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Short Communication

On a Characterization of Two-Sided Exponential Distribution and Its Stability

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Let $X_1, X_2, \ldots$ be independent identically distributed random variables with $F(x) = P(X_j < x)$, $\varphi(t) = Ee^{itX_j}$, $\mu = E|X_j| < +\infty$ and let N be independent of X_j, $j = 1, 2, \ldots$ with geometric distribution, i.e.,

$$P(N = k) = pq^{k-1}, \quad k = 1, 2, \ldots \ (0 < p < 1;\ q = 1 - p).$$

The random variable $Z = X_1 + \cdots + X_N$ is called the geometric compounding of X_j's.

The notation $G_\alpha(x)$ means $P(\alpha Z < x)$ and $\varphi_{\alpha Z}(t)$ means $Ee^{it\alpha Z}$. $\hat{F}_0(x)$ and $\hat{\varphi}_{\alpha Z}$ will denote the distribution function and the characteristic function, respectively, of the two-sided exponential distribution.

Characterization problems of the distributions and their stability have attracted much attention. Results of this nature may be found in [1, 3, 5, 6]. In [5], Renyi characterized the exponential distribution proving the following two assertions:

(a) $\lim_{p \to 0} G_p(x) = 1 - e^{-x}$.

(b) $G_p(x) = F(x) \Leftrightarrow F(x) = 1 - e^{-x}$ (with $X_j > 0$).

In [6], we estimated the stable degree of this theorem with the following metrics

$$\lambda(F_1; F_2) = \min_{T > 0} \max \left\{ \max_{|t| \le T} \frac{1}{2} |\varphi_1(t) - \varphi_2(t)|; \frac{1}{T} \right\},$$

$$\rho(F_1; F_2) = \sup_x |F_1(x) - F_2(x)|$$

for two distribution functions $F_1(x)$, $F_2(x)$ and characteristic functions $\varphi_1(t)$, $\varphi_2(t)$.

This paper presents some results concerning a characterization of two-sided exponential distribution and its stability. First, we get the following characteristic theorem.

Theorem 1. *Under the stated assumptions, a necessary and sufficient condition for $\sqrt{p}Z$ having a two-sided exponential distribution is that X_j, $j = 1, 2, \ldots$ have a two-sided exponential distribution, i.e.,*

$$G_{\sqrt{p}}(x) = \hat{F}_0(x) \Leftrightarrow F(x) = \hat{F}_0(x)\,.$$

This theorem can be proved by considering characteristic functions.

The stability of Theorem 1 will be considered with the metrics λ and ρ mentioned above and divided in two cases, when

(a) $F(x)$ is an ε-two-sided exponential distribution function, in the sense that $\exists T(\varepsilon) > 0,\ T(\varepsilon) \to +\infty$ when $\varepsilon \to 0$, such that

$$|\varphi(t) - \hat{\varphi}_0(t)| \le \varepsilon,\ \forall t\colon |t| \le T(\varepsilon)\,, \tag{1a}$$

(b) $G_{\sqrt{p}}(x)$ is an ε-two-sided exponential distribution function, in the sense that $\exists T(\varepsilon) > 0,\ T(\varepsilon) \to +\infty$ when $\varepsilon \to 0$, such that

$$|\varphi_{\sqrt{p}Z}(t) - \hat{\varphi}_0(t)| \le \varepsilon, \quad \forall t\colon |t| \le T(\varepsilon)\,. \tag{1b}$$

Further, we establish some lemmas.

Lemma 1. *For an arbitrary number α, we have the following inequalities*

$$\mu_\alpha z = E|\alpha Z| < +\infty\,, \tag{2}$$

$$|\varphi(t) - 1| \le \mu|t|\,, \quad \forall t \in R\,, \tag{3}$$

$$|\varphi_{\alpha Z}(t) - 1| \le \mu_{\alpha Z}|t|\,, \quad \forall t \in R\,. \tag{4}$$

Lemma 2. *If*

$$|\varphi(t) - \hat{\varphi}_0(t)| < \varepsilon, \quad \forall t\colon |t| \le T \tag{5}$$

(with some $T > 0$), then we have

$$|\varphi_{\sqrt{p}Z}(t) - \hat{\varphi}_0(t)| < \frac{\varepsilon}{p}\,, \quad \forall t\colon |t| \le \frac{1}{\sqrt{p}}\,T\,. \tag{6}$$

Lemma 3. *If*

$$|\varphi_{\sqrt{p}Z}(t) - \hat{\varphi}_0(t)| < \varepsilon\,, \quad \forall t\colon |t| \le T \tag{7}$$

(with $0 < \varepsilon < p/q$ and some $T > 0$), then we have

$$|\varphi(t) - \hat{\varphi}_0(t)| < \frac{\varepsilon}{p - q\varepsilon}\,, \quad \forall t\colon |t| \le \sqrt{p}\,T\,. \tag{8}$$

Considering the stability of Theorem 1(a), we get the following two theorems.

Theorem 2. *Assume that $F(x)$ is an ε-two-sided exponetial distribution function. Then we have*

$$\lambda(G_{\sqrt{p}};\hat{F}_0) \leq \max\Big\{\frac{\varepsilon}{2p};\frac{\sqrt{p}}{T(\varepsilon)}\Big\}, \tag{9}$$

where $T(\varepsilon)$ is number mentioned in (1a).

Theorem 3. *Assume that $F(x)$ is an ε-two-sided exponential distribution function with the number $T(\varepsilon)$ in (1a) satisfying condition $T(\varepsilon) = O(\varepsilon^{-\alpha})$ for some $\alpha > 0$ when $\varepsilon \to 0$. Then*

$$\rho(G_{\sqrt{p}};\hat{F}_0) \leq K_1\varepsilon^{\alpha} - K_2\varepsilon.\ln\varepsilon, \tag{10}$$

where $K_1 > 0$, $K_2 > 0$ are constants independent of ε.

These theorems follow from applying Lemmas 1 and 2.

Considering the stability of Theorem 1(b), we get the following two theorems:

Theorem 4. *Assume $G_{\sqrt{p}}(x)$ is an ε-two-sided exponential distribution function with $0 < \varepsilon < p/q$. Then we have*

$$\lambda(F;\hat{F}_0) \leq \max\Big\{\frac{\varepsilon}{2(p-q\varepsilon)};\frac{1}{\sqrt{p}\,T(\varepsilon)}\Big\}, \tag{11}$$

where $T(\varepsilon)$ is a number mentioned in (1b).

Theorem 5. *Assume $G_{\sqrt{p}}(x)$ is an ε-two-sided exponential distribution function with the number $T(\varepsilon)$ in (1b) satisfying condition $T(\varepsilon) = O(\varepsilon^{-\alpha})$ for some $\alpha > 0$ (when $\varepsilon \to 0$). Then*

$$\rho(F;\hat{F}_0) \leq H_1\,\varepsilon^{\alpha} - H_2\,\varepsilon\ln\varepsilon, \tag{12}$$

where $H_1 > 0$, $H_2 > 0$ are some constants independent of ε.

The proofs of Theorems 4 and 5 are based on Lemmas 1 and 3.

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