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Short Communication

An Elementary Proof of a Theorem of Izumiya–Marar

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Recently, Izumiya and Marar [2] proved the following.

Theorem 1. *Let N be a closed 2-manifold, P a 3-manifold, and $f : N \to P$ a smooth stable mapping, then we have the following formula*

$$\chi(f(N)) = \chi(N) + T(f) + \frac{1}{2}C(f),$$

where $T(f)$ is the number of triple points of f and $C(f)$ the number of cross-caps.

This result is interesting because it relates the topology of the image to that of the source space, while most results in topology concern only one space. This result was later reformulated in terms of a generic wavefront in a 3-manifold [3]. But it is readily seen that the theorem given above implies that in [3] because a generic wavefront in a 3-manifold is nothing else but the image of a stable mapping on a closed surface. The authors of [2, 3] used a method introduced in [4] to prove their results. This paper gives an elementary and intuitive proof of the above theorem.

Proof. The image of a stable mapping from a closed surface to a 3-manifold has only three types of singularities: cross-caps (Fig. 1), normal crossing (Fig. 2), and triple points (Fig. 3) (see [1]). Their preimages consist of one, two, and three points, respectively. Let $A \subset f(N)$ be the set of all singular points, then A is closed and $f : N - f^{-1}(A) \to f(N) - A$ is a diffeomorphism. Now, let $X \subset A$ be the set of cross-caps and Y the set of triple points. Then $A - (X \cup Y)$ decomposes into connected components with or without boundaries, i.e., the normal crossing segments and the normal crossing circles. Denote the set of normal crossing segments by Z and the set of normal crossing circles by W. Because N is compact, X, Y, Z, and W are finite. Denote $a = \#X = C(f)$, $b = \#Y = T(f)$, $c = \#Z$, and $d = \#W$.

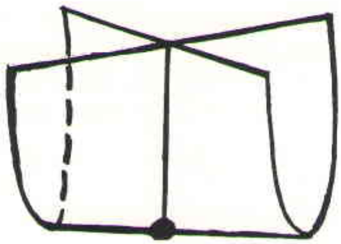

Fig. 1

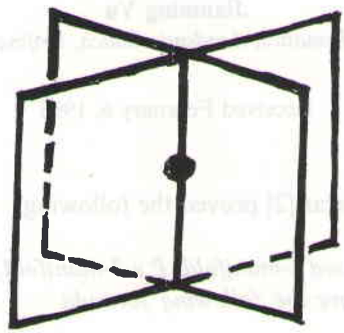

Fig. 2

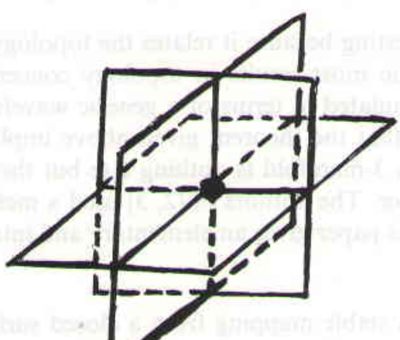

Fig. 3

Now we construct a triangulation on $f(N)$. If W is not empty, we take two points from each normal crossing circle, thus getting a set S. Note that $\# S = 2d$ and $W - S$ consists of $2d$ normal crossing segments. Now, we take $X \cup Y \cup S$ to be the vertices and $Z \cup (W - S)$ to be 1-simplexes. If $f(N) - A$ decomposes into

2-simplexes, we are finished. If this is not the case, we may add as many points and segments in $f(N) - A$ as necessary to get a triangulation on $f(N)$. Suppose the number of points added is r, the number of segments added is s, and the number of 2-simplexes in the resulting triangulation is t. Then we have

$$\chi(f(N)) = (a + b + 2d + r) - (c + 2d + s) + t$$
$$= (a + b - c) + (r - s + t).$$

Now, it is obvious that f induces a triangulation on N by pull-pack of the triangulation on $f(N)$. In this induced triangulation, there are $a + 3b + 4d + r$ vertices, $2c + 4d + s$ 1-simplexes, and t 2-simplexes. Hence, we have

$$\begin{aligned}\chi(N) &= (a + 3b + 4d + r) - (2c + 4d + s) + t \\ &= (a + 3b - 2c) + (r - s + t) \\ &= \chi(f(N)) + 2b - c. \qquad (1)\end{aligned}$$

Each normal crossing segment in Z has two end points. The two points are different because of stability. They are either cross-caps or triple points. Note that a cross-cap belongs to only one segment, while a triple point belongs to six segments. Therefore, the following relation holds:

$$2c = a + 6b. \qquad (2)$$

From (1) and (2), the formula is proved. ∎

Remark. Using the same idea, one can easily prove another result in [2].

Theorem 2. *Let N be a closed n-manifold, P a $(2n-1)$-manifold, $n \geq 3$. If $f\colon N \to P$ is a smooth stable mapping, then we have*

$$\chi(f(N)) = \chi(N) + \frac{1}{2}C(f),$$

where $C(f)$ is the number of cross-caps of f.

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References

1. M. Golubitsky and V. Guillemin, *Stable Mappings and Their Singularities*, Springer-Verlag, New York, 1973.
2. S. Izumiya and W. L. Marar, The Euler characteristic of the image of a stable mapping from a closed n-manifold to a $(2n-1)$-manifold, *Preprint Series in Mathematics*, Vol. 138, Hokkaido University, Japan, 1992.
3. S. Izumiya and W. L. Marar, The Euler characteristic of a generic wavefront in a 3-manifold, *Proceedings of the AMS* **118** (4) (1993) 1347–1350.
4. W. L. Marar, The Eucler characteristic of the disentanglement of the image of a corank 1 map germ, *Singularity Theory and its Applications*, D. Mond and J. Montaldi (eds.), Lecture Notes in Mathematics, vol. 1462, Springer-Verlag, 1991, pp. 212–220.