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Short Communication

On Non-Linear Approximation of Multivariate Functions with a Mixed Smoothness

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Let W be a compact subset in the normed linear space X. The non-linear n-width $\delta_n(W,X)$ [1] is defined by

$$\delta_n(W,X) := \inf_{F,M} \sup_{x\in W} \|x - M(F(x))\|,$$

where the infimum has taken over all continuous mappings F from W into $\mathbf{R}^n$ and all continuous mappings M from $\mathbf{R}^n$ into X. This non-linear n-width characterizes the best method of n-parametrized non-linear approximations. Such approximations are the rational approximation, approximation by splines with free knots, one-sided approximation, wavelet compression, etc.

The non-linear n-width δ_n is very close to the well-known Aleksandrov n-width

$$a_n(W,X) := \inf_{F,K} \sup_{x\in W} \|x - F(x)\|,$$

where the infimum has taken over all compact sets $K \subset X$ of topological dimensions $\le n$ and all continuous mappings from W into K, namely, for any compact subset W in the Banach space X, there hold the following inequalities [4]

$$\delta_{2n+1}(W,X) \le a_n(W,X) \le \delta_n(W,X).$$

The widths δ_n and a_n of classes of univariate and multivariate smooth functions were investigated in [1–6] (see [4] also for a brief historical survey on results in this direction).

In the present note we give some estimates of the asymptotic degrees of δ_n and a_n of classes of multivariate functions with a bounded mixed derivative or with a bounded difference.

Let $L_p(\mathbf{T}^d)$, $1 < p < \infty$, be the Banach space of functions f defined in the d-dimensional torus $\mathbf{T}^d := [-\pi, \pi]^d$, equipped by the norm

$$\|f\|_{L_p(\mathbf{T}^d)} := \left((2\pi)^{-d} \int_{\mathbf{T}^d} |f(x)|^p \, dx\right)^{1/p}.$$

In the classes of smooth functions on $\mathbf{T}^d$ which will be defined below, for the sake of simplicity, we consider only functions f with zero mean value at each variable, i.e.,

$$\int_{-\pi}^{\pi} f(x) \, dx_j = 0, \quad j = 1, \ldots, d.$$

For $\alpha \in \mathbf{R}^d$ with $\alpha_j > 0$, let BW_p^α be the set of all functions f on $\mathbf{T}^d$ such that the mixed fractional derivative in the sense of Weil $f^{(\alpha)}$ satisfies the condition

$$\|f^{(\alpha)}\|_{L_p(\mathbf{T}^d)} \leq 1,$$

and let BH_p^α be the set of all functions f on $\mathbf{T}^d$ such that the mixed higher-order difference Δ_h^r, $h \in \mathbf{T}^d$ satisfies the condition

$$\|\Delta_h^r f\|_{L_p(\mathbf{T}^d)} \leq \prod_{j=1}^{d} |h_j|^{\alpha_j}$$

for some $r \in \mathbf{N}^d$ with $r_j > \alpha_j$. Without loss of generality, we can assume that

$$\alpha_1 = \cdots = \alpha_r = \alpha_{r+1} < \alpha_{r+2} \leq \cdots \leq \alpha_d$$

for some non-negative integer r smaller than d.

Theorem. *Let* $1 < p, q < \infty$ *and* $\alpha \in \mathbf{R}^d$ *be a vector such that* $\alpha_j > \max\{0, 1/p - 1/q, 1/p - 1/2\}$. *Then we have*
(i) $a_n(BW_p^\alpha, L_q(\mathbf{T}^d)) \asymp \delta_n(BW_p^\alpha, L_q(\mathbf{T}^d)) \asymp n^{-\alpha_1} (\log n)^{r\alpha_1}$.
(ii) $a_n(BH_p^\alpha, L_q(\mathbf{T}^d)) \asymp \delta_n(BH_p^\alpha, L_q(\mathbf{T}^d)) \asymp n^{-\alpha_1} (\log n)^{r(\alpha_1 + (1/2))}$.

This theorem shows that as in the univariate case, the asymptotic degree of δ_n and a_n of the class BW_p^α and BH_p^α in the space $L_q(\mathbf{T}^d)$ does not depend on the relation between p and q. To prove the results of the present note we use particularly the Littlewood-Palley theorem, Lemma 2.4 of [4], methods of discretization and Aleksandrov widths of finite-dimensional sets [6].

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