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An Embedding Theorem of a $\mathscr{P}$-Regular Semigroup

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Abstract. In this paper, an embedding theorem of a $\mathscr{P}$-regular semigroup is given in terms of a $\mathscr{C}$-partial band and a regular *-semigroup.

1. Introduction

Yamada and Sen introduced the concept of $\mathscr{P}$-regularity [6] in a regular semigroup as a generalization of both the concept of "orthodox" and the concept of "(special) involution" [5]. In the area of $\mathscr{P}$-regular semigroups, which showed some popularity almost from its beginning, there is a large body of the semigroup literature. In the previous paper for $\mathscr{P}$-regular semigroups, such as [6–12], we have seen that the $\mathscr{C}$-set of a $\mathscr{P}$-regular semigroup plays an important part in studies on $\mathscr{P}$-regular semigroups. In [10], Zheng introduced the concept of $\mathscr{C}$-partial bands and gave an analog of the Hall semigroup of a band. In [9, 11], the author discussed further the properties of $\mathscr{C}$-partial bands. In this paper, an embedding theorem of a $\mathscr{P}$-regular semigroup is given in terms of a $\mathscr{C}$-partial band and a regular *-semigroup. Unless otherwise defined, our notation will be that of [1, 2].

2. Preliminary Results and Definitions

Let S be a regular semigroup and E_S the set of idempotents of S. Let $P \subseteq E_S$. Then (S, P) is called a *$\mathscr{P}$-regular semigroup* if it satisfies the following:

(1) $P^2 \subseteq E_S$,

(2) for any $q \in P$, $qPq \subseteq P$,

(3) for any $a \in S$, there exists $a^+ \in V(a)$ such that $aP^1a^+ \subseteq P$ and $a^+P^1a \subseteq P$ ($P^1 = P \cup \{1\}$ and $V(a)$ denotes the set of all inverses of a).

Hereafter, (S, P) will be denoted by $S(P)$. A subset P of E_S satisfying (1)–(3) is called a C-set of S. In (3), a^+ is called a *$\mathscr{P}$-inverse* of a and $V_{\mathscr{P}}(a)$ denotes the set of all *$\mathscr{P}$-inverses* of a. The class of $\mathscr{P}$-regular semigroups thus includes both the class of orthodox semigroups and the class of regular *-semigroups, which has been first shown in [6, 8], and the terminology "regular *-semigroup" has appeared in [3, 4].

Let $S_1(P_1)$ and $S_2(P_2)$ be $\mathscr{P}$-regular semigroups. A homomorphism f of $S_1(P_1)$ into $S_2(P_2)$ is called a *$\mathscr{P}$-homomorphism* if $P_1 f = P_2 \cap S_1(P_1)f$. A $\mathscr{P}$-homomorphism $f: S_1(P_1) \to S_2(P_2)$ is called a *$\mathscr{P}$-isomorphism* if, f is bijective, in such a case, denoted by $S_1(P_1) \overset{\mathscr{P}}{\cong} S_2(P_2)$. The kernel of a $\mathscr{P}$-homomorphism f of $S_i(P_1)$ into $S_2(P_2)$ means the congruence $\ker f$ on $S_1(P_1)$ induced by f, that is, $(x,y) \in \ker f$ if and only if $xf = yf$.

Let E be a partial groupoid. We shall use notation $\exists ef$ if the product ef of $e, f \in E$ is defined in E. A partial groupoid E is called a partial band if it satisfies the following axioms:

(1) Let $e, f, g \in E$, $\exists ef$, $\exists fg$. If one of $(ef)g$ and $e(fg)$ is defined in E, so are both of them, and $(ef)g = e(fg)$, denoted by efg.

(2) for any $e \in E$, $ee = e$.

Let E be a partial band, $P \subseteq E$. $E(P)$ is called a $\mathscr{P}$-partial band, if it satisfies the following:

(P1) for any $q, p \in P$, $\exists qp$;

(P2) for any $q, p \in P$, $\exists qpq$, and $qpq \in P$. For any $q_1, \ldots, q_n$, $e \in P$, $q_1(\cdots(q_{n-1}(q_n e q_n)q_{n-1})\cdots)q_1$ is denoted by $q_1 \cdots q_n e q_n \cdots q_1$;

(P3) for any $q, x, e \in P$, $(qxq)e(qxq) = qxqeqxq$.

A partial band $(P, .)$ is called a *$\mathscr{C}$-partial band* if there exists a $\mathscr{P}$-partial band $(E(P), \circ)$ such that the restriction of $\circ$ to P is right. For example, if $S(P)$ is a $\mathscr{P}$-regular semigroup, then $E(P)$ is a $\mathscr{P}$-partial band, and the C-set P is a $\mathscr{C}$-partial band, where $E = E_S$, and any band is a $\mathscr{C}$-partial band.

Let P_1 and P_2 be two $\mathscr{C}$-partial bands. A partial isomorphism θ from P_1 onto P_2 is called a strong isomorphism, if, for any $x, y \in P_1$, $(xyx)\theta = (x\theta)(y\theta)(x\theta)$. If there exists a strong isomorphism $\theta: P_1 \to P_2$ such that $\theta^{-1}: P_2 \to P_1$ is also a strong isomorphism, then P_1 and P_2 are said to be *isomorphic* and are denoted by $P_1 \cong P_2$.

We may find the notation above from [10].

Now let P be a $\mathscr{C}$-partial band. If $p \in P$, then $pPp = \{pep: e \in P\}$, which is denoted by $\langle p \rangle$. Let $\mathscr{U} = \{(q,p) \in P \times P : \langle q \rangle\}$. If $(q,p) \in \mathscr{U}$, then $H_{q,p}$ denotes the set of all strong isomorphisms from $\langle q \rangle$ onto $\langle p \rangle$ such that the inverse mappings are strong isomorphisms from $\langle p \rangle$ onto $\langle q \rangle$.

If $(q,p) \in \mathscr{U}$ and $\theta \in H_{q,p}$, we define $\theta_l \in \mathscr{PT}(P/\mathscr{L})$ (the semigroup of partial mappings of set $P/\mathscr{L}$) and $\theta_r \in \mathscr{PT}(P/\mathscr{R})$ by the formulae

$$L_x\theta_l = L_{x\theta},\ R_x\theta_r = R_{x\theta},\ (x \in \langle q \rangle).$$

Define

$$H_P = \{(\rho_q\theta_l, \lambda_p\theta_r^{-1}) \colon \theta \in H_{q,p}, (q,p) \in \mathscr{U}\},$$

where $\rho_q \in \mathscr{T}(P/\mathscr{L})$ (the semigroup of all transformations on $P/\mathscr{L}$) and $\lambda_p \in \mathscr{T}(P/\mathscr{R})$ are defined by

$$L_x\rho_q = L_{qxq},\ R_x\lambda_p = R_{pxp},\ (x \in P).$$

Lemma 1. [10, Result 1] *H_P is a subsemigroup of $\mathscr{T}(P/\mathscr{L}) \times \mathscr{T}^*(P/\mathscr{R})$, and H_P is $\mathscr{P}$-regular, with the C-set $P^* = \{(\rho_q, \lambda_q)\colon q \in P\}$ isomorphic to P, where the mapping $\alpha\colon P^* \to P$ defined by $(\rho_q, \lambda_q)\alpha = q$ is a strong isomorphism.*

We use the terminology *$\mathscr{P}$-congruence* for a usual congruence ρ on a $\mathscr{P}$-regular semigroup $S(P)$. If a $\mathscr{P}$-congruence ρ on $S(P)$ satisfies the following:

for any $q \in P$ and $a \in S$, $q\rho a$ implies $q\rho a^+$ for all $a^+ \in V_P(a)$,

then ρ is called a strong $\mathscr{P}$-congruence [8].

Lemma 2. [8, Theorem 4.6] *Let $S(P)$ be a $\mathscr{P}$-regular semigroup. Then the transitive closure γ^* of the relation γ defined by*

$$\gamma = \{(a, b) \in S \times S\colon V_P(a) \cap V_P(b) \neq \emptyset\}$$

is the least strong $\mathscr{P}$-congruence on $S(P)$.

Let $S(P)$ be a $\mathscr{P}$-regular semigroup. We define, for each $a \in S$, $\rho_a \in \mathscr{T}(P/\mathscr{L})$, and $\lambda_a \in \mathscr{T}(P/\mathscr{R})$

$$L_x\rho_a = L_{a^+xa}, \qquad R_x\lambda_a = R_{axa^+} (x \in P),$$

where a^+ is an arbitrary $\mathscr{P}$-inverse of a.

Lemma 3. [11, Theorem 4.1] *Let $S(P)$ be a $\mathscr{P}$-regular semigroup and ξ the mapping from $S(P)$ into $\mathscr{T}(P/\mathscr{L}) \times \mathscr{T}^*(P/\mathscr{R})$ defined by $a\xi = (\rho_a, \lambda_a)$. Then ξ is a homomorphism whose kernel is the maximum idempotent-separating congruence μ on $S(P)$.*

Lemma 4. [7, Lemma III.1.2] *Let $S(P)$ be a $\mathscr{P}$-regular semigroup and γ, μ the least strong $\mathscr{P}$-congruence on $S(P)$ and an idempotent-separating congruence on $S(P)$, respectively. Then $\gamma \cap \mu = 1_S$.*

Lemma 5. [7, Proposition II.2.1(1)] *Let $S(P)$ be a $\mathscr{P}$-regular semigroup and γ the least strong $\mathscr{P}$-congruence on $S(P)$. Then, for any $e \in E_S$, $e\gamma a$, $a \in S(P)$ implies $a \in E_S$.*

3. Embedding Theorem

Let P be a $\mathscr{C}$-partial band and T a regular *-semigroup with the set of projections P_1. Let γ_1 be the minimum strong $\mathscr{P}$-congruence on H_P. Then H_P/γ_1 is a regular *-semigroup with the set of projections $P^*/\gamma_1 = \{(\rho_q, \lambda_q)\gamma_1\colon q \in P\}$. Let $\Psi\colon T \to H_P/\gamma_1$ be a $\mathscr{P}$-homomorphism whose range contains all projections of H_P/γ_1. We denote the spined product

$$S = \{(x, t) \in H_P \times T\colon x\gamma_1^\natural = t\Psi\}$$

of H_P and T with respect to H_P/γ_1, $\gamma_1^\natural$ and Ψ by $\mathscr{Z}(P, T, \Psi)$.

Theorem 1.
(1) *$S = \mathscr{Z}(P, T, \Psi)$ is a $\mathscr{P}$-regular semigroup having $\{((\rho_q, \lambda_q), p) \in P^* \times P_1: (\rho_q, \lambda_q)\gamma_1 = p\Psi\}$ as its C-set.*
(2) *If Ψ is an idempotent-separating $\mathscr{P}$-homomorphism, then the C-set of S is isomorphic to P.*
(3) *If γ is the minimum strong $\mathscr{P}$-congruence on S, then $S/\gamma \overset{\mathscr{P}}{\cong} T$.*

Proof. (1) Obviously, $H_P \times T$ is regular. If $(x, t) \in S$, then $x\gamma_1^\sharp = t\Psi$. Taking $x^+ \in V_{P^*}(x)$, denoting a unique $\mathscr{P}$-inverse of t in T by $t^\sharp$, it easily follows that $x^+\gamma_1^\sharp \in V_{P^*/\gamma_1}(x\gamma_1^\sharp))$. Since the range of Ψ contains the projections of H_{P/γ_1} and since Ψ is a $\mathscr{P}$-homomorphism, $P_1\Psi = P^*/\gamma_1$. Thus, $t^\sharp\Psi \in V_{P^*/\gamma_1}(t\Psi)$. Therefore, $x^+\gamma_1^\sharp$ and $t^\sharp\Psi$ are both $\mathscr{P}$-inverses of the single element $x\gamma_1^\sharp = t\Psi$ of the regular *-semigroup H_{P/γ_1}, which implies that $x^+\gamma_1^\sharp = t\Psi$, and so $(x^+, t^\sharp) \in S$. Thus, S is regular.

Let

$$\bar{P} = \{((\rho_q, \lambda_q), p) \in P^* \times P_1: (\rho_q, \lambda_q)\gamma_1 = p\Psi\}.$$

Then $\bar{P} \subseteq E_S$. We easily deduce that $\bar{P}^2 \subseteq E_S$ and $\bar{q}\bar{P}^1\bar{q} \subseteq \bar{P}$ for any $\bar{q} \in \bar{P}$. By the proof above, we have $V_{\bar{P}}(x, t) \supseteq V_{P^*}(x) \times \{t^\sharp\}$ for any $(x, t) \in S$. Thus, $S(\bar{P})$ is $\mathscr{P}$-regular. Since $P^*/\gamma_1 = P_1\Psi$ and T is a regular *-semigroup, $V_{\bar{P}}(x, t) \subseteq V_{P^*}(x) \times \{t^\sharp\}$. Hence, $V_{\bar{P}}(x, t) = V_{P^*}(x) \times \{t^\sharp\}$.

(2) By Lemma 1, the mapping α: $P^* \to P$ defined by $(\rho_q, \lambda_q)\alpha = q$ is a strong isomorphism. We may show that the mapping ζ: $\bar{P} \to P$ defined by $(x, t)\zeta = x\alpha$ is a strong isomorphism. To prove that ζ^{-1} is also strong, let $(x, t)\zeta(y, u)\zeta \in P$, where $(x, t), (y, u) \in \bar{P}$. Then $x\alpha.y\alpha \in P$ and $(xy)\gamma_1^\sharp = (tu)\gamma_1^\sharp$. Since α is a partial isomorphism, $(x\alpha.y\alpha)\alpha = xy$. Thus, $xy \in P^*$ and so $(xy)\gamma_1^\sharp$ is a projection of H_{P/γ_1}. Hence, there exists $v \in P_1$ such that $(xy)\gamma_1^\sharp = v\Psi$, so $(tu)\Psi = v\Psi$. But $tu \in E_T$ and so, since Ψ is idempotent-separating, $tu = v \in P_1$. Hence, $(xy, tu) \in \bar{P}$ and we conclude that $\bar{P}$ and P are isomorphic.

(3) Obviously, the mapping $\pi : (x, t) \mapsto t$ is a $\mathscr{P}$-homomorphism from S onto T. If γ is the minimum strong $\mathscr{P}$-congruence on S, then $\gamma \subseteq \ker\pi$, and so there exists a $\mathscr{P}$-homomorphism f from S/γ onto T such that $\gamma^\sharp f = \pi$. To prove that $\gamma = \ker\pi$, we only need to show that $\ker\pi \subseteq \gamma$. Let $(x, t), (y, u) \in S$, satisfying $((x, t), (y, u)) \in \ker\pi$. Then $(x, t)\pi = (y, u)\pi$, i.e., $t = u$ and so $t\Psi = u\Psi$. Hence, $x\gamma_1 = y\gamma_1$, that is, $(x, y) \in \gamma_1$. Thus, by Lemma 2, there exist $x_0 = x, x_1, \ldots, x_n = y \in H_P$ such that $V_{P^*}(x_i) \cap V_{P^*}(x_{i+1}) \neq \emptyset$, $i = 0, 1, \ldots, n-1$. Since $x_0\gamma_1 = x\gamma_1 = t\Psi$, $x_1\gamma_1 = \cdots = x_{n-1}\gamma_1 = t\Psi$, which implies that $(x, t), \ldots, (x_{n-1}, t) \in S$. But

$$(V_{P^*}(x_i) \times \{t^\sharp\}) \cap (V_{P^*}(x_{i+1}) \times \{t^\sharp\}) \neq \emptyset$$

i.e., $V_{\bar{P}}(x_i, t) \cap V_{\bar{P}}(x_{i+1}, t) \neq \emptyset$, $i = 0, 1, \ldots, n-1$. Hence, $((x, t), (y, u)) \in \gamma$. Thus, f: $S/\gamma \to T$ is a $\mathscr{P}$-isomorphism.

Let $S(P)$ be a $\mathscr{P}$-regular semigroup and $a \in S$, $a^+ \in V_P(a)$. Denote $q = aa^+$, $p = a^+a$. Then it easily follows that $\rho a = \rho_q\theta_l$, where ρ_a is as in Lemma 3 and θ is the mapping $\theta : x \mapsto a^+xa$ from $\langle q\rangle$ onto $\langle p\rangle$. Dually, $\lambda_a = \lambda_p\theta_r^{-1}$. Hence, the homomorphism $\xi : a \mapsto (\rho_a, \lambda_a)$ in Lemma 3 is a $\mathscr{P}$-homomorphism from $S(P)$ to H_P. ■

Proposition 2. *The mapping $\eta : S \to H_P \times S/\gamma$ defined by $a\eta = ((\rho_a, \lambda_a), a\gamma)$ is an injective $\mathscr{P}$-homomorphism.*

Proof. By Lemma 4, $\gamma \cap \mu = 1_S$. However, by Lemma 3, since $\mu = \ker \xi$, η is an injective homomorphism. Clearly, $H_P \times S/\gamma$ is a $\mathscr{P}$-regular semigroup with the C-set $P^* \times P/\gamma$, where $P/\gamma = \{q\gamma\colon q \in P\}$. It easily follows that $P\eta \subseteq (P^* \times P/\gamma) \cap S\eta$. Conversely, if $((\rho_q, \lambda_q), p\gamma) \in (P^* \times P\gamma) \cap S\eta$, then $q, p \in P$, and there exists $a \in S$ such that $a\eta = ((\rho_q, \lambda_q), p\gamma)$. Thus, $(\rho_a, \lambda_a) = (\rho_q, \lambda_q)$ and $a\gamma = p\gamma$. Since $p \in P$, by Lemma 5, $a \in E_S$. But $(a, q) \in \ker \xi = \mu$, we have that $a = q$. Hence, $((\rho_q, \lambda_q), p\gamma) = q\eta \in P\eta$. Therefore, η is a $\mathscr{P}$-homomorphism. ■

If γ_1 is the minimum strong $\mathscr{P}$-congruence on H_P, then $\xi\gamma_1^\sharp$ is a $\mathscr{P}$-homomorphism from S into H_P/γ_1.

Proposition 3. *The mapping $\alpha : S/\gamma \to H_P/\gamma_1$ defined by $(a\gamma)\alpha = a\xi\gamma_1^\sharp$ is a $\mathscr{P}$-homomorphism whose range contains the projections of H_{P/γ_1}.*

Proof. If $a\gamma = b\gamma$, then there exist $a_0 = a, a_1, \ldots, a_n = b$ such that $V_P(a_i) \cap V_P(a_{i+1}) \neq \emptyset$, $i = 0, 1, \ldots, n-1$. Taking $b_i \in V_P(a_i) \cap V_P(a_{i+1})$, then it easily follows that

$$(\rho_{b_i}, \lambda_{b_i}) \in V_{P^*}(\rho_{a_i}, \lambda_{a_i} \cap V_{P^*}(\rho_{a_{i+1}}, \lambda_{a_{i+1}}) \neq \emptyset,\ i = 0, 1, \ldots, n-1.$$

Thus, $(\rho_a, \lambda_a)\gamma_1 = (\rho_b, \lambda_b)\gamma_1$, that is $a\xi\gamma_1^\sharp = b\xi\gamma_1^\sharp$, and so α is a mapping. It is easy to see that α is a $\mathscr{P}$-homomorphism whose range contains the projections of H_{P/γ_1}. ■

Theorem 4 (Embedding theorem). *Let $S(P)$ be a $\mathscr{P}$-regular semigroup. Then $S(P)$ can be embedded into $\mathscr{L}(P, S/\gamma, \alpha)$.*

Proof. Since $S(P)$ is $\mathscr{P}$-regular, P is a $\mathscr{C}$-partial band. Since S/γ is a regular *-semigroup with projections $P/\gamma = \{q\gamma\colon q \in P\}$, it follows from Theorem 1(1) that the spined product $\mathscr{L}(P, S/\gamma, \alpha)$ of H_P and S/γ with respect to H_{P/γ_1}, $\gamma_1^\sharp$ and α is a $\mathscr{P}$-regular semigroup with the C-set $\bar{P} = \{((\rho_q, \lambda_q), p\gamma) \in P^* \times P/\gamma\colon (\rho_q, \lambda_q)\gamma_1 = p\gamma\alpha\}$. Since, for any $((\rho_a, \lambda_a), a\gamma) \in \operatorname{ran}(\eta)$ in Proposition 2, we have that $(\rho_a, \lambda_a)\gamma_1 = a\xi\gamma_1 = a\gamma\alpha$, $((\rho_a, \lambda_a), a\gamma) \in \mathscr{L}(P, S/\gamma, \alpha)$. It follows from Proposition 2 that $\eta\colon S \to \mathscr{L}(P, S/\gamma, \alpha)$ is an injective $\mathscr{P}$-homomorphism. ■

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