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The Property $(\tilde{\Omega})$ and Holomorphic Functions*

Le Mau Hai
Department of Mathematics, Pedagogic College 1 of Hanoi, Hanoi, Vietnam

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Abstract. It is shown that a nuclear Frechet space E has the property $(\tilde{\Omega})$ if and only if every holomorphic function on $\Lambda_1^*(\alpha)$ with values in E^* is of uniform type.

1. Introduction

Let E be a Frechet space with a fundamental system of semi-norms $\|\cdot\|_k$. We say that E has the property

$$(\underline{DN}) \text{ if } \exists p\ \forall q\ \exists s,\ C > 0,\ \varepsilon > 0\text{: } \|\cdot\|_q^{1+\varepsilon} \le C\|\cdot\|_s\|\cdot\|_p^{\varepsilon},$$

$$\left.\begin{array}{l}(\tilde{\Omega}) \text{ if } \forall p\ \exists q,\ d > 0\ \forall k\ \exists C > 0 \\ (\bar{\bar{\Omega}}) \text{ if } \forall p\ \exists q\ \forall d > 0\ \forall k\ \exists C > 0\end{array}\right\}\text{: } \|\cdot\|_q^{*1+d} \le C\|\cdot\|_k^*\|\cdot\|_p^{*d}.$$

Here, for each subset B of E and $y^* \in E^*$, the strongly dual space of E, we put

$$\|y\|_B^* = \sup\{|y(x)|\colon x \in B\}$$

and, for every p, we write

$$\|\cdot\|_p^* = \|\cdot\|_{U_p}^*, \quad \text{where} \quad U_p = \{x \in E\colon \|x\|_p \le 1\}.$$

The properties $(\underline{DN})$, $(\tilde{\Omega})$, $(\bar{\bar{\Omega}})$, and others were introduced and investigated by Vogt in [9, 10]. Recently in [4], Meise and Vogt have proved that if a nuclear Frechet space E has the property $(\tilde{\Omega})$, then every scalar holomorphic function on E is of uniform type. Here, a holomorphic function f from a locally convex space E to a locally convex space F is of uniform type if there exists a continuous semi-norm ρ on E such that f can be holomorphically factorized through the canonical map $\omega_\rho : E \to E_\rho$, where we denote the Banach space associated to ρ by E_ρ.

Now let E and F be locally convex spaces and $f\colon E \to F$ a holomorphic

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function. We say that f has Dirichlet representation if there exist $\{u_k\} \subset E^*$, the strongly dual space of E, and $\{y_k\} \subset F$ such that

$$\mathrm{Exp}: f(x) = \sum_{k\geq 1} y_k \exp u_k(x) \quad \text{for} \quad x \in E,$$

where the series is convergent to f in the compact open topology of $H(E,F)$.

In the present paper, we shall prove the following.

Main Theorem. *Let E be a nuclear Frechet space. Then the following conditions are equivalent*:

(i) *E has the property $(\bar{\bar{\Omega}})$.*

(ii) *Every holomorphic function on $\Lambda_1^*(\alpha)$ with values in E^* is of uniform type for every exponent sequence $\alpha = (\alpha_n)$ for which $\Lambda_1(\alpha)$ is nuclear, where*

$$\Lambda_1(\alpha) = \left\{ (\xi_j) \in \mathbf{C}^N: \sum_{j\geq 1} |\xi_j| r^{\alpha_j} < \infty \quad \text{for} \quad 0 \leq r < 1 \right\}.$$

(iii) *E has the property $(\tilde{\Omega})$ and every holomorphic function f on $\Lambda_1^*(\alpha)$ with values in E^* has Dirichlet representation*:

$$(\mathrm{Exp}): f(y) = \sum_{k\geq 1} \xi_k \exp x_k(y)$$

which is absolutely convergent in $H(\Lambda_1^*(\alpha), E^*)$ for every α as in (ii).

(iv) *Every holomorphic function f from E to $\Lambda_1(\alpha)$ has Dirichlet representation*:

$$(\mathrm{Exp}): f(x) = \sum_{k\geq 1} \xi_k \exp u_k(x)$$

which is absolutely convergent in $H(E, \Lambda_1(\alpha))$ for every α as in (ii).

The proof of the Main Theorem is given in Sec. 2. Moreover, in this section, we prove that, if E is a Frechet–Hilbert–Schwartz space with the property (H_u) and D is a pseudoconvex neighborhood of $E/\|\cdot\|_\rho$ in E_ρ, then there exists $\beta \geq \rho$ such that $\mathrm{Im}\, \omega_{\beta\rho} \subseteq D$, where $\omega_{\beta\rho}$ is the canonical map from E_β to E_ρ.

2. Proof of Main Theorem

(i) implies (ii). Let $f \in H(\Lambda_1^*(\alpha), E^*)$, where α as in (ii). Without loss of generality, we may assume that $f(0) = 0$. Then f induces a continuous linear map $\hat{f}$ from $H(E^*)$ to $H_0(\Lambda_1^*(\alpha))$, the space of holomorphic functions φ on $\Lambda_1^*(\alpha)$ with $\varphi(0) = 0$, by the formula

$$(\hat{f}\varphi)(y) = \varphi(f(y)) \quad \text{for} \quad \varphi \in H(E^*) \quad \text{and} \quad y \in \Lambda_1^*(\alpha).$$

Let D_1 denote the open polydisc in $\Lambda_1^*(\alpha)$ given by

$$D_1 = \{y = (y_j) \in \Lambda_1^*(\alpha) : \sup |y_j| < 1\}$$

and let R be the restriction map from $H_0(\Lambda_1^*(\alpha))$ to $H_0(D_1)$. It is known [5] that

$$H(D_1) \cong \Lambda_1(\beta(\alpha)),$$

where $\beta(\alpha)$ is the increasing arrangement of the family

$$((\alpha|m)) = \left\{\sum \alpha_j m_j \colon m \in M\right\}$$

and

$$M = \{m = (m_j) \in N_0^N \colon m_j = 0 \quad \text{for almost all } j \in N\}.$$

Now we consider a map $u : E \to H(E^*)$ given by $u(t)(g) = g(t)$. Then the map $R\hat{f}u : E \to H(D_1) \cong \Lambda_1(\beta(\alpha))$ is a continuous linear map. By [9, Satz. 4.2], there exists a neighborhood U of $0 \in E$ such that $B = R\hat{f}u(U)$ is bounded in $H_0(D_1)$. Let $\delta : D_1 \to H_0^*(D_1)$ be the canonical map. Then $\delta^{-1}(B^0)$ is a neighborhood of 0 in D_1 and $\sup\{|f(y)(x)| : x \in U, \delta(y) \in B^0\} = \sup\{|\hat{f}(x)(y)| : x \in U, \delta(y) \in B^0\} = \sup\{|\delta(y)(\hat{f}(x))| : x \in U, \delta(y) \in B^0\} \leq 1$. Hence, f is bounded at $0 \in D_1$. Similarly, it follows that f is bounded at every point of $\Lambda_1^*(\alpha)$. Write

$$\Lambda_1^*(\alpha) = \bigcup_{n \geq 1} K_n,$$

where $\{K_n\}$ is an exhaustion sequence of compact sets in $\Lambda_1^*(\alpha)$ such that $nK_n \subseteq K_{n+1}$ for $n \geq 1$. Since f is locally bounded for each $n \geq 1$, there exists a neighborhood U_n of $0 \in \Lambda_1^*(\alpha)$ such that $f(K_n + U_n)$ is bounded. Put

$$U = \bigcap_{n \geq 1} \left(K_n + \left(\frac{1}{n}\right) U_{n+1}\right).$$

Then U is a neighborhood of $0 \in \Lambda_1^*(\alpha)$ and

$$f(nU) \subseteq f(nK_n + U_{n+1}) \subseteq f(K_{n+1} + U_{n+1})$$

for $n \geq 1$. This implies that f is of uniform type.

(ii) implies (i). By [9, Satz. 4.2], it suffices to show that every continuous linear map $T : E \to \Lambda_1(\alpha)$ is of uniform type. Indeed, by the hypothesis $T^* : \Lambda_1^*(\alpha) \to E^*$ is of uniform type and, hence, T is of uniform type.

(ii) implies (iii). Since (ii) implies (i), $E \in (\bar{\Omega})$ and, hence, $E \in (\tilde{\Omega})$. Now, given $f \in H(\Lambda_1^*(\alpha), E^*)$, by (ii), there exists a continuous semi-norm ρ on $\Lambda_1^*(\alpha)$ and a holomorphic function g on $(\Lambda_1^*(\alpha))_\rho$ with values in E^* such that

$$g\omega_\rho = f.$$

Choose a continuous semi-norm β on $\Lambda_1^*(\alpha)$ such that $\beta \geq \rho$ and the canonical map T from $(\Lambda_1^*(\alpha))_\beta$ into $(\Lambda_1^*(\alpha))_\rho$ is in the form

$$T(x) = \sum_{j \geq 1} \lambda_j u_j(x) e_j,$$

where

$$a = \sum_{j\geq 1} |\lambda_j| < \infty, \quad \|u_j\| \leq 1, \quad \|e_j\| \leq 1.$$

Consider the Taylor expansion of g at $0 \in (\Lambda_1^*(\alpha))_\beta$

$$g(x) = \sum_{n\geq 0} P_n g(x),$$

where

$$P_n g(x) = \frac{1}{2\pi i} \int_{|t|=r} \frac{g(tx)}{t^{n+1}}\, dt$$

for all $n \geq 0$ and $r > 0$.

By [3], there exist complex number sequences $\{\xi_k\}$, $\{\alpha_k\}$ such that, for $z \in C$, we can write

$$z = \sum_{k\geq 1} \xi_k \exp \alpha_k z$$

and

$$C_r = \sum_{k\geq 1} |\xi_k| \exp |\alpha_k| r < +\infty$$

for all $r \geq 0$.

Formally, we have

$$\begin{aligned}
(gT)(x) &= g(Tx)\\
&= \sum_{n\geq 0} P_n g\left(\sum_{j\geq 1} \lambda_j u_j(x) e_j\right)\\
&= \sum_{n\geq 0} \sum_{j_1, j_1, \ldots, j_n \geq 1} \lambda_{j_1} \lambda_{j_2} \ldots \lambda_{j_n} \widehat{P_n g}(e_{j_1}, \ldots, e_{j_n}) u_{j_1}(x) \ldots u_{j_n}(x)\\
&= \sum_{n\geq 0} \sum_{j_1, j_1, \ldots, j_n \geq 1} \lambda_{j_1} \lambda_{j_2} \ldots \lambda_{j_n} \widehat{P_n g}(e_{j_1}, \ldots, e_{j_n})\\
&\quad \times \left(\sum_{k\geq 1} \xi_k \exp \alpha_k u_{j_1}(x) \ldots \sum_{k\geq 1} \xi_k \exp \alpha_k u_{j_n}(x)\right)\\
&= \sum_{n\geq 0} \sum_{\substack{j_1, j_2, \ldots, j_n \geq 1\\ k_1, \ldots, k_n \geq 1}} \lambda_{j_1} \lambda_{j_2} \ldots \lambda_{j_n} \xi_{k_1} \ldots \xi_{k_n}\\
&\quad \times \widehat{P_n g}(e_{j_1}, \ldots, e_{j_n}) \exp[\alpha_{k_1} u_{j_1}(x) + \cdots + \alpha_{k_n} u_{j_n}(x)],
\end{aligned}$$

where $\widehat{P_n g}$ denotes the continuous symmetric n-linear map associated to $P_n g$.

It remains to be checked that the right-hand side is absolutely convergent in $H(\Lambda_1^*(\alpha), E^*)$. For each $r > 0$, we take $\delta > C_r ea$, where

$$C_r = \sum_{k\geq 1} |\xi_k| \exp r|\alpha_k| < \infty. \tag{1}$$

Now, let B be an arbitrary bounded subset of E. We have

$$\|\widehat{P_n g}(e_{j_1}, \ldots, e_{j_n})\|_B^* \leq n^n/n!\delta^n \|g\|_{B,\delta}^*, \tag{2}$$

where

$$\|g\|_{B,\delta}^* = \sup\{\|g(x)\|_B^* : \|x\| \leq \delta\}.$$

Without loss of generality, by the nuclearity of $\Lambda_1^*(\alpha)$, we may assume that g is bounded on every bounded set in $(\Lambda_1^*(\alpha))_\rho$.

From (1) and (2), we have

$$\sum_{n\geq 0} \sum_{\substack{j_1, j_2, \ldots, j_n \geq 1 \\ k_1, \ldots, k_n \geq 1}} |\lambda_{j_1}||\lambda_{j_2}| \ldots |\lambda_{j_n}||\xi_{k_1}| \ldots |\xi_{k_n}| \|\widehat{P_n g}(e_{j_1}, \ldots, e_{j_n})\|_B^*$$

$$\times \exp[r|\alpha_{k_1}| + \ldots + r|\alpha_{k_n}|]$$

$$\leq \sum_{n\geq 0} C_r^n a^n n^n/n!\delta^n \|g\|_{B,\delta}^* < \infty \quad \text{for } \|x\| \leq r.$$

(iii) implies (i). By [9, Satz. 4.2], it suffices to show that every continuous linear map T from $\Lambda_1^*(\alpha)$ to E^* is bounded on a neighborhood of $0 \in \Lambda_1^*(\alpha)$. Write T in Dirichlet representation

$$T(y) = \sum_{j=1}^{\infty} \xi_j \exp y(x_j),$$

where $x_j \in \Lambda_1(\alpha)$, $y \in \Lambda_1^*(\alpha)$ and the series is absolutely convergent in $H(\Lambda_1^*(\alpha), E^*)$. Since E is a Frechet space, we can find a semi-norm p on E such that

$$\sum_{j\geq 1} \|\xi_j\|_p^* < +\infty.$$

By the hypothesis, that E has the property $(\tilde{\Omega})$, there exist q and a compact set B in E such that

$$\|\cdot\|_q^{*^{1+d}} \leq \|\cdot\|_B^* \|\cdot\|_p^{*^d}$$

for some $d > 0$ [4, Lemma 3.6].

Then, for every $k \geq 1$, we have

$$\sum_{j\geq 1} \|\xi_j\|_q^* \exp\|x_j\|_k \leq \sum_{j\geq 1} \|\xi_j\|_B^{*^{(1/1+d)}} \|\xi_j\|_p^{*^{(d/1+d)}} \exp\|x_j\|_k$$

$$\leq \sum_{j\geq 1} \|\xi_j\|_B^* \exp(1+d)\|x_j\|_k + \sum_{j\geq 1} \|\xi_j\|_p^* < +\infty.$$

Thus, T continuously maps $\Lambda_1^*(\alpha)$ to E_q^*. Hence, T is bounded on a neighborhood of $0 \in \Lambda_1^*(\alpha)$.

(i) implies (iv) as (i) implies (iii).

(iv) implies (i). By [9, Satz. 4.2], it suffices to show that every continuous linear map $f : E \to \Lambda_1(\alpha)$ is of uniform type. Write f in Dirichlet expansion form

$$(\text{Exp}) : f(x) = \sum_{k \geq 1} \xi_k \exp u_k(x),$$

where the series is absolutely convergent in $H(E, \Lambda_1(\alpha))$. Since $\Lambda_1(\alpha)$ has the property $(\underline{DN})$, we can find $p \geq 1$ such that

$$\forall q \; \exists s, C, \varepsilon > 0 : \| \cdot \|_q^{1+\varepsilon} \leq C \| \cdot \|_s \| \cdot \|_p^{\varepsilon}.$$

Since E is a Frechet space, we can find a semi-norm m on E such that

$$\sum_{k \geq 1} \|\xi_k\|_p \exp \|u_k\|_m^* < \infty. \tag{3}$$

It remains to be checked that

$$\sum_{k \geq 1} \|\xi_k\|_q \exp \|u_k\|_m^* < \infty \quad \text{for } q \geq p.$$

Given $q \geq p$, choose (for q) s, ε, $C > 0$ such that the property $(\underline{DN})$ is satisfied. Then

$$\begin{aligned}
&\sum_{k \geq 1} \|\xi_k\|_q \exp \frac{\varepsilon}{1+\varepsilon} \|u_k\|_m^* \\
&\leq C^{(1/1+\varepsilon)} \sum_{k \geq 1} \|\xi_k\|_s^{(1/1+\varepsilon)} \|\xi_k\|_p^{(\varepsilon/1+\varepsilon)} \exp \frac{\varepsilon}{1+\varepsilon} \|u_k\|_m^* \\
&\leq C^{(1/1+\varepsilon)} \sum_{k \geq 1} \left(\frac{\|\xi_k\|_s}{1+\varepsilon} + \frac{\varepsilon}{1+\varepsilon} (\|\xi_k\|_p \exp \|u_k\|_m^*) \right) \\
&\leq C^{(1/1+\varepsilon)} \left[\sum_{k \geq 1} \|\xi_k\|_s + \sum_{k \geq 1} \|\xi_k\|_p \exp \|u_k\|_m^* \right] < \infty.
\end{aligned}$$

This deduces from the following. Since the series $\sum_{k \geq 1} \xi_k \exp u_k(x)$ is absolutely convergent in $H(E, \Lambda_1(\alpha))$ and, hence, for s the series $\sum_{k \geq 1} \|\xi_k\|_s \exp|u_k(0)| < \infty$. This shows that $\sum_{k \geq 1} \|\xi_k\|_s < \infty$. Thus, f is bounded on U_m. The theorem is proved.

Proposition. *Let E be a Frechet–Hilbert–Schwartz space and let E have the property $(\tilde{H}_u)$, i.e., every holomorphic function on E is of uniform type. Then, for every continuous semi-norm ρ on E and every pseudoconvex neighborhood D of $E/\ker \rho$ in E_ρ, there exists a continuous semi-norm β on E with $\beta \geq \rho$ such that $\operatorname{Im} \omega_{\beta\rho} \subset D$, where $\omega_{\beta\rho} : E_\beta \to E_\rho$ is the canonical map.*

Proof. Since the topology of E is defined by Hilbert semi-norms, without loss of generality, we may assume that E_ρ is a Hilbert space. Choose a continuous semi-norm α on E such that $\alpha \geq \rho$ and the canonical map from E_α to E_ρ is compact. Let τ denote the linear metric topology on $H(D)$ generated by the uniform convergence on the sets

$$K_r = \left\{ \omega_{\alpha\rho}(z); \|z\| \leq r, \omega_{\alpha\rho}(z) \in D, \operatorname{dist}(\omega_{\alpha\rho}(z), \partial D) \geq \frac{1}{r} \right\}.$$

Since the canonical map $[H(D), \tau] \to H(E)$ is continuous and since

$$H(E)_{\text{bor}} \cong \operatorname{lim\,ind} H_b(E_q) \quad [4],$$

where $H(E)_{\text{bor}}$ denotes the bornological space associated to $H(E)$ and for each q by $H_b(E_q)$, we denote the Frechet space of holomorphic functions on E_q which are bounded on every bounded set in E_q, we can find a continuous semi-norm β on E such that $\beta \geq \alpha$ and $H(D) \subseteq H(E_\beta)$. It remains to be checked that $\operatorname{Im} \omega_{\beta\rho} \subseteq D$. In the converse case, there exists $z \in E_\beta$ such that $\omega_{\beta\rho}(z) \in \partial D$. Choose a sequence $\{z_n\} \subset E/\ker \beta$ which converges to z. Since E_ρ is a separable Hilbert space, we can find $f \in H(D)$ such that

$$\sup |f \omega_{\beta\rho}(z_n)| = \infty.$$

This is impossible because $f \omega_{\beta\rho} \in H(E_\beta)$.

The proposition is proved. ■

Remarks. In [6], Ha and Khue have proved that a nuclear Frechet space E has the property (H_u) if and only if every holomorphic function on E can be written in Dirichlet representation.

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