Vietnam Journal of Mathematics **25**:2 (1997) 133–140

Vietnam Journal
of
MATHEMATICS

On Invertibility of Linear Subspaces Generating Clifford Algebras*

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Received November 27, 1995

Abstract. Let $\mathscr{A}$ be a universal Clifford algebra induced by an m-dimensional real linear space. A linear subspace L of $\mathscr{A}$ is said to be invertible if every nonzero element of it is invertible. In this paper, we obtain the necessary and sufficient condition for some subspaces of $\mathscr{A}$ to be invertible. A generalized Cauchy–Riemann operator, which linearizes the Laplace operator is presented.

1. Symbol of Clifford Numbers

Let V_m be an m-dimensional $(m \geq 1)$ real linear space with a basis $\{e_1, \ldots, e_m\}$. Consider the 2^m-dimensional real linear space $\mathscr{A}$ with a basis

$$E = \{e_\emptyset, e_1, \ldots, e_m, e_{12}, \ldots, e_{m-1m}, \ldots, e_{12\ldots m}\}.$$

A product of two elements $e_A, e_B \in E$ is given by

$$e_A e_B = (-1)^{\sharp(A\cap B)}(-1)^{p(A,B)} e_{A\vartriangle B}; \quad A, B \subset \{1, 2, \ldots, m\}, \tag{1}$$

where

$$\begin{cases} p(A,B) = \sum\limits_{j\in B} p(A,j), \\ p(A,j) = \sharp\{i \in A: i > j\}, \\ A \vartriangle B = (A\backslash B) \cup (B\backslash A), \end{cases}$$

and $\sharp A$ denotes the number of elements of A. Clearly,

$$p(A+B) + p(B+A) + \sharp(A\cap B) = \sharp(A\times B). \tag{2}$$

Every element $a = \sum\limits_A a_A e_A$ in $\mathscr{A}$ is called a Clifford number. The product of two

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* This work was supported in part by the National Basic Research Programme in Natural Science, Vietnam.

Clifford numbers $a = \sum_A a_A e_A$, $b = \sum_B b_B e_B$ is defined by

$$ab = \sum_A \sum_B a_A b_B e_A e_B.$$

It is an easy matter to check that, in this way, $\mathscr{A}$ becomes a linear, associative, noncommutative algebra over $\mathbb{R}$. It is called the Clifford algebra over V_m.

It follows from the multiplication rule (1) that $e_\emptyset$ is the identity element and that

$$e_i e_j + e_j e_i = 0 \quad \text{for } i \neq j, \quad e_j^2 = -1 \quad (j = 1, 2, \ldots, m),$$

$$e_{k_1 k_2 \ldots k_t} = e_{k_1} e_{k_2} \ldots e_{k_t}, \quad 1 \leq k_1 < k_2 < \cdots < k_t \leq m.$$

The involution for basic vectors is given by

$$\bar{e}_A = \bar{e}_{k_1 \ldots k_s} = (-1)^{s(s+1)/2} e_{k_1 \ldots k_s}.$$

For any $a = \sum_A a_A e_A \in \mathscr{A}$, let $\bar{a} = \sum_A a_A \bar{e}_A$.

Setting $e_0 = e_\emptyset$ and reindexing the vectors $\{e_{12}, \ldots, e_{m-1m}, \ldots, e_{12\ldots m}\}$ in the basis E, we rewrite

$$E = \{e_0, e_1, e_2, \ldots, e_m, e_{m+1}, \ldots, e_{n-1}\},$$

where $n = 2^m$.

Let $a = \sum_{i=0}^{n-1} a_i e_i$. Denote by $\sigma(a)$ an $(n \times n)$-matrix with elements in the set $\{\pm a_0, \pm a_1, \ldots, \pm a_{n-1}\}$ defined by the formula

$$ax = (e_0, e_1, \ldots, e_{n-1}) \sigma(a) (x_0, x_1, \ldots, x_{n-1})^T, \text{ for all } x = \sum_{j=0}^{n-1} x_j e_j \in \mathscr{A}, \tag{3}$$

where M^T denotes the transpose of the matrix M.

Definition 1. *$\sigma(a)$ defined by* (3) *is called the symbol of a. The set of all symbols of Clifford numbers is denoted by $\Sigma(\mathscr{A})$.*

We therefore obtain a matrix representation of $\mathscr{A}$. This representation is an isomorphism between $\mathscr{A}$ and $\Sigma(\mathscr{A})$, and $\sigma(e_0) = I$, where I is the identity $(n \times n)$-matrix. It is easy to see that $\Sigma(\mathscr{A}) \not\subseteq \mathscr{M}_n$ (the algebra of real $(n \times n)$-matrices).

We now find symbols of basic vectors $e_k \in \mathscr{A}$ $(0 \leq k \leq n-1)$.

Let k be fixed, $\sigma(e_k) = (\lambda_{ij})$, and $x = \sum_{j=0}^{n-1} x_j e_j$. Then, by (3) we have

$$\sum_{j=0}^{n-1} (e_k e_j) x_j = e_k x = \sum_{j=0}^{n-1} \left(\sum_{i=0}^{n-1} e_i \lambda_{ij} \right) x_j.$$

Since x is arbitrarily chosen, this implies

$$e_k e_j = \sum_{i=0}^{n-1} e_i \lambda_{ij} \quad \text{for all } j \in \{0, \ldots, n-1\}.$$

Hence, $e_k = \sum_{i=0}^{n-1} \lambda_{ij}(e_i \bar{e}_j)$, i.e.,

$$\lambda_{ij} = \begin{cases} 1, & \text{if } e_k = e_i \bar{e}_j, \\ -1, & \text{if } e_k = -e_i \bar{e}_j, \\ 0, & \text{otherwise.} \end{cases} \tag{4}$$

Lemma 1. *$\sigma(ab) = \sigma(a)\sigma(b)$ and $\sigma(a+b) = \sigma(a) + \sigma(b)$ for all $a, b \in \mathscr{A}$.*

Proof. By Definition 1, we have

$$(ab)x = (e_0, e_1, \ldots, e_{n-1})\sigma(ab)(x_0, x_1, \ldots, x_{n-1})^T.$$

On the other hand,

$$(ab)x = a(bx) = (e_0, e_1, \ldots, e_{n-1})\sigma(a)(\eta_0, \eta_1, \ldots, \eta_{n-1})^T,$$

where

$$(\eta_0, \eta_1, \ldots, \eta_{n-1})^T = \sigma(b)(x_0, x_1, \ldots, x_{n-1})^T.$$

Hence,

$$(e_0, e_1, \ldots, e_{n-1})\sigma(ab)(x_0, x_1, \ldots, x_{n-1})^T$$

$$\equiv (e_0, e_1, \ldots, e_{n-1})\sigma(a)\sigma(b)(x_0, x_1, \ldots, x_{n-1})^T,$$

which gives $\sigma(ab) = \sigma(a)\sigma(b)$. The second relation is checked in the same way. ∎

Corollary 1. *Every one-sided, invertible Clifford number is invertible.*

Proof. Let $a \in \mathscr{A}$ be left invertible and let $a^{(l)}$ be its left inverse, i.e., $aa^{(l)} = e_0$. Then

$$I = \sigma(e_0) = \sigma(aa^{(l)}) = \sigma(a)\sigma(a^{(l)}).$$

Hence, $\sigma(a)\sigma(a^{(l)}) = I$ and this follows that $\sigma(a^{(l)})\sigma(a) = I$, which gives

$$aa^{(l)} = a^{(l)}a = e_0.$$ ∎

Lemma 2. (See [1]) $(\sigma(a))^T = \sigma(\bar{a})$.

Proof. Write $\sigma(e_k) = (\sigma_{ij}^{(k)})$, $\sigma(\bar{e}_k) = (\bar{\sigma}_{ij}^{(k)})$. From (4), we find

$$\sigma_{ij} = \begin{cases} 1, & \text{if } e_k = e_i \bar{e}_j, \\ -1, & \text{if } e_k = -e_i \bar{e}_j, \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad \bar{\sigma}_{ji} = \begin{cases} 1, & \text{if } \bar{e}_k = e_j \bar{e}_i, \\ -1, & \text{if } \bar{e}_k = -e_j \bar{e}_i, \\ 0, & \text{otherwise.} \end{cases}$$

By (2), we have $\overline{e_Ae_B} = \bar{e}_B\bar{e}_A$ for $e_A, e_B \in E$. Hence,

$$\bar{\sigma}_{ji} = \begin{cases} 1, & \text{if } \bar{e}_k = e_i\bar{e}_j, \\ -1, & \text{if } e_k = -e_i\bar{e}_j, \\ 0, & \text{otherwise,} \end{cases}$$

which gives $\sigma(e_k)^T = \sigma(\bar{e}_k)$.

Then, for any number $a = \sum_{k=0}^{n-1} a_k e_k \in \mathscr{A}$, we get

$$(\sigma(a))^T = \left(\sum_{k=0}^{n-1} a_k\sigma(e_k)\right)^T = \sum_{k=0}^{n-1} a_k(\sigma(e_k))^T = \sum_{k=0}^{n-1} a_k\sigma(\bar{e}_k) = \sigma(\bar{a}). \qquad \blacksquare$$

2. Invertibility of Subspaces in $\mathscr{A}$

For every Clifford number $a = \sum_A a_A e_A$, we write $|a| = \left(\sum_A a_A^2\right)^{1/2}$.

Definition 2. *A linear subspace X of $\mathscr{A}$ is said to be right invertible (left invertible, invertible) if every nonzero element in X is right invertible (left invertible, invertible).*

Corollary 1 shows that, in any Clifford algebra, every one-sided, invertible subspace is invertible. Therefore, in the sequel, we shall only deal with invertible subspaces.

It is well known that (see [1]), for Clifford numbers of the form $a = \sum_{i=0}^{m} a_i e_i \neq 0$, $a^{-1} = \bar{a}/|a|^2$. Hence, $L(e_0, \ldots, e_m) \overset{\text{def}}{=} \text{lin}\{e_0, \ldots, e_m\}$ is invertible.

Let $\{e_{m+1}, e_{m+2}, \ldots, e_{m+s}\}$ be s distinct basic elements of $\mathscr{A}$, where $e_{m+k} \notin \{e_0, \ldots, e_m\}$ for all $k \in \{1, \ldots, s\}$. Define

$$L(e_0, \ldots, e_{m+s}) = \text{lin}\{e_0, \ldots, e_{m+s}\}. \tag{5}$$

Theorem 1. *$L(e_0, \ldots, e_{m+s})$ $(s > 0)$ is invertible if and only if the following conditions simultaneously hold:*

(i) $m = 2 (\text{mod}\, 4)$,

(ii) $s = 1$,

(iii) $e_{m+1} = e_{12\ldots m}$.

Proof. Sufficiency.

Let $a = \sum_{k=0}^{m} a_k e_k + \alpha e_{m+1}$, where $e_{m+1} = e_{12\ldots m}$. Then

$$\bar{a} = \sum_{l=0}^{m} a_l\bar{e}_l + \alpha\bar{e}_{m+1} = \sum_{l=0}^{m} a_l\bar{e}_l + \alpha(-1)^{m(m+1)/2}e_{m+1} = \sum_{l=0}^{m} a_l\bar{e}_l - \alpha e_{m+1}.$$

Hence, by (2), one gets

$$
\begin{aligned}
a\bar{a} &= \left(\sum_{k=0}^{m} a_k e_k + \alpha e_{m+1}\right)\left(\sum_{l=0}^{m} a_l \bar{e}_l - \alpha e_{m+1}\right) \\
&= \sum_{k=0}^{m} a_k^2 e_0 + \sum_{k\neq l} a_k a_l (e_k \bar{e}_l + e_l \bar{e}_k) \\
&\quad + \alpha \sum_{l=0}^{m} a_l e_{m+1} \bar{e}_l - \alpha \sum_{k=0}^{m} a_k e_k e_{m+1} - \alpha^2 e_{m+1} e_{m+1} \\
&= \sum_{k=0}^{m} a_k^2 e_0 + \alpha a_0 e_{m+1} e_0 - \alpha a_0 e_0 e_{m+1} \\
&\quad - \sum_{k=1}^{m} \alpha a_k (e_{m+1} e_k + e_k e_{m+1}) - \alpha^2 (-1)^{m+p(\{1,2,\dots,m\},\{1,2,\dots,m\})} e_0 \\
&= \sum_{k=0}^{m} a_k^2 e_0 - \alpha^2 (-1)^{m(m+1)/2} e_0 - \sum_{k=1}^{m} \alpha a_k \\
&\quad \times ((-1)(-1)^{p(\{1,2,\dots,m\},k)} + (-1)(-1)^{p(k,\{1,2,\dots,m\})}) e_{\{1,\dots,m\}\setminus\{k\}} \\
&= \sum_{k=1}^{m} \alpha a_k ((-1)^{p(\{1,2,\dots,m\},k)} + (-1)^{m-p(\{1,2,\dots,m\},k)-1}) e_{\{1,\dots,m\}\setminus\{k\}} \\
&\quad + \left(\sum_{k=0}^{m} a_k^2 + \alpha^2\right) e_0 = \left(\sum_{k=0}^{m} a_k^2 + \alpha^2\right) e_0 = |a|^2 e_0.
\end{aligned}
$$

Similarly, we find $\bar{a}a = |a|^2 e_0$. Hence, if $a \neq 0$, there exists $a^{-1} = \bar{a}/|a|^2$.

Necessity. Suppose $s > 1$. Without loss of generality, one can assume $e_{m+1} = e_A$ with $1 < \sharp A = q < m$.

Let $q = 0 (\mathrm{mod}\, 4)$ or $q = 3 (\mathrm{mod}\, 4)$. Choosing $a = e_0 + e_A$ and $b = e_0 - e_A$, we find

$$ab = e_0 + e_A - e_A - e_A e_A = e_0 - (-1)^{q(q+1)/2} e_0 = e_0 - e_0 = 0.$$

Hence, the nonzero numbers a and b are not invertible.

Let $q = 1 (\mathrm{mod}\, 4)$ and $i \in A \cap \{1, \dots, m\}$. Choosing $a = e_i + e_A$ $(\neq 0)$ and $b = e_i - e_A$ $(\neq 0)$, by (2), we find

$$
\begin{aligned}
ab &= (e_i + e_A)(e_i - e_A) = e_i e_i + e_A e_i - e_i e_A - e_A e_A \\
&= -e_0 + ((-1)(-1)^{p(A,i)} - (-1)(-1)^{p(i,A)}) e_{A\setminus\{i\}} - (-1)^{q(q+1)/2} e_0 \\
&= -e_0 + ((-1)^{p(i,A)} - (-1)^{q-p(i,A)-1}) e_{A\setminus\{i\}} + e_0 = 0.
\end{aligned}
$$

Hence, a and b are not invertible.

Finally, we deal with the case $q = 2 (\mathrm{mod}\, 4)$. Since $q < m$, there is at least one $j \in \{1, \dots, m\} \setminus A$. Choosing $a = e_j + e_A$ and $b = e_j - e_A$, by (2), we have the following equalities:

$$\begin{aligned} ab &= (e_j + e_A)(e_j - e_A) = e_je_j + e_Ae_j - e_je_A - e_Ae_A \\ &= -e_0 + ((-1)^{p(A,j)} - (-1)^{p(j,A)})e_{A\cup\{j\}} - (-1)^{q(q+1)/2}e_0 \\ &= -e_0 + ((-1)^{p(A,j)} + (-1)^{q-p(A,j)+1})e_{A\cup\{j\}} + e_0 = 0. \end{aligned}$$

Therefore, for $s > 1$, there are noninvertible numbers in $L(e_0, \dots, e_{m+s})\backslash\{0\}$, i.e., $L(e_0, \dots, e_{m+s})$ is not invertible.

Consider now the case $s = 1$. Let $e_{m+1} = e_B$ and $\sharp B = q > 1$. There are three distinct cases to deal with: $q = 0(\bmod 4)$ or $q = 3(\bmod 4)$, $q = 1(\bmod 4)$, and $q = 2(\bmod 4)$.

If $q = 0(\bmod 4)$ or $q = 3(\bmod 4)$, then $a = e_0 + e_B \neq 0$ is not invertible. Indeed, in this case, $a(e_0 - e_B) = e_0 - e_Be_B = 0$ with $e_0 - e_B \neq 0$.

If $q = 1(\bmod 4)$ and $c = e_i + e_B$ $(i \in B)$, then $c(e_i - e_B) = 0$, i.e., c is not invertible.

Finally, consider the case $q = 2(\bmod 4)$. If $q < m$, then there is $j \in \{1, \dots, m\}\backslash B$. It is easy to check that $bd = 0$ for $b = e_j + e_B$ and $d = e_j - e_B$, i.e., b and d are not invertible. Thus, $q = m$ and $e_{m+1} = e_{12\dots m}$. The proof is complete. ■

Corollary 2. *Every quaternion* ($\mathscr{A} = \text{lin}\{e_0, e_1, e_2, e_{12}\}$) *is invertible.*

Proof. Indeed, every quaternion is a Clifford algebra induced by a 2-dimensional real linear space with a certain basis $\{e_1, e_2\}$. Hence, $m = 2 = 2(\bmod 4)$.

3. Remarks on Monogenic Functions

Let $m = 4p + 2$ $(p \in \mathbb{N})$. Consider the differential operator

$$D = \sum_{i=0}^{m+1} e_i\partial_{x_i}, \quad \text{where} \quad e_{m+1} = e_{12\dots m} \tag{6}$$

and the conjugate operator of D

$$\bar{D} = \sum_{i=0}^{m+1} \bar{e}_i\partial_{x_i}. \tag{7}$$

Actions of D and $\bar{D}$ on functions from the left and from the right are governed by the rules (see [1])

$$Df = \sum_{i,A} e_ie_A\partial_{x_i}f_A, \qquad fD = \sum_{i,A} e_Ae_i\partial_{x_i}f_A$$

and

$$\bar{D}f = \sum_{i,A} \bar{e}_ie_A\partial_{x_i}f_A, \qquad f\bar{D} = \sum_{i,A} e_A\bar{e}_i\partial_{x_i}f_A$$

for all $f(x) = \sum_A e_Af_A(x)$. Here, the functions $f_A(x)$ are real-valued.

Lemma 3. *$D\bar{D} = \bar{D}D = \Delta_{m+2}e_0$, where Δ_{m+2} denotes the Laplacian in $\mathbb{R}^{m+2}$.*

Proof. Since $e_{m+1} = e_{12\ldots m}$ and $m = 4p+2$ ($p \in \mathbb{N}$), we find $\bar{e}_{m+1} = -e_{m+1}$. For every $i \in \{1,\ldots,m\}$, we have

$$\begin{aligned} e_i\bar{e}_{m+1} + e_{m+1}\bar{e}_i &= -e_ie_{m+1} - e_{m+1}e_i \\ &= -\left((-1)(-1)^{p(i,\{1,2,\ldots,m\})} + (-1)(-1)^{p(\{1,2,\ldots,m\},i)}\right)e_{\{1,2,\ldots,m\}\setminus\{i\}} \\ &= ((-1)^{i-1} + (-1)^{m-i})e_{\{1,2,\ldots,m\}\setminus\{i\}} = 0, \end{aligned}$$

because $(i-1)+(m-i) = m-1$ is odd.

Therefore,

$$\begin{aligned} D\bar{D} &= \left(\sum_{i=0}^{m} e_i\partial_{x_i} + e_{1\ldots m}\partial_{x_{m+1}}\right)\left(\sum_{j=0}^{m} \bar{e}_j\partial_{x_j} + \bar{e}_{1\ldots m}\partial_{x_{m+1}}\right) \\ &= \sum_{i,j=0}^{m} e_i\bar{e}_j\partial_{x_i}\partial_{x_j} + \sum_{i=0}^{m}(e_i\bar{e}_{1\ldots m} + e_{1\ldots m}\bar{e}_i)\partial_{x_i}\partial_{x_{m+1}} + e_{1\ldots m}\bar{e}_{1\ldots m}\frac{\partial^2}{\partial x_{m+1}^2} \\ &= \sum_{i=0}^{m} e_i\bar{e}_i\frac{\partial^2}{\partial x_i^2} + e_{1\ldots m}\bar{e}_{1\ldots m}\frac{\partial^2}{\partial x_{m+1}^2} = \Delta_{m+2}e_0. \end{aligned}$$

Similarly, one can check the equality $\bar{D}D = \Delta_{m+2}e_0$.

Lemma 3 permits us to introduce the so-called left (right) monogenic function in a certain open domain $\Omega \subset \mathbb{R}^{m+2}$ as for the case $\Omega \subset \mathbb{R}^k$ ($k < m+1$) in [1, 2]. ∎

Definition 3. *A function $f \in \mathscr{C}^1(\Omega;\mathscr{A})$ is said to be left (right) monogenic in Ω if and only if $Df = 0$ ($fD = 0$) in Ω.*

Following all the procedures for monogenic functions as in [1], we can obtain the main function theoretic results as Cauchy's integral formula, Morera's Theorem, Taylor expansion theorems, and Laurent series for pointwise singularities, etc.

Remark. Thus, for the case $m = 4p+2$ ($p \in \mathbb{N}$), the theory of monogenic functions can be extended to $\mathbb{R}^{m+2}$. If $p = 0$, we get regular functions of a quaternionic variable (see [3–7]) as a particular case of monogenic functions in Clifford analysis.

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