such that

$$\int_{-T}^{T}\left\{\alpha_1\frac{d}{dt}\left(\ln\frac{u_1^2(t)}{u_1^1(t)}\right)-\alpha_2\frac{d}{dt}\left(\ln\frac{u_2^2(t)}{u_2^1(t)}\right)\right\}dt\le M,\quad\text{for any } T>0.$$

By (1.5), it follows that

$$\int_{-\infty}^{+\infty}\varepsilon_2\big(u_1^1(t)-u_1^2(t)\big)+\varepsilon_2\big(u_2^2(t)-u_2^1(t)\big)dt\le M.$$

Consequently, $u_1^1(t)-u_1^2(t)\to 0$ and $u_2^2(t)-u_2^1(t)\to 0$, as $t\to\pm\infty$. Since $0<u_{iL}^j\le u_{iM}^j<+\infty$ $(i,j=1,2)$, it follows that $\dfrac{u_1^2(t)}{u_1^1(t)}\to 1$, $\dfrac{u_2^2(t)}{u_2^1(t)}\to 1$ as $t\to\pm\infty$. Therefore, from (3.4), we get

$$\begin{aligned}0&=\int_{-\infty}^{+\infty}\left\{\alpha_1\frac{d}{dt}\left(\ln\frac{u_1^2(t)}{u_1^1(t)}\right)-\alpha_2\frac{d}{dt}\left(\ln\frac{u_2^2(t)}{u_2^1(t)}\right)\right\}dt\\&\ge\int_{-\infty}^{+\infty}\varepsilon_2\Big[\big(u_1^1(t)-u_1^2(t)\big)+\big(u_2^2(t)-u_2^1(t)\big)\Big]dt\ge 0.\end{aligned}$$

Consequently, $u^1\equiv u^2$. This is also a contradiction. Therefore, (ii) does not happen.

Suppose (iii) happens. We can, without loss of generality, assume that $i=1$, $j=1$, i.e., we have $u_1^1(t_0)=u_1^2(t_0)$, $u_1^1(t)<u_1^2(t)$ and $u_2^1(t)>u_2^2(t)$ for $t>t_0$. From (3.3) we get

$$\begin{aligned}&-\alpha_1\frac{d}{dt}\left(\ln\frac{u_1^2(t)}{u_1^1(t)}\right)+\alpha_2\frac{d}{dt}\left(\ln\frac{u_2^2(t)}{u_2^1(t)}\right)\\&=\big(\alpha_1a_{11}(t)-\alpha_2a_{21}(t)\big)\big(u_1^2(t)-u_1^1(t)\big)\\&\quad+\big(-\alpha_1a_{21}(t)+\alpha_2a_{22}(t)\big)\big(u_2^1(t)-u_2^2(t)\big)\\&\ge\varepsilon_2\Big[\big(u_1^2(t)-u_1^1(t)\big)+\big(u_2^1(t)-u_2^2(t)\big)\Big],\end{aligned}$$

for $t\ge t_0$. By the same argument given before, we get $u^1(t)=u^2(t)$ for $t\ge t_0$. It follows that $u^1(t)=u^2(t)$ for any $t\in\mathbf{R}$, a contradiction. Therefore, (iii) does not happen. Since the possibilities (i), (ii) and (iii) are exhaustive, the theorem is proved. ■

Theorem 4. *Suppose the system (1.1) satisfies all the conditions in Theorem 3. Then the solution u^0 in Theorem 3 satisfies*

$$u_i^0(t) - u_i(t) \to 0, \quad \text{as } t \to +\infty; \quad i = 1, 2,$$

for any positive solution $u(t)$ of (1.1).

Proof. Let $x = (x_1, x_2)$, $x_i > 0$; $i = 1, 2$. Let us denote by $u(t, x)$ the solution of the system (1.1) defined by the initial condition $u(0, x) = x$, $U_i(t, x)$, the solution of (1.3) given by $U_i(0, x) = x_i$.

It is enough to show that $u_i(t, x) - u_i^0(t) \to 0$, as $t \to +\infty$ $(i = 1, 2)$. From (1.4), it follows that there exists $\gamma_i > 0$ $(i = 1, 2)$ such that

$$A_i(t) - \gamma_i a_{ii}(t) - a_{i3-i}(t)\big(U_{3-i}^0(t) + \gamma_i\big) > 0; \quad i = 1, 2. \tag{3.5}$$

Let us fix $i = 1, 2$. It is not hard to prove that $U_i(t, x) - U_i^0(t) \to 0$, as $t \to +\infty$. Therefore, there exists $t_0 > 0$ such that

$$U_i(t, x) < U_i^0(t) + \gamma_i, \quad \text{for } t \geq t_0. \tag{3.6}$$

We claim that

$$u_i(t, x) \geq \gamma_i^* = \min\{u_i(t_0, x), \gamma_i\}, \quad \text{for } t \geq t_0.$$

If it is false, let us define $g_i(t) = \gamma_i^* - u_i(t, x)$. Then there exists $t_1 > t_0$ such that $g_i(t_1) > 0$. Since $g_i(t_0) \leq 0$, there exists $t_2 > t_0$ such that $g_i(t_2) > 0$, $g_i'(t_2) > 0$. It implies

$$\begin{aligned} 0 < &- A_i(t_2) + a_{ii}(t_2)u_i(t_2, x) + a_{i3-i}(t_2)u_{3-i}(t_2, x) \\ \leq &- A_i(t_2) + a_{ii}(t_2)\gamma_i + a_{i3-i}(t_2)u_{3-i}(t_2, x). \end{aligned} \tag{3.7}$$

By Lemma 1, it follows that $u_i(t, x) < U_i(t, x)$ for $t > 0$. From (3.6) and (3.7), we have

$$0 < - A_i(t_2) + a_{ii}(t_2)\gamma_i + a_{i3-i}(t_2)\big(U_{3-i}^0(t_2) + \gamma_i\big),$$

which contradicts (3.5). Hence, the claim is proved.

It is not hard to see that $u_i(t, x) \leq \max\left\{x_i, \dfrac{A_{iM}}{a_{iiL}}\right\} := \Gamma_i$ for $t \geq 0$. Therefore, by the claim, we have $0 < \gamma_i^* \leq u_i(t, x) \leq \Gamma_i < +\infty$ for $t \geq t_0$.

Using the similar argument as in proving Theorem 3, we get

$$u_i(t, x) - u_i^0(t) \to 0, \quad \text{as } t \to +\infty; \quad i = 1, 2.$$

The theorem is proved. ■

4. Almost Periodicity

In this section we assume in addition that $A_i(t)$, $a_{ij}(t)$ $(i, j = 1, 2)$ are almost periodic. Suppose $f = (f^1, \ldots, f^n)$: $\mathbf{R} \to \mathbf{R}^n$; $n \geq 1$, is continuous. Let us recall that f is almost periodic if for each $\varepsilon > 0$, there exists a positive number $\ell = \ell(\varepsilon)$

such that each interval $(\alpha, \alpha + \ell)$, $\alpha \in R$, contains at least a number $\tau = \tau(\varepsilon)$ satisfying $\sup_{t \in \mathbf{R}} \|f(t+\tau) - f(t)\| \leq \varepsilon$, where $\|f(t)\| = \max_{1 \leq i \leq n} \{|f^i(t)|\}$. We recall Bochner's criterion for almost periodicity: $f(t)$ is almost periodic if and only if for every sequence of numbers $\{\tau_m\}_1^\infty$, there exists a subsequence $\{\tau_{m_k}\}_{k=1}^\infty$ such that the sequence of translates $\{g(t + \tau_{m_k})\}_{k=1}^\infty$ converges uniformly on $(-\infty, +\infty)$ (see, for example, [3]).

Lemma 3. *For $i = 1, 2$, the solution $U_i^0(t)$ of (1.3) is almost periodic.*

Proof. Let us fix $i = 1, 2$. Take $\varepsilon' > 0$. By Bochner's criterion, it follows that $(A_i(t), a_{ii}(t))$ is almost periodic. Therefore, there exists a positive number ℓ such that each interval $(\alpha, \alpha + \ell)$, $\alpha \in R$, contains at least a number $\tau = \tau(\varepsilon')$ such that

$$\sup_{t \in \mathbf{R}} |A_i(t+\tau) - A_i(t)| \leq \varepsilon', \quad \sup_{t \in \mathbf{R}} |a_{ii}(t+\tau) - a_{ii}(t)| < \varepsilon'. \tag{4.1}$$

Take an arbitrary τ as above. Define $W_i(t) = \dfrac{1}{U_i^0(t)}$. From (1.3), it follows that

$$\begin{aligned} \frac{d}{dt}\left[W_i(t) - W_i(t+\tau)\right] &= a_{ii}(t) - a_{ii}(t+\tau) - A_i(t)\left[W_i(t) - W_i(t+\tau)\right] \\ &\quad + \left[A_i(t+\tau) - A_i(t)\right] W_i(t+\tau). \end{aligned} \tag{4.2}$$

Consider the following equation

$$Z' = a_{ii}(t) - a_{ii}(t+\tau) + \left(A_i(t+\tau) - A_i(t)\right) W_i(t+\tau) - A_i(t) Z. \tag{4.3}$$

Since $A_{iL} > 0$, it is not hard to see that if $Z(t)$ is a bounded solution of (4.3) defined on $(-\infty, +\infty)$, then

$$\begin{aligned} &\inf_{t \in \mathbf{R}} \left\{ \frac{a_{ii}(t) - a_{ii}(t+\tau) + \left(A_i(t+\tau) - A_i(t)\right) W_i(t+\tau)}{A_i(t)} \right\} \leq Z(t) \\ &\leq \sup_{t \in \mathbf{R}} \left\{ \frac{a_{ii}(t) - a_{ii}(t+\tau) + \left(A_i(t+\tau) - A_i(t)\right) W_i(t+\tau)}{A_i(t)} \right\}; \quad t \in \mathbf{R}. \end{aligned}$$

Therefore, from (4.1), it follows that

$$|Z(t)| \leq \frac{\varepsilon'\left(1 + \dfrac{1}{U_{iL}^0}\right)}{A_{iL}}, \quad \text{for any } t \in \mathbf{R}.$$

Since $\dfrac{1}{U_i^0(t)} - \dfrac{1}{U_i^0(t+\tau)}$ is a bounded solution of (4.3), we have

$$\left| \frac{1}{U_i^0(t)} - \frac{1}{U_i^0(t+\tau)} \right| \leq \varepsilon' \left(\frac{1 + \dfrac{1}{U_{iL}^0}}{A_{iL}} \right).$$

Consequently,

$$|U_i^0(t) - U_i^0(t+\tau)| \le \varepsilon' \frac{\left(1 + \frac{1}{U_{iL}^0}\right)(U_{iM}^0)^2}{A_{iL}}.$$

Therefore if $\varepsilon = \varepsilon' \frac{\left(1 + \frac{1}{U_{iL}^0}\right)(U_{iM}^0)^2}{A_{iM}}$, then $|U_i^0(t) - U_i^0(t+\tau)| \le \varepsilon$ and we can take $\ell(\varepsilon) = \ell(\varepsilon')$. This proves that $U_i^0(t)$ is almost periodic. The theorem is proved.

In proving the following theorem, we use the idea from [1]. ■

Theorem 5. *Suppose $A_i(t)$, $a_{ij}(t)$ $(i,j = 1, 2)$ are as in Theorem 4 and, in addition, they are almost periodic. Then the solution $u^0(t)$ in Theorem 4 is almost periodic.*

Proof. Let $\{\tau_m\}_{m=1}^{\infty}$ be an arbitrary sequence of numbers. Since $A_i(t)$, $a_{ij}(t)$, $U_i^0(t)$ $(i, j = 1, 2)$ are almost periodic, there exists a subsequence $\{\tau_{m_k}\}_{k=1}^{\infty}$ of $\{\tau_m\}_{m=1}^{\infty}$ such that $A_i(t+\tau_{m_k})$, $a_{ij}(t+\tau_{m_k})$, $U_i^0(t+\tau_{m_k})$ converge uniformly to functions $A_i{}^*(t)$, $a_{ij}^*(t)$, $U_i^{0*}(t)$ respectively on $(-\infty, +\infty)$. It is not hard to see that $A_{iL}^* = A_{iL}$, $A_{iM}^* = A_{iM}$, $a_{ijL}^* = a_{ijL}$, $a_{ijM}^* = a_{ijM}$, $U_{iL}^{0*} = U_{iL}^0$ and $U_{iM}^{0*} = U_{iM}^0$ $(i, j = 1, 2)$. Furthermore, it is also not hard to prove that for each $i = 1, 2$, $U_i^{0*}(t)$ is a solution of

$$U_i' = U_i\big(A_i^*(t) - a_{ii}^*(t)U_i\big), \tag{4.4}$$

defined on $(-\infty, +\infty)$. Since $0 < A_{iL}^* \le A_{iM}^* < +\infty$ and $0 < a_{iiL}^* \le a_{iiM}^* < +\infty$ for $i = 1, 2$, it follows that U_i^{0*} is the unique solution of (4.4) such that $0 < U_{iL}^{0*} < U_{iM}^{0*} < +\infty$. Since $A_i(t+\tau_{m_k}) - a_{i3-i}(t+\tau_{m_k})U_{3-i}(t+\tau_{m_k})$ converges uniformly to $A_i^*(t) - a_{i3-i}^*(t)U_{3-i}^{0*}(t)$, as $k \to +\infty$ $(i = 1, 2)$, on $(-\infty, +\infty)$, it follows from (1.4) that

$$A_i^*(t) - a_{i3-i}^*(t)U_{3-i}^{0*}(t) \ge \varepsilon_1; \quad i = 1, 2; \quad t \in \mathbf{R}. \tag{4.5}$$

Similarly, from (1.5), it follows that

$$\alpha_i a_{ii}^*(t) - \alpha_{3-i} a_{3-ii}^*(t) \ge \varepsilon_2; \quad i = 1, 2; \quad t \in \mathbf{R}. \tag{4.6}$$

By Theorems 2 and 3, it follows that

$$u_i' = u_i\big[A_i^*(t) - a_{ii}^*(t)u_i - a_{i3-i}^*(t)u_{3-i}\big]; \quad i = 1, 2 \tag{4.7}$$

has a unique solution u^{0*} defined on $(-\infty, +\infty)$ such that

$$\eta_i \le u_i^{0*}(t) \le \Delta_i,$$

where η_i, Δ_i are positive numbers satisfying

$$\eta_i \le \min\left\{\varepsilon_1/a_{iiM}^*, \inf_{t \in R} U_i^{0*}(t)\right\} = \min\left\{\varepsilon_1/a_{iiM}, \inf_{t \in R} U_i^*(t)\right\},$$

$$\Delta_i = U_{iM}^{0*} = U_{iM}^0.$$

Let us denote $S = [\eta_1, \Delta_1] \times [\eta_2, \Delta_2]$. We claim that $u^0(t + \tau_{m_k})$ converges to $u^{0*}(t)$, uniformly as $t \to \infty$, which will show that $u^0(t)$ is almost periodic. Suppose the claim is false. Then there exist a subsequence $\{\tau_{m_{k_\ell}}\}$ of $\{\tau_{m_k}\}$, a sequence of numbers $\{S_\ell\}$, and a fixed number $\alpha > 0$ such that

$$\|u^0(S_\ell + \tau_{m_{k_\ell}}) - u^{0*}(S_\ell)\| \geq \alpha, \quad \text{for all } \ell.$$

Since A_i, a_{ij}, U_j^0 $(i, j = 1, 2)$ are almost periodic, we may assume, without loss of generality, that $A_i(t + \tau_{m_{k_\ell}} + S_\ell)$, $a_{ij}(t + \tau m_{k_\ell} + S_\ell)$, $U_i^0(t + \tau_{m_{k_\ell}} + S_\ell)$ converge uniformly to $\hat{A}_i(t)$, $\hat{a}_{ij}(t)$, $\hat{U}_i^0(t)$ respectively as $\ell \to \infty$ on $(-\infty, +\infty)$. Hence, $A_i^*(t + S_\ell) \to \hat{A}_i(t)$, $a_{ij}^*(t + S_\ell) \to \hat{a}_{ij}(t + S_\ell)$, $U_i^{0*}(t + S_\ell) \to \hat{U}_i^0(t)$, uniformly with respect to t in $(-\infty, +\infty)$ as $\ell \to +\infty$ and $\hat{A}_{iL} = A_{iL}$, $\hat{A}_{iM} = A_{iM}$, $\hat{a}_{ijL} = a_{ijL}$, $\hat{a}_{ijM} = a_{ijM}$, $\hat{U}_{iL}^0 = U_{iL}^0$ and $\hat{U}_{iM}^0 = U_{iM}^0$ $(i, j = 1, 2)$.

Since $u^0(t) \in S$ for all t in $(-\infty, +\infty)$, we can assume without loss of generality that $u^0(S_\ell + \tau_{m_{k_\ell}}) \to (\xi_0, \eta_0)$ as $\ell \to \infty$, where $(\xi_0, \eta_0) \in S$. Similarly, we may assume that $u^{0*}(S_\ell) \to (\xi_0^*, \eta_0^*)$ as $\ell \to \infty$. Clearly $\|(\xi_0, \eta_0) - (\xi_0^*, \eta_0^*)\| \geq \alpha$.

For each ℓ $(\ell = 1, 2, \ldots)$, $u^0(t + \tau m_{k_\ell} + S_\ell)$ is a solution of the system

$$u_i' = u_i\left[A_i(t + \tau_{m_{k_\ell}} + S_\ell) - a_{ii}(t + \tau_{m_{k_\ell}} + S_\ell)u_i - a_{i3-i}(t + \tau_{m_{k_\ell}} + S_\ell)u_{3-i}\right]; \quad i = 1, 2. \tag{4.8}$$

Consider the solution $\hat{u}^0$ of

$$u_i' = u_i\left[\hat{A}_i(t) - \hat{a}_{ii}(t)u_i - \hat{a}_{i3-i}(t)u_{3-i}\right]; \quad i = 1, 2, \tag{4.9}$$

having the initial value $\hat{u}^0(0) = (\xi_0, \eta_0)$.

We have two systems (4.8) and (4.9) where the right side of (4.8) converges uniformly to the right side of (4.9) on any compact subset of $\mathbf{R}^3$, as $\ell \to +\infty$. Also, the initial values satisfy the property $u^0(\tau_{m_{k_\ell}} + S_\ell) \to (\xi_0, \eta_0)$, as $\ell \to +\infty$. Hence, it follows that $u^0(t + \tau_{m_{k_\ell}} + S_\ell) \to \hat{u}^0(t)$ uniformly on compact subintervals of the domain of $\hat{u}^0(t)$. This implies that $\hat{u}^0(t) \in S$ for all $t \in \mathbf{R}$.

Now recall that $u^{0*}(t)$ is the unique solution of (4.7) with $u^{0*}(t) \in S$ for all t. For each integer ℓ, $u^{0*}(t + S_\ell)$ is a solution of

$$u_i' = u_i(A_i^*(t + S_\ell) - a_{ii}^*(t + S_\ell)u_i - a_{i3-i}^*(t + S_\ell)u_{3-i}); \quad i = 1, 2, \tag{4.10}$$

with $u^{0*}(S_\ell) \to (\xi_0^*, \eta_0^*)$, as $\ell \to \infty$.

Since $A_i^*(t + S_\ell) \to \hat{A}_i(t)$, $a_{ij}^*(t + S_\ell) \to \hat{a}_{ij}(t)$ $(i, j = 1, 2)$ as $\ell \to \infty$ uniformly with respect to t in $(-\infty, +\infty)$, it follows that if $\hat{u}^{0*}(t)$ is the solution of (4.9) with $\hat{u}^{0*}(0) = (\xi_0^*, \eta_0^*)$, then $u^{0*}(t + S_\ell) \to \hat{u}^{0*}(t)$ as $\ell \to \infty$ uniformly on any compact subintervals of the domain of $\hat{u}^{0*}$. By the same argument given before, we have $\hat{u}^{0*}(t) \in S$ for any $t \in \mathbf{R}$. We also have $\hat{u}^0(t) \in S$ for any $t \in R$. Using the same argument in proving that (4.7) has the unique solution $u^{0*}(t)$ in S for $t \in \mathbf{R}$, we may see that (4.9) has a unique solution defined on $(-\infty, +\infty)$ which is in S for any $t \in (-\infty, +\infty)$. Therefore, we must have $\hat{u}^0 \equiv \hat{u}^{0*}$. But $\hat{u}^0(0) = (\xi_0, \eta_0)$, $\hat{u}^{0*}(0) = (\xi_0^*, \eta_0^*)$ and $\|(\xi_0, \eta_0) - (\xi_0^*, \eta_0^*)\| \geq \alpha$, a contradiction. This proves the theorem. ■

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