Vietnam Journal of Mathematics **25**:3 (1997) 211–228

Vietnam Journal
of
MATHEMATICS

Homotopy Invariance of Entire Current Cyclic Homology

Do Ngoc Diep and Nguyen Van Thu
Institute of Mathematic, P.O. Box 631, Bo Ho 10000, Hanoi, Vietnam

Received February 29, 1996

Abstract. De Rham currents are dual to differential forms with compact support. We construct in this paper the dual to the entire cyclic cohomology of project limit of ideals with ad-invariant trace and prove its homotopy invariance.

Introduction

Classical homology theory of de Rham currents is just dual to the cohomology of differential forms with compact support. This means that from the well-known Stokes theorem, one can consider cycles as functionals over differential forms on manifolds. If we restrict ourselves to consider the differential forms with compact support, then currents are the continuous functionals over projective limits of differential forms by restrictions to supports. Besides the classical cycles there are also other currents, which are, for example, in the form of dense flows. Nevertheless, the de Rham current homology was well developed and has many applications in many problems from geometry, physics, etc. Cohomology of differential forms on manifolds was well quantized by Connes et al. as cyclic cohomology. In [3, 4], Khalkhali realized the so-called entire cyclic cohomology HE* as infinite cyclic cohomology. He proved two main properties of this theory: homotopy invariance and Morita invariance. We use his results in such a way by restricting to some non-commutative analogy of differential forms with compact support as ideals with ad-invariant trace. We then use the inductive and projective topologies to form non-commutative analogs of de Rham currents. Our main technical point is to use the Cuntz–Quillen theory [1, 2] of non-commutative differential forms over algebras. We pass this machinery to inductive limits. The main reason by what we can construct this homology is the point that, with the trace we can define a scalar product on each ideal.

Our main results are the construction of the cyclic homology of entire currents and the proof of its homotopy invariance (Theorem 5.4).

1. Basic Operators

Let A be an involutive Banach algebra, $\{A_\lambda\}_{\lambda\in I}$ the family of ideals with trace, i.e., with a map $\tau_\lambda: A_\lambda \to \mathbf{C}$, satisfying the following four conditions:

(1) τ_λ is a continuous linear map, $\|\tau_\lambda\| = 1$,
(2) $\tau_\lambda(a^*a) \geq 0$ (the map $a \mapsto a^*$, such that $a^{**} = a$, for all $\lambda \in I$),
(3) $\tau_\lambda(a^*a) = 0$ if and only if $a = 0$, for every $\lambda \in I$,
(4) τ_λ is ad_A-invariant, i.e., $\tau_\lambda(xa) = \tau_\lambda(ax)$, $\forall x \in A$ and $\forall a \in A_\lambda$.

For every λ, the map τ_λ defines, therefore, a scalar product over A_λ by the formula

$$\tau_\lambda(a^*b) = \langle a, b\rangle_{\tau_\lambda},$$

for all $a, b \in A_\lambda$. Next, we have a family $\{(A_\lambda, \tau_\lambda)\}$ with the natural ordering $\lambda \leq \mu$, following inclusions $A_\lambda \subseteq A_\mu$. The inclusion map $A_\lambda \hookrightarrow A_\mu$ of Hilbert ideals and restrictions $\tau_\lambda = \tau_\mu|_{A_\lambda}$ define a morphism of pairs $\{(A_\lambda, \tau_\lambda)\}$, such that for every triple $\tau, \mu, \gamma \in I$, $\lambda \leq \mu \leq \gamma$, the diagram

$$\begin{array}{ccc} (A_\lambda, \tau_\lambda) & \longrightarrow & (A_\mu, \tau_\mu) \\ \uparrow & & \uparrow \\ (A_\gamma, \tau_\gamma) & =\!= & (A_\gamma, \tau_\gamma) \end{array}$$

is commutative.

We have an inverse system $\{(A_\lambda, \tau_\lambda)\}$, and then the projective limit $\varprojlim A_\lambda = P$.

Now, we consider the system $\{C^n(A_\lambda, \tau_\lambda)\}$ consisting of $(n+1)$-linear maps $\phi\colon (\bar{A}_\lambda)^{n+1} \to C$. By definition, $\bar{A}_\lambda$ is the completion of A_λ with respect to the scalar product $\langle a, b\rangle_{\tau_\lambda} := \tau_\lambda(a^*b)$ and the definition of $\{C^n(A_\lambda, \tau_\lambda)\}$ is automatically extended to the same one for the completion $\bar{A}$ of A. For $\lambda \leq \mu$, there is a well-defined morphism $D^\mu_\lambda\colon C^n(\bar{A}_\lambda, \tau_\lambda) \to C^n(\bar{A}_\mu, \tau_\mu)$. Following the well-known Riesz' theorem for $\bar{A}_\lambda \otimes \cdots \otimes \bar{A}_\lambda$, for every $\phi \in C^n(\bar{A}_\lambda, \tau_\lambda)$, there is an element $z^\lambda_\phi \in \bar{A}_\lambda \otimes \cdots \otimes \bar{A}_\lambda$, such that

$$\phi(x) = \langle x, z^\lambda_\phi\rangle_{\tau_\lambda}.$$

Because $\tau_\lambda = \tau_\mu|_{A_\lambda}$, we have $D^\mu_\lambda \phi(x) = \langle x, z^\mu_\phi\rangle_{\tau_\lambda}$, for all $x \in \prod \bar{A}_\mu$. By virtue of uniqueness of the Riesz representative element, $z^\mu_\Phi - z^\lambda_\Phi$ belongs to the orthogonal complement of $\bar{A}_\lambda \otimes \cdots \otimes \bar{A}_\lambda$ in $\bar{A}_\mu \otimes \cdots \otimes \bar{A}_\mu$ This means that z^μ_ϕ must be projected onto z^λ_ϕ by D^μ_λ.

We have therefore a direct system which is denoted by $\{C^n(\bar{A}_\lambda, \tau_\lambda), D^\mu_\lambda\}_{\lambda \in I}$, and then the direct limit $\varinjlim C^n(\bar{A}_\lambda, \tau_\lambda) = Q$. Recall that $Q \subset \oplus_{\lambda \in I} C^n(\bar{A}_\lambda, \tau_\lambda)$ consists of all sequences $(\phi^n_\lambda)_{\lambda \in I}$ such that $D^\gamma_\lambda(\phi^n_\lambda) = D^\gamma_\mu \circ D^\mu_\lambda(\phi^n_\lambda)$, for all $\lambda, \mu, \gamma \in I$ satisfying $\lambda \leq \mu \leq \gamma$.

Recall that the standard operators b', b were also considered in [3]. We have extended them to the corresponding operators, denoted by the same letters

$$b', b\colon C^n(\bar{A}_\lambda, \tau_\lambda) \to C^{n+1}(\bar{A}_\lambda, \tau_\lambda),$$

following formulas

$$b'\phi^n_\lambda(a^0_\lambda, \ldots, a^{n+1}_\lambda) = \sum_{j=1}^{n} (-1)^j \phi^n_\lambda(a^0_\lambda, \ldots, a^j_\lambda a^{j+1}_\lambda, \ldots a^{n+1}_\lambda)$$

and

$$b\phi_\lambda^n(a_\lambda^0,\ldots,a_\lambda^{n+1}) = \sum_{j=0}^{n}(-1)^j\phi_\lambda^n(a_\lambda^0,\ldots,a_\lambda^j a_\lambda^{j+1},\ldots,a_\lambda^{n+1}) + (-1)^{n+1}\phi_\lambda^n(a_\lambda^{n+1}a_\lambda^0,\ldots,a_\lambda^n).$$

Definition 1.1. *Let A be an involutive algebra, $C_0^n(A) = \varinjlim C^n(\bar{A}_\lambda,\tau_\lambda)$. Set $C_0^n(A) = 0$ for $n \le 0$; the above defined operators b, b' acting on this complex are called the Hochchild boundary operators.*

Proposition 1.1. *The operators b' b, defined on $C^n(\bar{A}_\lambda,\tau_\lambda)$ can be extended, and do, to the operators, denoted by the same letters b', b on $\varinjlim C^n(\bar{A}_\lambda,\tau_\lambda)$, for all fixed n.*

Proof. Let $(\phi_\lambda^n)_{\lambda\in I}$ be a cochain in $\varinjlim C^n(\bar{A}_\lambda,\tau_\lambda)$. For every $\phi_\lambda^n \in C^n(\bar{A}_\lambda,\tau_\lambda)$, in view of Ręsz' theorem, there is an element $z_{\phi_\lambda^n}$ in $(\bar{A}_\lambda)^{n+1}$, such that $\phi_\lambda^n(x) = \langle x, z_{\phi_\lambda^n}\rangle_{\tau_\lambda}$. Because of the assumptions $\tau_\lambda = \tau_\mu|_{A_\lambda}$, we have $\phi_\mu^n(x) = \langle x, z_{\phi_\lambda^n}\rangle_{\tau_\lambda}$, for x in $(\bar{A}_\lambda)^{n+1}$, and

$$\phi_\mu^n|_{(\bar{A}_\lambda)^{n+1}} = \phi_\lambda^n = \phi_\gamma^n|_{(\bar{A}_\lambda)^{n+1}},$$

for $\lambda \le \mu, \gamma$.

Now we have

$$\begin{aligned}(b\phi)_\lambda^{n+1}(x) &= \sum_{j=0}^{n}(-1)^j\phi_\lambda^n(x_j) + (-1)^{n+1}\phi_\lambda^n(x_{n+1})\\ &= \sum_{j=0}^{n}(-1)^j\langle x_j, z_{\phi_\lambda^n}\rangle_{\tau_\lambda} + (-1)^{n+1}\langle x_{n+1}, z_{\phi_\lambda^n}\rangle_{\tau_\lambda},\end{aligned}$$

where by definition, $x = (a_\lambda^0,\ldots,a_\lambda^{n+1})$, $x_j = (a_\lambda^0,\ldots,a_\lambda^j a_\lambda^{j+1},\ldots,a_\lambda^{n+1})$, $x_{n+1} = (a_\lambda^{n+1}a_\lambda^0,\ldots,a_\lambda^n)$ and

$$(b\phi)_\mu^{n+1}(x) = \sum_{j=0}^{n}(-1)^j\langle x_j, z_{\phi_\lambda^n}\rangle_{\tau_\mu} + (-1)^{n+1}\langle x_{n+1}, z_{\phi_\lambda^n}\rangle_{\tau},$$

$$(b\phi)_\gamma^{n+1}(x) = \sum_{j=0}^{n}(-1)^j\langle x_j, z_{\phi_\lambda^n}\rangle_{\tau_\gamma} + (-1)^{n+1}\langle x_{n+1}, z_{\phi_\lambda^n}\rangle_{\tau_\gamma},$$

satisfying

$$D_\lambda^\gamma[(b\phi)_\lambda^{n+1}] = D_\mu^\gamma \circ D_\lambda^\mu[(b\phi)_\lambda^{n+1}].$$

Recall that in terms of "z_ϕ"-elements, D_λ^μ projects z_{ϕ_λ} onto the well-defined component z_{ϕ_μ}. We have therefore $[(b\phi)_\lambda^{n+1}]_{\lambda\in I}$ as an element of the direct limit. By analogy, we have b' as a homomorphism between $\varinjlim C^n(\bar{A}_\lambda,\tau_\lambda)$ and $\varinjlim C^{n+1}(\bar{A}_\lambda,\ \tau_\lambda)$. ■

Proposition 1.2. *The homomorphism λ: $C^n(\bar{A}_\lambda, \tau_\lambda) \to C^n(\bar{A}_\lambda, \tau_\lambda)$ can be extended to the corresponding operators, denoted also by*

$$\lambda: \varinjlim C^n(\bar{A}_\lambda, \tau_\lambda) \to \varinjlim C^n(\bar{A}_\lambda, \tau_\lambda).$$

The proof is straightforward.

Recall that in [3], the operator λ (do not confuse with index λ) is by definition a homomorphism between $C^n(\bar{A}_\lambda, \tau_\lambda)$ and $C^n(\bar{A}_\lambda, \tau_\lambda)$, which corresponds $\phi_\lambda^n \mapsto (\lambda\phi)_\lambda^n$,

$$(\lambda\phi)_\lambda^n(a_\lambda^0, \ldots, a_\lambda^n) = (-1)^n \phi_\lambda^n(a_\lambda^n, a_\lambda^0, \ldots, a_\lambda^{n-1}).$$

It is easy to check that λ is a homomorphism between $\varinjlim C^n(\bar{A}_\lambda, \tau_\lambda)$ and $\varinjlim C^n(\bar{A}_\lambda, \tau_\lambda)$.

By analogy, we have that $N = 1 + \lambda + \cdots + \lambda^n$ is a homomorphism between $\varinjlim C^n(\bar{A}_\lambda, \tau_\lambda)$ and $\varinjlim C^n(\bar{A}_\lambda, \tau_\lambda)$.

Proposition 1.3. *The operator*

$$S: C^{n+1}(\widetilde{\bar{A}}_\lambda, \tau_\lambda) \to C^n(\widetilde{\bar{A}}_\lambda, \tau_\lambda)$$

can be extended to the corresponding operator

$$S: \varinjlim C^{n+1}(\widetilde{\bar{A}}_\lambda, \tau_\lambda) \to \varinjlim C^n(\widetilde{\bar{A}}_\lambda, \tau_\lambda),$$

where $\widetilde{\bar{A}}_\lambda$ is the corresponding algebra with a formally joined unity element.

We recall the well-known operator S from [3], $\phi_\lambda^n \mapsto (S\phi)_\lambda^n$ with

$$(S\phi)_\lambda^n(a_\lambda^0, \ldots, a_\lambda^n) = \phi_\lambda^{n+1}(1, a^0, \ldots, a_\lambda^n).$$

It is not hard to see that S is a homomorphism between

$$\varinjlim C^{n+1}(\tilde{\bar{A}}_\lambda, \tau_\lambda) \quad \text{and} \quad \varinjlim C^n(\tilde{\bar{A}}_\lambda, \tau_\lambda).$$

Proposition 1.4. *$Q = \varinjlim C^n(\bar{A}_\lambda, \tau_\lambda)$ is a closed vector subspace of the direct sum of Hilbert spaces $\oplus_{\lambda \in I} C^n(\bar{A}_\lambda, \tau_\lambda)$, which is therefore also a Hilbert space.*

Proof. It is easy to see that Q is a vector subspace of $\oplus_{\lambda \in I} C^n(\bar{A}_\lambda, \tau_\lambda)$. Let $(a_\lambda^m)_{\lambda \in I} \in Q$, $\lim_{m\to\infty} a_\lambda^m = a_\lambda$. There is an element $z_{a_\lambda^m}$ in $(\bar{A}_\lambda)^{m+1}$, such that

$$a_\lambda^m(x) = \langle x, a_\lambda^m \rangle_{\tau_\lambda}.$$

We have

$$\lim_{m\to\infty} \langle x, z_{a_\lambda^m} \rangle_{\tau_\lambda} = \langle x, z_{a_\lambda} \rangle_{\tau_\lambda},$$

such that

$$\lim_{m\to\infty} z_{a_\lambda}^m = z_{a_\lambda}.$$

By analogy,

$$\lim_{m\to\infty} \langle x, z^m_{a_\lambda}\rangle_{\tau_\mu} = \langle x, z_{a_\lambda}\rangle_{\tau_\mu},$$

$$\lim_{m\to\infty} \langle x, z^m_{a_\lambda}\rangle_{\tau_\gamma} = \langle x, z_{a_\lambda}\rangle_{\tau_\gamma},$$

$$a_\lambda(x) = \langle x, z_{a_\lambda}\rangle_{\tau_\lambda},$$

$$a_\mu(x) = \langle x, z_{a_\lambda}\rangle_{\tau_\mu},$$

$$a_\gamma(x) = \langle x, z_{a_\lambda}\rangle_{\tau_\gamma}.$$

Because

$$a_\mu|_{(\bar{A}_\lambda)^{n+1}} = a_\lambda = a_\gamma|_{(\bar{A}_\lambda)^{n+1}},$$

we have $(a_\lambda)_{\lambda\in I} \in Q = \varinjlim C^n(\bar{A}_\lambda, \tau_\lambda)$.

Next, we see that Q is a vector closed subspace in the direct sum of Hilbert space $\oplus_{\lambda\in I} C^n(\bar{A}_\lambda, \tau_\lambda)$, which is therefore also a Hilbert space. ■

2. Entire Cyclic Current Homology

Definition 2.1. *Let A be an involutive Banach algebra and A_λ an ideal with trace τ_λ. The space $C^0_n(A) = \mathrm{Hom}\,(\varinjlim C(\bar{A}_\lambda, \mathbb{C}))$ of continuous functionals is called the space of cyclic currents (with respect to the direct limit topology).*

Let f be an element in $C^0_n(A)$, i.e., a continuous functions (with respect to the direct limit topology). This means that there exists a fixed λ such that f is a continuous linear functional on the Hilbert space $\bar{A}_\lambda^{\otimes(n+1)}$. Conversely, every functional of this form can be trivially extended by zero to other orthogonal component, to a continuous functional on the direct limit (with respect to the direct limit topology). Following Riesz' theorem, there exists an element z_f in $\varinjlim C^n(\bar{A}_\lambda, \tau_\lambda)$, such that $f(x) = \langle x, z_f\rangle_{\tau_\lambda}$, for all $x \in \varinjlim C^n(\bar{A}_\lambda, \tau_\lambda)$. Now, we can define continuous homomorphisms $b'^*, b^*, \lambda^*, S^*$ and N^*, which are just the adjoint ones of b', b, λ, S and N with respect to this Hilbert structure. Since $b^2 = b'^2 = 0$, we have $b^{*2} = b'^{*2} = 0$.

By analogy, we have $N^*(1-\lambda^*) = (1-\lambda^*)N^* = 0$. It is easy to see that $B^* = [NS(1-\lambda)]^* = (1-\lambda^*)S^*N^*$, acting from $C^0_n(\tilde{A})$ to $C^0_n(\tilde{A})$, for all n.

Definition 2.2. (The cyclic bicomplex) *Let A be an involutive Banach algebra. Define $C^0(A)$ as the following chain bicomplex in the upper half-plane,*

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longleftarrow & C^0_1(A) & \longleftarrow & C^0_1(A) & \longleftarrow & C^0_1(A) & \longleftarrow & \cdots \\
 & & \downarrow{\scriptstyle -b'^*} & & \downarrow{\scriptstyle b'^*} & & \downarrow{\scriptstyle -b'^*} & & \\
\cdots & \xleftarrow[1-\lambda^*]{} & C^0_0(A) & \xleftarrow[N^*]{} & C^0_0(A) & \xleftarrow[1-\lambda^*]{} & C^0_0(A) & \longleftarrow & \cdots,
\end{array}$$

where on the even columns the differentials are b^* and on the odd columns the differentials are $-b'^*$.

Hence, we have the *total complex*

$$\mathrm{Tot}(C^0(A))^{ev} := \mathrm{Tot}(C^0(A))^{odd} := \oplus_{n\geq 0} C^0_n(A),$$

of $C^0(A)$, which is periodic of period two

$$\oplus_{n\geq 0} C^0_n(A) \overset{\partial}{\rightleftharpoons} \oplus_{n\geq 0} C^0_n(A),$$

where $\partial = d_1 + d_2$ is the total differential.

Definition 2.3. *The periodic cyclic current homology of involutive algebra A, denoted by $\mathrm{HP}^0_*(A)$, is defined to be the total homology of the cyclic total complex $C^0(A)$.*

There are hence only two groups $\mathrm{HP}^0_{ev}(A)$, $\mathrm{HP}^0_{odd}(A)$. To each algebra A we associate an extension by adjoining the formal unity $\tilde{A} = A \oplus \mathbf{C}$. If A_λ is an ideal in A, $\tilde{A}_\lambda = A_\lambda \oplus \mathbf{C}$ is subalgebra with unity.

One can associate to $\tilde{A}$ the double (b^*, B^*)-complex with

$$B^0_{n,m}(\tilde{A}) = (C^0_{n-m}(\tilde{A}), d_1 + d_2) = (C^0_{n-m}(\tilde{A}), b^* + B^*).$$

By the same arguments as in [3], the operators b and B map reduced cochains to cochains of the same type. Hence, we also have the same property of b^*, B^*. The corresponding reduced subcomplex of the (b^*, B^*)-bicomplex is denoted by $B^0(\tilde{A})_{red}$.

Let $f_n \in C^0_n(A)$. There is an element $z_{f_n} \in \varinjlim C^n(\tilde{A}_\lambda, \tau_\lambda)$, such that

$$f_n(x) = \langle x, z_{f_n} \rangle_{\tau_\lambda}$$

and

$$\|f_n\| = \|z_{f_n}\|.$$

Definition 2.4. *An even* (or *odd*) *chain* $(f_n)_{n\geq 0}$ *in* $C^0(A)$ *is called entire if the radius of convergence of the power series*

$$\sum_{n\geq 0} \frac{n!}{(\frac{n}{2})!} \|f_n\| z^n,$$

$z \in C$ *is infinite.*

It is easy to see that *the total differential of the cyclic bicomplex sends entire chains to chains of the same type.* Hence, we have a periodic complex of entire chains

$$C^0_e(A) \underset{\partial}{\overset{\partial}{\rightleftharpoons}} C^0_e(A),$$

where $\partial = d_1 + d_2$.

Definition 2.5. *The entire cyclic current homology* $\mathrm{HE}_*(A)$ *of a Banach algebra A is the homology of the above complex* $C_e^0(A)$ *of entire current chains.*

Definition 2.6. *An even (resp., odd) chain* $(f_{2n})_{n\geq 0}$ *(resp.,* $(f_{2n+1})_{\geq 0}$*) in* $B^0(\tilde{A})_{red}$ *is called entire if the radius of convergence of the power series* $\sum_{n\geq 0}\frac{(2n!)}{n!}\|f_{2n}\|z^n$ *(resp.* $\sum_{n\geq 0}\frac{(2n+1)!}{n!}\|f_{2n+1}\|z^n$*) is infinite.*

It is easy to check that the operators b^*, B^* sends an entire chain to a chain of the same type.

Thus, we have a periodic complex of entire chains

$$C_e^0(\tilde{A})_{red} \underset{\partial}{\overset{\partial}{\rightleftarrows}} C_e^0(\tilde{A})_{red},$$

where the total differential $\partial = b^* + B^*$. In [3], it was shown that the operator Θ is an isomorphism of cochain complexes.

It is easy to see that the operator Θ can be extended to an operator, denoted also by

$$\Theta\colon \varinjlim \operatorname{Tot} C(\bar{A}_\lambda) \to \varinjlim \operatorname{Tot}(\bar{\tilde{A}}_\lambda)_{red},$$

between direct limits. Now, we have the operator Θ^*, which is just the adjoint one of Θ and is also an isomorphism of chain complexes.

Clearly, *the map* Θ^* *sends entire chains to chains of the same type.* Hence, we have the entire cyclic current homology of $B_e^0(\tilde{A})_{red}$ of an involutive Banach algebra A.

3. Normalized Current Cycles

In this section, we define the notion of a normalized cycle in the cyclic double complex and in the double (b^*, B^*)-complex. We can also prove "the normalized lemma". This lemma plays an important role in our proof of homotopy invariance of entire cyclic current homology.

Definition 3.1. *Let A be an involutive Banach algebra. A chain f in* $C^0(A)$ *is called cyclic if* $(1-\lambda^*)(f) = 0$.

Definition 3.2. *Let A be an involutive Banach algebra. An even (resp., odd) cycle* $(f_n)_{n\geq 0}$ *in the cyclic bicomplex is defined to be normalized if* f_{2n} *(resp.,* f_{2n+1}*) is a cyclic chain, for all n.*

Proposition 3.1. (Normalization lemma) *Let A be an involutive Banach algebra. For every entire cycle in the cyclic bicomplex* $C^0(A)$ *of A, there is a normalized entire cycle, homologous with it.*

Proof. Let $(f_n)_{n\geq 0}$ be an even entire cycle in $C^0(A)$. Pose

$$\theta_{2m} = f_{2m} - \frac{1}{2m+1}N^*(f_{2m}),$$

we have $N^*\theta_{2m} = 0$. Recall from [3] the operator N', which is defined as follows:

$$N' = -\frac{1}{n+1}(1 + 2\lambda + 3\lambda^2 + \cdots + (n+1)\lambda^n),$$

satisfying the well-known relation

$$(1-\lambda)N' + \frac{1}{n+1}N = 1.$$

These operators and relations are well extended to the corresponding completions and we also have the the relations for adjoint operators. Using the formula

$$(1-\lambda^*)N'^* + \frac{1}{n+1}N^* = 1,$$

we have

$$(1-\lambda^*)N'^*(\theta_{2m}) = \theta_{2m}.$$

Hence, $\tilde{\theta}_{2m} = N'^*(\theta_{2m})$ and therefore, $(1-\lambda^*)\tilde{\theta}_{2m} = \theta_{2m}$. Now, we can define $(f'_m)^{ev}_{m\geq 0}$ in $C^0(A)$ by formulas

$$f'_{2m-1} = f_{2m-1} - b'^*\tilde{\theta}_{2m}; f'_{2m} = f_{2m} - \theta_{2m}.$$

We show that it is a normalized cycle. Really,

$$\begin{aligned}(1-\lambda^*)(f'_{2m}) &= (1-\lambda^*)(f_{2m}) - (1-\lambda^*)\theta_{2m}\\ &= (1-\lambda^*)(f_{2m}) - (1-\lambda^*)(f_{2m} - \frac{1}{2m+1}N^*(f_{2m})\\ &= (1-\lambda^*)(f_{2m}) - (1-\lambda^*)(f_{2m})\\ &\quad + \frac{1}{2m+1}(1-\lambda^*)N^*(f_{2m})\\ &= 0.\end{aligned}$$

Let us show that f' is homologous to f. To see this, we define the chain $\psi = (\psi_m)_{m\geq 0}$, as $\psi_{2m-1} = 0$, $\psi_{2m} = \tilde{\theta}_{2m}$. We have $(\partial\psi)_{2m} = \theta_{2m}$, $(\partial\psi)_{2m-1} = -b'^*\psi_{2m}$, such that $f' = f + \partial\psi$, i.e., f' is homologous to f.

It is easy to check that f' and ψ are entire chains. Indeed, we can use the same formula as (2.3) in [4]. Since

$$\tilde{\theta}_{2m} = N'^*\theta_{2m},$$

we have

$$\|\tilde{\theta}_{2m}\| \leq 2(m+1)\|f_{2m}\|.$$

It follows that $\tilde{\theta}_{2m}$ and hence, ψ and f' are entire chains. ■

Definition 3.3. *Let A be an involutive Banach algebra. An even (resp., odd) current $(f_{2n})_{n\geq 0}$ (resp. $(f_{2n})_{n\geq 0}$) in $B^0(\tilde{A})_{red}$ is defined to be normalized if $B_0^*(f_{2n})$ (resp., $B_0^*(f_{2n+1})$) is a cyclic chain, for all n.*

It is easy to see that *the map Θ^* sends normalized cycles to cycles of the same type.* Thus, for every entire cycle in $B^0(\tilde{A})_{red}$, there is a normalized entire homologous cycle.

4. Infinite-Dimensional Differential Forms

Let A be an involutive Banach algebra, we have $C_n^0(A) \cong \varprojlim \Omega_n(\bar{A}_\lambda)$. Thus, to each $(f_n)_{n\geq 0}$ in $C^0(A)$ corresponds an element $(z_{f_n})_{n\geq 0}$ in $\varprojlim \Omega_n(\bar{A}_\lambda)$, denoted by $z_{f_n} = a_\lambda^0 da_\lambda^1 \dots da_\lambda^n$ (see in [2] for the case without projective limits), and then for the operators $b^*, b'^*, \lambda^*, S^*$ corresponding to the operators b, b', λ, S, one also has the same relations as in [2]

$$b^*(a_\lambda^0, \dots, a_\lambda^{n+1}) = \sum_{j=0}^{n} (-1)^j (a_\lambda^0, \dots, a_\lambda^j a_\lambda^{j+1}, \dots, a_\lambda^{n+1})$$
$$+ (-1)^{n+1} (a_\lambda^{n+1} a_\lambda^0, \dots a_\lambda^n).$$

Let us recall the definition of differential forms.

Definition 4.1. *A non-commutative, n-dimensional differential form over A is an element, formally written as $\omega_n = a_\lambda^0 da_\lambda^1 \dots da_\lambda^n$ in $\varprojlim \Omega_n(\bar{A}_\lambda)$, which satisfies the two conditions that follow:*

For every n-dimensional cycle $\int$ in $\mathrm{Hom}(\Omega_n(\bar{A}_\lambda), \mathbf{C})$

$$\int_\phi d\omega = 0 \quad \text{and} \quad \int_\phi [\omega_1, \omega_2] = 0,$$

for every $\omega, \omega_1, \omega_2$ in $\varprojlim \Omega_n(\bar{A}_\lambda)$, where d denotes the differential operator as usual (see, for example, [4]*).*

Proposition 4.1. *There is a one-to-one correspondence between n-dimensional differential forms and n-cyclic cycles f_n in $C_n^0(A)$.*

Proof. Let ω_n be an n-dimensional differential form. Using the previous isomorphism, we can define the corresponding chain f_n in $C_n^0(A)$.

It is easy to see that f_n is a cyclic cycle. For every n-dimensional cycle $\int_\phi$, we have

$$
\begin{aligned}
\int_\phi (1-\lambda^*)(\omega_n) &= \int_\phi (1-\lambda^*)(a_\lambda^0 da_\lambda^1 \dots da_\lambda^n) \\
&= \int_\phi (a_\lambda^0 da_\lambda^1 \dots da_\lambda^n - (-1)^n a_\lambda^n da_\lambda^0 \dots da_\lambda^{n-1}) \\
&= \int_\phi (-1)^n (da_\lambda^0 \dots da_\lambda^{n-1} a_\lambda^n + (-1)^{n-1} a_\lambda^0 da_\lambda^1 \dots da_\lambda^n \\
&\quad + a_\lambda^n da_\lambda^0 \dots da_\lambda^{n-1} - da_\lambda^0 \dots da_\lambda^{n-1} a_\lambda^n) \\
&= \int_\phi (-1)^{n-1} d(a_\lambda^0 da_\lambda^1 \dots da_\lambda^{n-1} a_\lambda^n) \\
&\quad + \int_\phi (-1)^{n-1} [a_\lambda^n, da_\lambda^0 \dots da_\lambda^{n-1}] \\
&= 0,
\end{aligned}
$$

so that $(1-\lambda^*)(f_n) = 0$. We have to calculate the boundary b^* of f_n. For every n-dimensional cycle $\int$ and the corresponding cocycle $\phi_n^{\int}$, we have

$$
\begin{aligned}
\int b^*(\omega_n) &= b\phi_n^{\int}(a_\lambda^0, \dots, a_\lambda^{n+1}) \\
&= \int \sum_{j=0}^n (-1)^j a_\lambda^0 da_\lambda^1 \dots d(a_\lambda^j a_\lambda^{j+1}) \dots da_\lambda^{n+1} \\
&\quad + \int (-1)^{n+1} a_\lambda^{n+1} a_\lambda^0 da_\lambda^1 \dots da_\lambda^n \\
&= (-1)^n \int a_\lambda^0 da_\lambda^1 \dots da_\lambda^n a_\lambda^{n+1} + (-1)^{n+1} \int a_\lambda^{n+1} a_\lambda^0 da_\lambda^1 \dots da_\lambda^n \\
&= (-1)^n \int [a_\lambda^0 da_\lambda^1 \dots da_\lambda^n, a_\lambda^{n+1}] \\
&= 0.
\end{aligned}
$$

Conversely, given an n-dimensional cyclic cycle f_n in $C_n^0(A)$, we can construct an n-dimensional differential form ω_n in $\varprojlim \Omega_n(\tilde{A}_\lambda)$, which satisfies the indicated two conditions $\int d\omega = 0$ and $\int [\omega_1, \omega_1] = 0$ for every n-dimensional cycle.

In this way, we have a canonical one-to-one correspondence between cyclic n-cycle in $C_n^0(A)$ and n-dimensional differential forms in $\varprojlim \Omega_n(\tilde{A}_\lambda)$. ∎

Definition 4.2. *A non-commutative even (resp., odd)* infinite-dimensional differential form *over an involutive Banach algebra A is by definition an element ω in $\oplus_{n\geq 0} \varprojlim (\Omega_n \widetilde{A}_\lambda)_{red}$, such that $\int_\phi [\omega_1, \omega_2] = (-1)^{deg \omega_n} \int_\phi d\omega_1 d\omega_2$, for all infinite dimensional cycle $\int_\phi$ on A, see also [4] for the case without projective limits.*

Proposition 4.2. *Let A be an involutive Banach algebra and f_{2n} in $C^0_{2n}(\tilde{A})_{red}$ (resp., f_{2n+1} in $C^0_{2n+1}(\tilde{A})_{red}$) be such that, for every n,*
(1) $b^*(f_{2n}) = B^0_0(f_{2n-2})$,
(2) $B^*_0(f_{2n})$ is cyclic.
Then, there is a differential form $(\omega_{2n})_{n\geq 0}$ in $\oplus_{n\geq 0} \varprojlim (\Omega_{2n}\widetilde{\tilde{A}}_\lambda)_{red}$, such that

$$\int \omega_{2n} = \psi^{\int}_{2n}(a^0_\lambda, \ldots, a^{2n}_\lambda),$$

$$\int da^1_\lambda \ldots da^{2n}_\lambda = B_0\psi^{\int}_{2n}(a^1_\lambda, \ldots, a^{2n}_\lambda),$$

for every infinite-dimensional cycle $\int$ and the corresponding normalized cocycle $(\psi^{\int}_{2n})_{n\geq 0}$ in $\oplus_{n\geq 0} \varinjlim C^{2n}(\bar{\tilde{A}}_\lambda, \tau_\lambda)$. The differential form $\omega = (\omega_{2n})_{n\geq 0}$ is an infinite-dimensional differential form in $\oplus_{n\geq 0} \varinjlim (\Omega_{2n}\bar{\tilde{A}}_\lambda)_{red}$.

Proof. It is easy to check that $\omega = (\omega_{2n})_{n\geq 0}$ is an infinite-dimensional differential form in $\oplus_{n\geq 0} \varinjlim (\Omega_{2n}\tilde{A}_\lambda)_{red}$.

Conversely, given an even (resp., odd) infinite-dimensional differential form $\omega = (\omega_m)_{m\geq 0}$, such that

$$\int \omega_m = \psi^{\int}_m(a^0_\lambda, \ldots, a^m_\lambda),$$

$$\int da^1_\lambda \ldots da^m_\lambda = B_0\psi^{\int}_m(a^1_\lambda, \ldots, a^m_\lambda),$$

one can define chains f_m. Proposition 4.1 shows that $f = (f_{2m})_{m\geq 0}$ satisfies conditions (1) and (2) of Proposition 4.2.

In the even case, given a normalized even cycle $f = (f_{2n})_{n\geq 0}$ in the (b^*, B^*)-bicomplex, define $\varphi = (\varphi_{2n})_{n\geq 0}$ by the formula

$$\varphi_{2n} = \lambda_{2n} f_{2n},$$

where $\lambda_{2n} = (-1)^n (2n)!$.

It is easy to see that $\varphi = (\varphi_{2n})_{n\geq 0}$ satisfies (1) and (2) in (4.2) if and only if $f = (f_{2n})_{n\geq 0}$ is a normalized cycle in the (b^*, B^*)-bicomplex. We can then define a bijective correspondence between normalized cycles in $B^0(\tilde{A})_{red}$ and infinite-dimensional differential forms in $\varprojlim (\Omega\tilde{\bar{\tilde{A}}}_\lambda)_{red}$, where $\varprojlim (\Omega\tilde{\bar{\tilde{A}}}_\lambda)_{red} = \oplus_{m\geq 0} \varprojlim (\Omega_m\tilde{\bar{A}}_\lambda)_{red}$. ■

5. Homotopy Invariance

In this section, we prove that any continuous derivation of an involutive Banach algebra induces the zero homomorphism on entire cyclic current homology groups. We generalize the notion of Lie derivative and then we prove the homotopy invariance of entire cyclic current homology.

Definition 5.1. *Let A be an involutive Banach algebra. For every ideal A_λ in algebra A, the continuous linear map $\delta_\lambda : A_\lambda \to A_\lambda$ is called a derivation of a subalgebra A_λ if*

$$\delta_\lambda(a_\lambda b_\lambda) = a_\lambda \delta_\lambda(b_\lambda) + \delta_\lambda(a_\lambda) b_\lambda, \forall a_\lambda, b_\lambda \in A_\lambda.$$

It should think of derivations as infinitesimal homomorphisms, which act on chains and this action commutes with many of the operators of the theory. More precisely, given a derivation δ_λ, we define a continuous homomorphism $L^*_{\delta_\lambda}$, just as the adjoint of L_{δ_λ}, from [3].

The map $L^*_{\delta_\lambda}$ is called the *Lie derivative* associated to derivation δ_λ.

Theorem 5.1. *Let f be a reduced and normalized entire cycle in $B^0(\tilde{A})_{red}$, then there is a canonically defined reduced and entire chain ψ in $B^0(\tilde{A})_{red}$, such that $L^*_{\delta_\lambda}(f) = \partial(\psi)$, where ∂ is the total boundary operator.*

Proof. Let us prove the even case. Suppose $f = (f_{2n})_{n\geq 0}$ to be a normalized entire cycle and $\omega = (\omega_{2n})_{n\geq 0}$ the corresponding form. We can define an even form ω^j_{2n} in $\varprojlim (\Omega_{2n}\tilde{A}_\lambda)$

$$\omega^j_{2n} = a^0_\lambda da^1_\lambda \ldots \delta_\lambda a^j_\lambda da^{j+1}_\lambda \ldots da^{2n+1}_\lambda.$$

For every even infinite-dimensional cycle $\int$, also confer [4], we define a cochain in $\varprojlim C^{n+1}(\tilde{\tilde{A}}_\lambda, \tau_\lambda)$ by formula

$$\int a^0_\lambda da^1_\lambda \ldots \delta_\lambda a^j_\lambda \ldots da^{2n+1}_\lambda = \psi^j_{2n+1}(a^0_\lambda, \ldots, a^{2n+1}_\lambda).$$

Hence, we have a cochain by

$$\begin{aligned}\bar{\psi}_{2n+1}(a^0_\lambda, \ldots, a^{2n+1}_\lambda) &= \sum_{j=1}^{2n+1} (-1)^{j-1} \int a^0_\lambda da^1_\lambda \ldots \delta_\lambda a^j_\lambda \ldots da^{2n+1}_\lambda \\ &= \sum_{j=1}^{2n+1} (-1)^{j-1} \int \omega^j_{2n}.\end{aligned}$$

Following the well-known Riesz' theorem for $\varprojlim \Omega_{2n+1}\tilde{\tilde{A}}_\lambda$, there is an element $\bar{\omega}_{2n+1}$ in $\varprojlim \Omega_{2n+1}\tilde{\tilde{A}}_\lambda$, such that

$$\bar{\psi}_{2n+1}(x) = \langle x, \bar{\omega}_{2n+1}\rangle_{\tau_\lambda}$$

and

$$\bar{\psi}_{2n+1}(a^0_\lambda, \ldots, a^{2n+1}_\lambda) = \int \bar{\omega}_{2n+1}.$$

We want to show that $(\bar{\omega}_{2n+1})_{n\geq 0}$ is reduced and estimate the norm of its components. It is easy to see that

$$\int da^1_\lambda \ldots da^{2k}_\lambda = 0,$$

if $a_\lambda^i = 1$ for some i and for every even infinite-dimensional cycle $\int$. Hence, we have $\int \omega_1 d1\omega_2 = \pm\int \omega_1\omega_2 d1 \pm \int d\omega_1 d1 d\omega_2 = 0$, where ω_1, ω_2 in $\oplus_{n\geq 0} \varprojlim \Omega_{2n}\tilde{A}_\lambda$, for every even infinite-dimensional cycle, such that ω_{2n}^j is reduced. We have

$$\psi_{2n+1}^j(a_\lambda^0, \ldots, a_\lambda^{2n+1}) = \int a_\lambda^0 da_\lambda^1 \ldots \delta_\lambda a_\lambda^j \ldots da_\lambda^{2n+1},$$

using formula $d(ab) = adb + dab$,

$$\begin{aligned}\psi_{2n+1}^j(a_\lambda^0, \ldots, a_\lambda^{2n+1}) &= \int a_\lambda^0 da_\lambda^1 \ldots d(a_\lambda^{j-1}\delta_\lambda a_\lambda^j) \ldots da_\lambda^{2n+1} \\ &\quad - \int a_\lambda^0 da_\lambda^1 \ldots d(a_\lambda^{j-2} a_\lambda^{j-1}) \ldots da_\lambda^{2n+1} - \ldots \\ &\quad - \int (a_\lambda^0 a_\lambda^1) da_\lambda^2 \ldots d\delta_\lambda a_\lambda^j \ldots da_\lambda^{2n+1} \\ &= \lambda_{2n}\phi_{2n}^{\int}(a_\lambda^0, \ldots, a_\lambda^{j-1}\delta_\lambda a_\lambda^j, \ldots a_\lambda^{2n+1}) \\ &\quad - \lambda_{2n}\phi_{2n}^{\int}(a_\lambda^0, \ldots, a^{j-2} a_\lambda^{j-1}, \delta_\lambda a_\lambda^j, \ldots, a_\lambda^{2n+1}) \ldots \\ &\quad - \lambda_{2n}\phi_{2n}^{\int}(a_\lambda^0 a_\lambda^1, \ldots, \delta_\lambda a_\lambda^j, \ldots, a_\lambda^{2n+1}).\end{aligned}$$

We conclude that

$$\|\psi_{2n+1}^j \leq |\lambda_{2n}||j|\|\phi_{2n}^{\int}\|\|\delta_\lambda\|.$$

Because $\|\bar{\omega}_{2n+1}\| = \|\bar{\psi}_{2n+1}\|$, we have

$$\bar{\omega}_{2n+1}\| \leq |\lambda_{2n}|(n+1)(2n+1)\|\phi_{2n}^{\int}\|\|\delta_\lambda\|,$$

where $\phi_{2n}^{\int}$ is the corresponding cochain of infinite-dimensional cycle $\int$ which is a recuded entire cochain. We see that $\bar{\omega}_{2n+1}$ is a reduced entire cocycle.

We will next calculate $B_0^*(\bar{\omega}_{2n+1})$ and show that it is cyclic using the formula $1da_\lambda^0 = da_\lambda^0 - d1a_\lambda^0$. For every even infinite-dimensional cycle $\int$, we have

$$\begin{aligned}\int 1da_\lambda^0 \ldots \delta_\lambda a_\lambda^{j-1} \ldots da_\lambda^{2n} &= \int da_\lambda^0 \ldots \delta_\lambda a_\lambda^{j-1} \ldots da_\lambda^{2n} - \\ &= \int da_\lambda^0 \ldots \delta_\lambda a_\lambda^{j-1} \ldots da_\lambda^{2n}.\end{aligned}$$

Using (3.1) in [4], it is easy to see that

$$\int d1a_\lambda^0 \ldots \delta_\lambda a_\lambda^{j-1} \ldots da_\lambda^{2n} = 0.$$

For every even infinite-dimensional cycle $\int$, we have

$$\begin{aligned}\int B_0^*(\bar{\omega}_{2n+1}) &= B_0\bar{\psi}_{2n+1}(a_\lambda^0,\dots,a_\lambda^{2n}),\\ &= \sum_{j=1}^{2n+1}(-1)^{j-1}B_0\psi_{2n+1}^j(a_\lambda^0,\dots,a_\lambda^{2n})\\ &= \sum_{j=1}^{2n+1}(-1)^{j-1}\int da_\lambda^0\dots\delta_\lambda a_\lambda^{j-1}\dots da_\lambda^{2n}.\end{aligned}$$

Because

$$B_0\bar{\psi}_{2n+1} = NB_0\psi_{2n+1}^{2n+1},$$

(see [4]), we have

$$\int B_0^*(1-\lambda^*)(\bar{\omega}_{2n+1}) = (1-\lambda)B_0\bar{\psi}_{2n+1} = (1-\lambda)NB_0\psi_{2n+1}^{2n+1} = 0.$$

Thus, $B_0^*(\bar{\omega}_{2n+1})$ is cyclic.

Let us calculate the Hochschild boundary of $\bar{\omega}_{2n+1}$. We see that

$$\int b^*(\bar{\omega}_{2n+1}) = b\bar{\psi}_{2n+1}(a_\lambda^0,\dots,a^{2n+2}),$$

for every even infinite-dimensional cycle $\int$. It is easy to see that

$$\begin{aligned}b\bar{\psi}_{2n+1}(a_\lambda^0,\dots,a_\lambda^{2n+2}) &= \sum_{j=1}^{2n+1}(-1)^j\int da_\lambda^0\dots\delta_\lambda a_\lambda^j\dots da_\lambda^{2n+2}\\ &\quad-\sum_{j=1}^{2n+1}\int a_\lambda^0 da_\lambda^1\dots d\delta_\lambda a_\lambda^j\dots da_\lambda^{2n+2}.\end{aligned}$$

We have

$$\begin{aligned}\int \lambda_{2n+2}L_{\delta_\lambda}^*(\omega_{2n+2}) &= \lambda_{2n+2}L_{\delta_\lambda}\phi_{2n+2}^{\int}(a_\lambda^0,\dots,a_\lambda^{2n+2})\\ &= \lambda_{2n+2}\sum_{j=0}^{2n+2}\phi_{2n+2}^{\int}(a_\lambda^0,\dots,\delta_\lambda a_\lambda^j,\dots,a_\lambda^{2n+2})\\ &= \lambda_{2n+2}\sum_{j=0}^{2n+2}\int a_\lambda^0 da_\lambda^1\dots d\delta_\lambda a_\lambda^j\dots da_\lambda^{2n+2},\end{aligned}$$

for every even infinite-dimensional cycle $\int$, the corresponding cycle $\phi_{2n}^{\int}$. We define a cochain in $\varinjlim C^{2n+2}(\bar{\tilde{A}}_\lambda,\tau_\lambda)$ by the formula

$$\chi_{2n+2} = \int d(a_\lambda^0 da_\lambda^1\dots da_\lambda^{2n+1}\delta_\lambda a_\lambda^{2n+2}).$$

We deduce that $(B_0\bar{\psi}_{2n+3} - \lambda_{2n+2}L_{\delta_\lambda}\phi^{\int}_{2n+2} - \chi_{2n+2})(a^0_\lambda, \ldots, a^{2n+2}_\lambda) = b\bar{\psi}_{2n+1}(a^0_\lambda, \ldots, a^{2n+2}_\lambda)$ for every even infinite-dimensional cycle $\int$. Hence, we have

$$\int b^*(\bar{\omega}_{2n+1}) = (B_0\bar{\psi}_{2n+3} - \lambda_{2n+2}L_{\delta_\lambda}\phi^{\int}_{2n+2} - \chi_{2n+2})(a^0_\lambda, \ldots, a^{2n+2}_\lambda).$$

We define a differential form in $\varprojlim \Omega_{2n}\tilde{\tilde{A}}_\lambda$, following

$$\bar{\omega}_{2n+2} = d(da^0_\lambda \ldots da^{2n}_\lambda \delta_\lambda a^{2n+1}_\lambda).$$

For every even infinite-dimensional cycle $\int$, we have

$$\int d(da^0_\lambda \ldots da^{2n}_\lambda \delta_\lambda a^{2n+1}_\lambda) = \psi^*_{2n+1}(a^0_\lambda, \ldots, a^{2n+1}_\lambda),$$

where ψ^*_{2n+1} in $\varinjlim C^{2n+1}(\tilde{\tilde{A}}_\lambda, \tau_\lambda)$. Using Riesz' theorem, there is an odd differential form ω^*_{2n+1} in $\lim \Omega_{2n+1}\tilde{\tilde{A}}_\lambda$, such that $\psi^*_{2n+1}(x) = \langle x, \omega^*_{2n+1}\rangle_{\tau_\lambda}$. It is easy to see that

$$\int \omega^*_{2n+1} = \int a^0_\lambda da^1_\lambda \ldots da^{2n+1}_\lambda = 0,$$

if $a^i_\lambda = 1$ for some index $i \geq 1$. This means that ω^*_{2n+1} is reduced. For every even infinite-dimensional cycle $\int$, it is easy to check that $\int B^*_0(\omega^*_{2n+1}) = 0$. Because $\|\psi^*_{2n+1}\| = \|\omega^*_{2n+1}\|$, we have $\|\omega^*_{2n+1}\| \leq |\lambda_{2n+1}|\|\omega_{2n+2}\|\|\delta_\lambda\|$, so that ω^*_{2n+1} is an entire current cycle.

We also have to calculate the Hochschild boundary of ω^*_{2n-1} from the relation

$$\int b^*(\omega^*_{2n-1}) = b\psi^*_{2n-1}(a^0_\lambda, \ldots, a^{2n}_\lambda) = (\chi_{2n} - b\psi^{2n-1}_{2n-1})(a^0_\lambda, \ldots, a^{2n}_\lambda),$$

for every even infinite-dimensional cycle $\int$. Let

$$\tilde{\omega}_{2n+1} = \bar{\omega}_{2n+1} + \omega^*_{2n+1} + \omega^{2n+1}_{2n+1},$$

where ω^{2n+1}_{2n+1} in $\varprojlim \Omega_{2n+1}\tilde{\tilde{A}}_\lambda$, such that

$$\psi^{2n+1}_{2n+1}(x) = \langle x, \omega^{2n+1}_{2n+1}\rangle_{\tau_\lambda}.$$

For every even infinite-dimensional cycle $\int$, it is easy to see that

$$\begin{aligned}\int b^*(\tilde{\omega}_{2n+1}) &= b(\bar{\psi}_{2n+1} + \psi^*_{2n+1} + \psi^{2n+1}_{2n+1}) \\ &= B_0\bar{\psi}_{2n+3} - \lambda_{2n+2}L_{\delta_\lambda}\phi^{\int}_{2n+2}\end{aligned}$$

and

$$\int b^*(\tilde{\omega}_{2n+3}) = B\tilde{\psi}_{2n+3} = (2n+4)B_0\bar{\psi}_{2n+3},$$

so that

$$\int b^*(\tilde{\omega}_{2n+3}) = \frac{1}{2n+4}B\tilde{\psi}_{2n+3} - \lambda_{2n+2}L_{\delta_\lambda}\phi^{\int}_{2n+2}.$$

Finally, if we define a chain $\psi = (\psi_{2n+1})_{n\geq 0}$ in $\oplus_{n\geq 0} C^0_{2n+1}(\tilde{A})_{red}$, such that

$$\psi_{2n+1}(x) = \lambda^{-1}_{2n+2}\langle x, \widetilde{\omega}_{2n+3}\rangle_{\tau_\lambda}, \quad \psi_{2n+3}(x) = \langle x, \widetilde{\omega}_{2n+1}\rangle,$$

we thus have

$$\lambda_{2n+2} b^*(\psi_{2n+3}) - \frac{1}{2n+4}\lambda_{2n+4} B^*(\psi_{2n+1}) = -\lambda_{2n+2} L^*_{\delta_\lambda}(f_{2n+2}),$$

or

$$b^*(\psi_{2n+3}) + B^*(\psi_{2n+1}) = -L^*_{\delta_\lambda}(f_{2n+2}),$$

which is the required equation. It is easy to check that $\psi = (\psi_{2n+1})_{n\geq 0}$ is an entire chain. The theorem is proved. ■

Theorem 5.2. *Let $L^*_{\delta_\lambda}$ be the Lie derivative associated to derivation δ_λ, then $L^*_{\delta_\lambda}: \mathrm{HE}^0_*(A) \to \mathrm{HE}^0_*(A)$ is the zero homomorphism.*

Proof. Let f be an entire cycle, then f is homologous to a normalized one. Now recall the isomorphism $\Theta^*: \mathrm{Tot}\, B^0(\tilde{A})_{red} \to \mathrm{Tot}\, C^0(A)$ which is compatible with Lie derivatives and sends the normalized cycles to normalized cycles. By the previous Theorem 5.1, there is a reduced entire chain ψ, such that

$$L^*_{\delta_\lambda}\Theta^{*-1}(f) = \partial\psi,$$
$$L^*_{\delta_\lambda}(f) = \partial\Theta^*\psi$$

and

$$\|\Theta^*\psi\| \leq \|\Theta^*\|\|\psi\|.$$

This means that $\Theta^*\psi$ is an entire chain. ■

Definition 5.2. (General Lie derivatives) *Let A and B be involutive Banach algebras and $\varphi: A \to B$ a homomorphism. There are the natural extensions $\varphi: \bar{A}_\lambda \to \bar{B}_\lambda$ for all λ. A continuous linear map $\delta_\lambda: A \to B$, such that $\delta_\lambda: \bar{A}_\lambda \to \bar{B}_\lambda$ is called a continuous derivation, associated to φ if $\delta_\lambda(a_\lambda b_\lambda) = \varphi(a_\lambda)\delta_\lambda(b_\lambda) + \delta_\lambda(a_\lambda)\varphi(b_\lambda)$, for all a_λ, b_λ in A_λ.*

Given a continuous derivative δ_λ, the operator $L^*_{\delta_\lambda}: C^0_n(A) \to C^0_n(B)$, or $L^*_{\delta_\lambda}: \varprojlim \Omega_n\tilde{\bar{A}}_\lambda \to \varprojlim \Omega_n\tilde{\bar{B}}_\lambda$, defined by formula

$$L^*_{\delta_\lambda}(a^0_\lambda, \ldots, a^n_\lambda) = \sum_{j=0}^{n}(\varphi(a^0_\lambda), \ldots, \delta_\lambda a^j_\lambda, \ldots, \varphi(a^n_\lambda))$$

is called a *Lie derivative along* δ_λ. It is easy to check that the Lie derivative $L^*_{\delta_\lambda}$ commutes with the differentials; thus, $L^*_{\delta_\lambda}$ is a morphism of bicomplex.

Definition 5.3. *Let A and B be involutive Banach algebras and $\varphi_t: A \to B$, $0 \leq t \leq 1$ a one-parameter family of homomorphism between them, which are extended to $\varphi_t: \bar{A}_\lambda \to \bar{B}_\lambda$, for all $\lambda \in I$. Such a family will be called smooth, if*

(a) *each* $\varphi_t, t \in [0,1]$ *is a continuous homomorphism with uniformly bounded norms* $\|\varphi_t\| \leq M$,
(b) *for all* $a_\lambda \in A_\lambda$, *the map* $t \mapsto \varphi_t(a_\lambda)$ *from* $[0,1]$ *to* B_λ *is of class* C^1 *for every* $\lambda \in I$.

Moreover, we have the corresponding family of derivatives $\delta_t \colon \bar{A}_\lambda \to \bar{B}_\lambda$, for all $\lambda \in I$, by formula

$$\delta_t(a_\lambda) = \lim_{s \to 0} \frac{\varphi_{t+s}(a_\lambda) - \varphi_t(a_\lambda)}{s}.$$

It is easy to check the relations

$$\delta_t(a_\lambda b_\lambda) = \delta_t(a_\lambda)\varphi(b_\lambda) + \varphi(a_\lambda)\delta_t(b_\lambda).$$

This shows that δ_t is a derivation with respect to φ.

Theorem 5.3. *Let A and B be involutive Banach algebras,* $\varphi \colon A \to B$ *a continuous algebra homomorphism with a family of continuous derivation* $\delta_\lambda \colon A_\lambda \to B_\lambda$ *with respect to* φ, *extendable to* $\varphi \colon \bar{A}_\lambda \to \bar{B}_\lambda$ *and* $\delta_\lambda \colon \bar{A}_\lambda \to \bar{B}_\lambda$. *Let f be a normalized, reduced, entire cycle in* $C^0(\tilde{A})_{red}$, *then there exists a canonical reduced and entire chain* ψ *in* $C^0(\tilde{B})_{red}$, *such that* $L^*_{\delta_\lambda}(f) = \partial\psi$.

Proof. Using Theorem 5.1, it is easy to see that ψ is a reduced entire chain, such that $L^*_{\delta_\lambda}(f) = \partial\psi$. It is easy to check that the Lie derivation $L^*_{\delta_\lambda}$, associated to δ_λ, induces the zero homomorphism between the entire cyclic current homology groups $\mathrm{HE}^0_*(A)$ and $\mathrm{HE}^0_*(B)$. ■

Theorem 5.4. (Homotopy invariance) *Let A and B be involutive Banach algebras,* $\varphi_t \colon A \to B$, $t \in [0,1]$, *a smooth family of homomorphisms,* $\delta_\lambda \colon A_\lambda \to B_\lambda$ *the continuous derivation with respect to* φ_t. *Then* φ_0 *and* φ_1 *induce the same map between entire cyclic current homology groups.*

Proof. By our assumptions, there are constants M and N such that $\|\varphi\| \leq M$ and $\|\delta_\lambda\| \leq N$. Let $f = (f_n)_{n \geq 0}$ be an entire cycle, we can assume f to be normalized. Hence, we have the corresponding infinite-dimensional differential forms $\omega = (\omega_n)_{n \geq 0}$, such that

$$\begin{aligned}
\frac{d}{dt}\varphi_{t_*}(\omega_n) &= \frac{d}{dt}\varphi_{t_*}(a^0_\lambda da^1_\lambda \dots da^n_\lambda) \\
&= \sum_{j=0}^{n} (\varphi_t(a^0_\lambda, \dots, \delta_\lambda a^j_\lambda, \dots \varphi_t(a^n_\lambda)) \\
&= L^*_{\delta_\lambda}(\omega_n).
\end{aligned}$$

For every even infinite-dimensional cycle $\int$, we have

$$\begin{aligned}\int \frac{d}{dt}\varphi_{t_*}(f) &= \frac{d}{dt}\phi_n^{\int}(\varphi_t(a_\lambda^0),\dots,\varphi_t(a_\lambda^n))\\ &= \sum_{j=0}^{n}\phi_n^{\int}(\varphi_t(a_\lambda^0),\dots,\delta_\lambda a_\lambda^j,\dots,\varphi_t(a_\lambda^n))\\ &= L_{\delta_\lambda}\phi_n^{\int}(a_\lambda^0,\dots,a_\lambda^n).\end{aligned}$$

By Theorem 5.3, for each $t \in [0,1]$, there is a canonical entire chain $\psi^t = (\psi_n^t)_{n\geq 0}$, such that $L_{\delta_\lambda}^*(f) = \partial\psi^t$. We have

$$\begin{aligned}\varphi_{0*}(f) - \varphi_{1*}(f) &= \int_0^1 \frac{d}{dt}\varphi_{t_*}(f)dt\\ &= \int_0^1 L_{\delta_\lambda}^*(f)dt\\ &= \int_0^1 \partial\psi^t dt = \partial\int_0^1 \psi^t dt.\end{aligned}$$

It is easy to see that the integral exists and defines an entire chain. Indeed, because ψ^t is an entire chain and $\bar{\omega}^t = (\bar{\omega}^t)_{n\geq 0}$ is the corresponding smooth family of differential forms, for the fixed $a_\lambda^0,\dots,a_\lambda^n \in B_\lambda$, $\delta_t(a)$ is a continuous functions of t, and the integral $\int_0^1 \bar{\omega}^t\,dt$ exists and hence, the integral $\int_0^1 \psi^t\,dt$ defines the required chain.

It is easy to see that

$$\begin{aligned}\|\psi^t\| &\leq (n+2)(\|f_{2n}\| + \|f_{2n+2}\|)\|\varphi_t\|^{2n+1}\|\delta_\lambda\|\\ &\leq (n+2)(\|f_{2n}\| + \|f_{2n+2}\|)M^{2n+1}N,\end{aligned}$$

and there is also the similar formula for $\|\psi_{2n}^t\|$. We conclude finally that $(\int_0^1 \psi_n^t dt)_{n\geq 0}$ is an entire chain. The theorem is proved. ■

References

1. J. Cuntz and D. Quillen, Operators on noncommutative differential forms and cyclic homology, *Mathematische Institut Universitat Heidelberg*, 1992, p. 36, preprint.
2. J. Cuntz and D. Quillen, Algebra extensions and nonsingularity, *Mathematische Institut Universitat Heidelberg*, 1992, p. 43, preprint.
3. M. Khalkhali, On the entire cyclic cohomology of Banach algebras I: Morita invariance, *Mathematische Institut Universitat Heidelberg*, Forsclunggruppe "Topologie und nichtkommutative Geometrie", Heft 54, 1992, p. 24.
4. M. Khalkhali, On the entire cyclic cohomology of Banach algebras II: Homotopy invariance, *Mathematische Institut Universitat Heidelberg*, Forschunggruppe "Topologie und nichtkommutative Geometrie", Heft 55, 1992, p. 18.