Vietnam Journal of Mathematics **25**:3 (1997) 271–273

Vietnam Journal
of
MATHEMATICS

Short Communication

On a Problem of H. Freudenthal

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Received May 14, 1996

In 1963, Freudenthal [1] posed the following interesting:

Problem. *When is the inequality*

$$\sum_{i=1}^{n}|a_i| - \sum_{1\le i<j\le n}|a_i+a_j| + \sum_{1\le i<j\le n}|a_i+a_j+a_k| - \cdots + (-1)^{n-1}|a_1+a_2+\cdots+a_n| \ge 0 \qquad (1)$$

always true for all $a_1,\ a_2,\dots,\ a_n \in \mathbf{R}^m$*?*

Obviously, if $n=1$, (1) is true.
If $n=2$, (1) is the Minkowski inequality

$$|a+b| \le |a|+|b|.$$

If $n=3$, (1) is the Hlawka inequality

$$|a|+|b|+|c|-|a+b|-|a+c|-|b+c|+|a+b+c| \ge 0.$$

But if $n=4$, Luxemburg [2] gave a counter-example showing that (1) is not true, with $a_i = c$ $(i=1,2,3)$, $a_4=-2c$ $(c\ne 0)$.

When $n\ge 5$, the problem has not been settled. In this paper, by means of the elementary method, the problem is solved completely.

Let $A_n=(a_1,a_2,\dots,a_n)$, $a_i=(x_{i_1},x_{i_2},\dots,x_{i_m})\in\mathbf{R}^m$, $|a_i| = \left(\sum_{j=1}^{m} x_{ij}^2\right)^{1/2}$, and

$$f_n(A_n) = \sum_{k=1}^{n}(-1)^{k-1}\sum_{1\le i_1<i_2<\cdots i_k\le n}\left|\sum_{j=1}^{k} a_{i_j}\right|.$$

Then the problem becomes: When is $f_n(A_n) \geq 0$ always true? If $n = 4$, $m = 1$, letting $A_4 = (c, c, c, -2c)$ $(c \neq 0)$, we have $f_4(A_4) = -2|c| < 0$, so (1) is false, as mentioned above.

Now let

$$\begin{aligned} g_n(A_n) &= |a_n| - \sum_{i=1}^{n-1} |a_i + a_n| + \sum_{1 \leq i < j \leq n-1} |a_i + a_j + a_n| - \cdots \\ &\quad + (-1)^{n-1} |a_1 + a_2 + \cdots + a_n| \\ &= |a_n| + \sum_{k=1}^{n-1} (-1)^k \sum_{1 \leq i_1 < i_2 < \cdots i_k \leq n-1} \left| \sum_{j=1}^{k} a_{i_j} + a_n \right| . \end{aligned}$$

Lemma 1. *If $A_n = (a_1, a_2, \ldots, a_n)$, $a_j \in \mathbf{R}^m$, $j = 1, 2, \ldots, n$, then $f_n(A_n) = f_{n-1}(A_{n-1}) + g_n(A_n)$.*

Proof.

$$\begin{aligned} f_n(A_n) &= \sum_{k=1}^{n} (-1)^{k-1} \sum_{1 \leq i_1 < i_2 < \cdots i_k \leq n} \left| \sum_{j=1}^{k} a_{i_j} \right| \\ &= \sum_{k=1}^{n-1} (-1)^{k-1} \sum_{1 \leq i_1 < i_2 < \cdots i_k \leq n-1} \left| \sum_{j=1}^{k} a_{ij} \right| + |a_n| - \sum_{j=1}^{n-1} |a_j + a_n| \\ &\quad + \sum_{1 \leq i < j \leq n-1} |a_i + a_j + a_n| - \cdots + (-1)^{n-1} \left| \sum_{i=1}^{n-1} a_i + a_n \right| \\ &= f_{n-1}(A_{n-1}) + g_n(A_n) . \end{aligned}$$

■

Lemma 2. *If $A_n = (a_1, a_2, \ldots, a_n) \in \mathbf{R}^n$, $b \in \mathbf{R}$, and $b \geq \sum_{\substack{i=1 \\ a_i \leq 0}}^{n} |a_i|$, then $g_{n+1}(A_n, b) = 0$.*

Proof. Since $b \geq \sum_{\substack{i=1 \\ a_i \leq 0}}^{n} |a_i|$, then for $1 \leq k \leq n$, $\left| \sum_{\substack{i=1 \\ a_i \leq 0}}^{k} a_i + b \right| = \sum_{\substack{i=1 \\ a_i \leq 0}}^{n} a_i + b \geq 0$, and

$$\begin{aligned} g_{n+1}(A_n, b) &= |b| + \sum_{k=1}^{n} (-1)^k \sum_{1 \leq i_1 < i_2 < \cdots i_k \leq n} \left| \sum_{j=1}^{k} a_{i_j} + b \right| \\ &= b + \sum_{k=1}^{n} (-1)^k \sum_{1 \leq i_1 < i_2 < \cdots i_k \leq n} \left(\sum_{i=1}^{k} a_{i_j} + b \right) \\ &= b + \sum_{k=1}^{n} (-1)^k \left\{ \binom{n}{k} b + \binom{n-1}{k-1} \sum_{j=1}^{n} a_j \right\} \end{aligned}$$

$$= b\sum_{k=0}^{n}\binom{n}{k}(-1)^k - \sum_{j=1}^{n} a_j \sum_{k=0}^{n-1}\binom{n-1}{k}(-1)^k$$

$$= b\,.\,0 - \left(\sum_{j=1}^{n} a_j\right).\,0 = 0\,. \qquad \blacksquare$$

Lemma 3. *If* $C_{m+n} = (A_m, B_n) = (a_1, \ldots, a_m, b_1, \ldots, b_n) \in \mathbf{R}^{m+n}$ *and* $b_j \geq \sum_{\substack{i=1\\ a_i \leq 0}}^{n} |a_i|$, $j = 1, 2 \ldots, n$, *then* $f_{m+n}(C_{m+n}) = f_m(A_m)$.

Proof. By Lemma 1,

$$\begin{aligned} f_{m+n}(C_{m+n}) &= f_{m+n}(A_m, B_n) \\ &= f_{m+n-1}(A_m, B_{n-1}) + g_{m+n}(A_m, B_n) \\ &= f_m(A_m) + \sum_{k=1}^{n} g_{m+k}(A_m, B_k)\,. \end{aligned}$$

Since $b_j \geq \sum_{\substack{i=1\\ a_i \leq 0}}^{n} |a_i| \geq 0, j = 1, 2 \ldots, n$, then for $1 \leq k \leq n$, $\sum_{\substack{i=1\\ b_i \leq 0}}^{n} |b_i| = 0$, and

$$b_k \geq \sum_{\substack{i=1\\ a_i \leq 0}}^{n} |a_i| + \sum_{\substack{i=1\\ b_i \leq 0}}^{n} |b_i|\,.$$

By Lemma 2, $g_{m+k}(A_m, B_k) = 0$, and $f_{m+n}(C_{m+n}) = f_m(A_m)$. The lemma is thus proved. ■

We now pass to the proof of our result.

By Lemma 3, for all $c > 0$, if $A_n = (c, c, c, -2c, 2c, \ldots, 2c) \in \mathbf{R}^n$, then $f_n(A_n) = f_4(c, c, c, -2c) = -2c < 0$.

If $A_n = (a_1, a_2, \ldots, a_n)$, $a_i \in \mathbf{R}^m$, $i = 1, 2, \ldots, n$, let $a_1 = a_2 = a_3 = (c, 0, \ldots, 0)$, $-a_4 = a_5 = \cdots = a_n = (2c, 0, \ldots, 0)$, then also $f_n(A_n) = f_4(c, c, c, -2c) = -2c < 0$.

From the above, we have

Theorem. *For any* $a_1, a_2, \ldots, a_n \in \mathbf{R}^m$, *the inequality (1) is always true if and only if* $n = 1, 2, 3$.

References

1. H. Freudenthal, Problem 141, *Wisk Opgaven* **21** (1963) 137–139.
2. D. C. Mitrinovic, *Analytic Inequalities*, Springer-Verlag, New York, 1970.