Vietnam Journal of Mathematics **25**:2 (1997) 115–123

Vietnam Journal
of
MATHEMATICS

An Extension to the Mean Square Criterion of Gaussian Equivalent Linearization

Nguyen Dong Anh[1] **and W. Schiehlen**[2]
[1]*Institute of Mechanics, Hanoi, Vietnam*
[2]*Institut ß fur Mechanik, Universität Stuttgart, Pfaffenwaldring 9, 70569 Stuttgart, Germany*

Received November 10, 1995

Abstract. Within the scope of Gaussian equivalent linearization, a new mean square criterion of error sample function for determining the coefficients of the linearized equivalent equation is proposed to treat stationary response of nonlinear systems under zero mean Gaussian random. Application to the Duffing oscillator subjected to white noise is presented which shows a significant improvement over the corresponding accuracy of the classical Gaussian equivalent linearization for both weak and strong nonlinearity.

1. Introduction

There has been a large amount of extensive investigations into the response of nonlinear stochastic systems due to the fact that many excitations of engineering interest are basically random in nature. Since all real engineering systems are, more or less, nonlinear and for those systems where the exact solutions are known only for a number of special cases, it is necessary to develop approximate techniques to determine the response statistics of nonlinear systems under random excitation. One of the known approximate techniques is Gaussian equivalent linearization which was first proposed by Caughey [3] and has been developed by many authors (see, e.g., [2, 4, 6–9, 12]). It has been shown that the Gaussian equivalent linearization is presently the simplest tool widely used for analyzing nonlinear stochastic problems. However, the major limitation of this method is seemingly that its accuracy decreases as the nonlinearity increases and it can lead to unacceptable errors in the second moments [1, 5]. Further, if one needs more accurate approximate solutions, there is no way to obtain them using the conventional version of Gaussian equivalent linearization.

To obtain a series of approximate solutions in this excellent technique, a mean square criterion of error sample function is proposed for determining the coefficients of linearization. The criterion is based on the chosen sample functions of equation error. The proposed technique is then applied to an oscillator with nonlinearity under a zero mean Gaussian white noise. It is obtained that the technique yields a significant improvement over corresponding accuracy of the classical Gaussian equivalent linearization for both weak and strong nonlinearity.

2. Gaussian Equivalent Linearization (GEL)

First of all, we recall some basic ideas of the method of GEL. We consider the nonlinear stochastic equation

$$\ddot{z} + 2h\dot{z} + \omega_0^2 z + \varepsilon g(z,\dot{z}) = f(t), \tag{1}$$

where dots denote time differentiation, h, ω_0, ε are positive constants, and g is a nonlinear function which can be expanded into a polynomial series form. The excitation $f(t)$ is a zero mean Gaussian stationary process with the correlation function and spectral density given, respectively, by

$$R_f(\tau) = \langle f(t)f(t+\tau)\rangle, \qquad S_f(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R_f(\tau)e^{i\omega\tau}d\tau, \tag{2}$$

where $\langle\ \rangle$ denotes the expectation. For the sake of simplicity, we restrict to the case of stationary response of Eq. (1) if it exists.

Denote

$$e(z) = \ddot{z} + 2h\dot{z} + \omega_0^2 z + \varepsilon g(z,\dot{z}) - f(t). \tag{3}$$

Equation (1) yields

$$e(z) = 0. \tag{4}$$

Following the GEL methods, we introduce new linear terms in the expression of e

$$e(z) = \ddot{z} + (2h+\mu)\dot{z} + (\omega_0^2+\lambda)z + \varepsilon g(z,\dot{z}) - \mu\dot{z} - \lambda z - f(t). \tag{5}$$

Let $x(t)$ be a stationary solution of the linearized equation

$$\ddot{x} + (2h+\mu)\dot{x} + (\omega_0^2+\lambda)x - f(t) = 0. \tag{6}$$

Using (6), one gets from (5)

$$e(x) = \varepsilon g(x,\dot{x}) - \mu\dot{x} - \lambda x. \tag{7}$$

It is seen from (5) that $e(x)$ is an equation error which is different from zero. Thus, the problem reduces to the linearized Eq. (6) where the coefficients of linearization are to be found from an optimal criterion. There are some criteria for determining the coefficients (see, e.g., [11]). The most extensively used criterion is the mean square error criterion which requires that the mean square of equation error be minimum

$$\langle e^2(x)\rangle = \langle(\varepsilon g(x,\dot{x}) - \mu\dot{x} - \lambda x)^2\rangle \to \min_{\mu,\lambda}. \tag{8}$$

Thus, from

$$\frac{\partial}{\partial \mu}\langle e^2(x)\rangle = 0, \qquad \frac{\partial}{\partial \lambda}\langle e^2(x)\rangle = 0,$$

it follows that

$$\mu = \varepsilon\frac{\langle g\dot{x}\rangle}{\langle \dot{x}^2\rangle}, \qquad \lambda = \varepsilon\frac{\langle gx\rangle}{\langle x^2\rangle}. \tag{9}$$

Since the process $x(t)$ is a solution of the linearized Eq. (6) under Gaussian process excitation, one obtains that $x(t)$ and $\dot{x}(t)$ are Gaussian or normal processes. Hence, all higher moments $\langle gx\rangle$, $\langle g\dot{x}\rangle$ can be expressed in terms of second moments $\langle x^2\rangle$, $\langle \dot{x}^2\rangle$, and the relation (9) results in two algebraic equations for 4 unknowns μ, λ, $\langle x^2\rangle$, and $\langle \dot{x}^2\rangle$. To close the system (9), two other equations for second moments $\langle x^2\rangle$, $\langle \dot{x}^2\rangle$ can be derived from (6):

$$\begin{aligned}\langle \dot{x}^2\rangle &= \int_{-\infty}^{\infty} \frac{\omega^2 S_f(\omega)d\omega}{(2h+\mu)^2\omega^2 + (\omega^2-\omega_0^2-\lambda)^2},\\ \langle x^2\rangle &= \int_{-\infty}^{\infty} \frac{S_f(\omega)d\omega}{(2h+\mu)^2\omega^2 + (\omega^2-\omega_0^2-\lambda)^2}.\end{aligned} \tag{10}$$

So the classical version of GEL, as described above, supposes that the minimization of the equation error in mean square sense may give a minimization of the solution error. It should be noted that up to now there is no theoretical proof of GEL; its accuracy has been investigated only by the comparison of the solutions obtained by GEL with their exact solutions if available or with simulation solutions.

3. Mean Square Criterion of Error Sample Function

It is expected that the accuracy of GEL may be improved by an adequate extension of the classical mean square error criterion (8). An alternative approach to the problem is the following. Let $a(e, \alpha_2, \alpha_3, \ldots, \alpha_k)$ be an arbitrary function of the equation error e and parameters α_n $(n = 2, 3, \ldots, k)$. The function a is called the error sample function. Now, the mean square error criterion (8) can be extended to a mean square criterion of error sample function which requires that

$$\langle a^2(e, \alpha_2, \alpha_3, \ldots, \alpha_k)\rangle \to \min_{\alpha_2, \alpha_3, \ldots, \alpha_k, \mu, \lambda}. \tag{11}$$

Thus, one gets

$$\begin{aligned}&\frac{\partial}{\partial \alpha_n}\langle a^2(e, \alpha_2, \alpha_3, \ldots, \alpha_k)\rangle = 0,\\ &\frac{\partial}{\partial s}\langle a^2(e, \alpha_2, \alpha_3, \ldots, \alpha_k)\rangle = 0, \qquad s = \mu, \lambda.\end{aligned} \tag{12}$$

It is supposed that the error sample function a is such that the system of $k+1$ equation (12) allows us to define $k+1$ parameters μ, λ, $\alpha_2, \alpha_3, \ldots, \alpha_k$ as functions of response mean squares $\langle x^2 \rangle, \langle \dot{x}^2 \rangle$. Further, the latter can be definitely determined from (10). It seems that the extended version of the classical mean square error criterion may contain many useful advantages. First of all, one can get a series of approximate response mean squares by choosing different error sample functions $a(e, \alpha_2, \alpha_3, \ldots, \alpha_k)$. This property of the extended criterion is very important from the point of view of creating an approximate technique to solve nonlinear stochastic problems. Furthermore, some interesting related questions may occur, namely,

- Is there an error sample function $a(e, \alpha_2, \alpha_3, \ldots, \alpha_k)$ for which the criterion (11) gives exact response mean squares $\langle x^2 \rangle$ and $\langle \dot{x}^2 \rangle$?
- Are there error sample functions $a_k(e, \alpha_2, \alpha_3, \ldots, \alpha_k)$, $k = 2, 3, \ldots$, such that the corresponding approximate response mean squares $\langle x^2 \rangle_k$, $\langle \dot{x}^2 \rangle_k$ approach the exact ones?

So the problem of choosing optimal error sample functions is open and waiting for a solution. In the following section, a polynomial form of error sample functions will be proposed and investigated in detail.

4. Polynomial Error Sample Functions

Consider the case where error sample functions are polynomials of equation error

$$a_1 = e, \qquad a_k = e - \sum_{i=2}^{k} \alpha_i e^{2i-1}, \qquad k = 2, 3, \ldots. \tag{13}$$

Substituting (13) into (12) yields

$$\sum_{i=2}^{k} \langle e^{2i+2j-2} \rangle \alpha_i = \langle e^{2j} \rangle, \qquad j = 2, 3, \ldots \tag{14}$$

and

$$\frac{\partial}{\partial s}\langle e^2 \rangle - 2\sum_{i=2}^{k} \frac{\partial}{\partial s}\langle e^{2i} \rangle + \sum_{i=2}^{k}\sum_{j=2}^{k} \alpha_i \alpha_j \frac{\partial}{\partial s}\langle e^{2i+2j-2} \rangle = 0. \tag{15}$$

Since the mean square $\langle a^2 \rangle$ is definitely positive, the system of linear equations (14) gives a unique solution for unknown parameters α_j of the form

$$\alpha_j = \Delta_j / \Delta, \tag{16}$$

where Δ_j, Δ are known determinants obtained from the linear system (14) and depend on error even moments $\langle e^2 \rangle$, $\langle e^4 \rangle$, $\langle e^{4k-2} \rangle$. Using (16), one can rewrite (15) as follows

$$\frac{\partial}{\partial s}\langle e^2 \rangle - 2\sum_{i=2}^{k} \frac{\Delta_i}{\Delta} \frac{\partial}{\partial s}\langle e^{2i} \rangle + \sum_{i=2}^{k}\sum_{j=2}^{k} \frac{\Delta_i \Delta_j}{\Delta^2} \frac{\partial}{\partial s}\langle e^{2i+2j-2} \rangle = 0,$$

$$s = \mu, \lambda. \tag{17}$$

On the other hand, one gets from (7) the relation

$$\langle e^{2n}\rangle = \langle(\varepsilon g(x,\dot{x}) - \mu\dot{x} - \lambda x)^{2n}\rangle, \qquad n = 2, 3, \ldots \tag{18}$$

which shows that error even moments $\langle e^{2n}\rangle$ can be expressed in terms of response second moments $\langle x^2\rangle$ and $\langle \dot{x}^2\rangle$. So there are 4 algebraic equations (10), (17) for 4 unknowns μ, λ, $\langle x^2\rangle$, $\langle \dot{x}^2\rangle$. Denote by $\langle x^2\rangle_k$, $\langle \dot{x}^2\rangle_k$ the corresponding solutions obtained from the conditions $\langle a_k^2\rangle \to \min$. It might be expected that the approximate solutions $\langle x^2\rangle_k$, $\langle \dot{x}^2\rangle_k$ would approach the exact ones, respectively, as one has clearly the following series of inequalities:

$$\min_{\mu,\lambda}\langle e^2\rangle \geq \min_{\mu,\lambda,\alpha_i}\langle a_2^2\rangle \geq \min_{\mu,\lambda,\alpha_i}\langle a_3^2\rangle \geq \cdots \geq \min_{\mu,\lambda,\alpha_i}\langle a_k^2\rangle. \tag{19}$$

5. Three First Approximate Solutions

The approximate solution $\langle x^2\rangle_k$, $\langle \dot{x}^2\rangle_k$ is called kth approximate one. In this section, the equations used to determine the coefficients of linearization μ and λ corresponding to three first approximate solutions are derived in explicit form. Obviously, the first approximate solution $\langle x^2\rangle_1$, $\langle \dot{x}^2\rangle_1$ is identical to the one obtained from the classical mean square error criterion since $a_1 = e$. The second response mean squares $\langle x^2\rangle_2$, $\langle \dot{x}^2\rangle_2$ are found from the condition:

$$\langle(e - a_2 e^3)^2\rangle \to \min_{\mu,\lambda,\alpha_2}. \tag{20}$$

Setting in (14) and (15), $k = 2$ yields

$$\alpha_2 = \frac{\langle e^4\rangle}{\langle e^6\rangle} \tag{21}$$

and

$$\frac{\partial}{\partial s}\langle e^2\rangle - 2\frac{\langle e^4\rangle}{\langle e^6\rangle}\frac{\partial}{\partial s}\langle e^4\rangle + \frac{\langle e^4\rangle^2}{\langle e^6\rangle^2}\frac{\partial}{\partial s}\langle e^6\rangle = 0, \qquad s = \mu, \lambda. \tag{22}$$

The third response mean squares $\langle x^2\rangle_3$, $\langle \dot{x}^2\rangle_3$ are found from the condition:

$$\langle(e - \alpha_2 e^3 - \alpha_3 e^5)^2\rangle \to \min_{\mu,\lambda,\alpha_2,\alpha_3}. \tag{23}$$

Putting in (14) and (15), $k = 3$ gives

$$\begin{aligned}\alpha_2 &= (\langle e^4\rangle\langle e^{10}\rangle - \langle e^6\rangle\langle e^8\rangle)/\Delta,\\ \alpha_3 &= (\langle e^6\rangle^2 - \langle e^4\rangle\langle e^8\rangle)/\Delta,\\ \Delta &= \langle e^6\rangle\langle e^{10}\rangle - \langle e^8\rangle^2\end{aligned} \tag{24}$$

and

$$\frac{\partial}{\partial s}\langle e^2\rangle - 2\alpha_2\frac{\partial}{\partial s}\langle e^4\rangle + (\alpha_2^2 - 2\alpha_3)\frac{\partial}{\partial s}\langle e^6\rangle$$

$$+ 2\alpha_2\alpha_3\frac{\partial}{\partial s}\langle e^8\rangle + \alpha_3^2\frac{\partial}{\partial s}\langle e^{10}\rangle = 0, \qquad s = \mu,\ \lambda. \tag{25}$$

Hence, the second and third approximate coefficients of linearization μ, λ are found from Eqs. (22) and (25), respectively. To close the system of equations, one can use two additional equations (10).

6. Duffing Oscillator

In order to elucidate the extended version of GEL, consider the Duffing oscillator which is a single degree of freedom system with linear damping and cubic nonlinear spring, and has been applied to model many mechanical systems. The equation of motion of such a system is given by

$$\ddot{x} + 2h\dot{x} + \omega_0^2 x + \varepsilon\gamma x^3 = f(t). \tag{26}$$

Here, $f(t)$ is a Gaussian white noise excitation for which

$$\langle f(t)f(t+\tau)\rangle = \sigma^2\delta(\tau), \qquad S_f(\omega) = \frac{\sigma^2}{2\pi}. \tag{27}$$

It is easy to show that Eqs. (22) and (25) are satisfied for $\mu = 0$. So the equivalent linearized equation is to be

$$\ddot{x} + 2h\dot{x} + \omega_0^2 x + \varepsilon\gamma l\langle x^2\rangle x = f(t), \tag{28}$$

where l is an introduced nondimensional stiffness coefficient of linearization which is to be found as a positive root of the following algebraic equation:

- for second approximate solution (see Appendix)

$$L_2(I(l), l) \equiv \frac{d}{dl}I_2 - 2\frac{I_4}{I_6}\frac{d}{dl}I_4 + \frac{I_4^2}{I_6^2}\frac{d}{dl}I_6 = 0, \tag{29}$$

- for third approximate solution (see Appendix)

$$L_3(I(l), l) \equiv \frac{d}{dl}I_2 - 2\alpha_2 I_4 + (\alpha_2^2 - 2\alpha_3)\frac{d}{dl}I_6$$

$$+ 2\alpha_2\alpha_3\frac{d}{dl}I_8 + \alpha_3^2\frac{d}{dl}I_{10} = 0, \tag{30}$$

where

$$\alpha_2 = \frac{I_4 I_{10} - I_6 I_8}{I_6 I_{10} - I_8^2}, \qquad \alpha_3 = \frac{I_6^2 - I_4 I_8}{I_6 I_{10} - I_8^2}. \tag{31}$$

Using the linearized equation (28) the displacement mean square $\langle x^2\rangle$ can be

Table 1. Approximate mean squares of displacement, Duffing equation (26).

N	ε	$\langle x^2\rangle_e$	$\langle x^2\rangle_1$	Error %	$\langle x^2\rangle_2$	Error %	$\langle x^2\rangle_3$	Error %
1	0.1	0.8176	0.8054	−1.49	0.8149	−0.33	0.8264	1.08
1	1.0	0.4679	0.4343	−7.19	0.4458	−4.72	0.4607	−1.54
3	10.0	0.1889	0.1667	−11.8	0.1723	−8.79	0.1798	−4.82
4	100.0	0.0650	0.0561	−13.6	0.0581	−10.59	0.0608	−6.47

determined from the relation

$$\langle x^2\rangle = \frac{\sigma^2}{4h(\omega_0^2 + \varepsilon\gamma l\langle x^2\rangle)}. \tag{32}$$

The results obtained by the procedure proposed (Eqs. (29) and (30), and (32)) $\langle x^2\rangle_2$, $\langle x^2\rangle_3$ are compared in Table 1 with the values $\omega_0 = 1$, $\gamma = 1$, $\sigma^2 = 4h$, and for different values of ε. In addition, the results obtained by the classical GEL technique $\langle x^2\rangle_1$ are also shown. The numerical calculation shows that the mean squares of error sample functions (20) and (23) have a local minimum at the values λ obtained from Eqs. (29) and (30). It is seen from Table 1 that the solutions $\langle x^2\rangle_2$, $\langle x^2\rangle_3$ are much closer to the exact solutions $\langle x^2\rangle_e$ than the solutions $\langle x^2\rangle_1$.

7. Conclusions

The main question inherent in Gaussian equivalent linearization is how the coefficients of the linearized equation are found. Instead of the well-known mean square error criterion, a mean square criterion of error sample function has been proposed to determine these coefficients. An important property of this extended criterion is that it gives a possibility to obtain a series of approximate response mean squares, including the conventional one, as the first approximate solution. Further, a polynomial form of error sample functions is investigated in detail and three first corresponding approximate solutions are given. It is obtained that the technique proposed is as general and simple as within the scope of GEL. Application to Duffing oscillator shows a significant improvement over the corresponding accuracy of the classical GEL.

Acknowledgement. Support from the fundamental research project in natural science is acknowledged.

Appendix

For Duffing equation (26), the equation error is

$$e = \varepsilon(x^3 - l\langle x^2\rangle x). \tag{1}$$

Since x is a Gaussian process, one gets

$$\begin{aligned}\langle e^{2n}\rangle &= \langle[\varepsilon(x^3 - l\langle x^2\rangle x)]^{2n}\rangle \\ &= \left\langle \varepsilon^{2n}\sum_{i=0}^{2n} C_{2n}^i (x^3)^{2n-i} l^i \langle x^2\rangle^i x^i \right\rangle \\ &= \varepsilon^{2n}\sum_{i=0}^{2n} C_{2n}^i (6n-2i-1)!!\langle x^2\rangle^{3n-i}\langle x^2\rangle^i l^i \\ &= \varepsilon^{2n}\langle x^2\rangle^{3n} I_{2n}(l), \end{aligned} \tag{2}$$

where it is denoted

$$I_{2n}(l) = \sum_{i=0}^{2n} C_{2n}^i (6n-2i-1)!!l^i,$$
$$C_{2n}^i = \frac{(2n)!}{i!(2n-1)!}, \qquad (2n-1)!! = 1\cdot 3\cdot 5\cdots(2n-1). \tag{3}$$

Substituting (2) with $n = 1, 2, 3$ into (22) gives Eq. (29). Analogously, substituting (2) with $n = 1, 2, 3, 4, 5$ into (25), (24) gives Eqs. (30) and (31).

References

1. S. A. Assaf and L. D. Zirkie, Approximate analysis of non-linear stochastic systems, *Int. J. Control* **23** (1976) 477–496.
2. S. T. Atalik and S. Utku, Stochastic linearization of multi-degree-of freedom non-linear systems, *J. Earth Eng. Struct. Dynamics* **4** (1976) 420–441.
3. T. K. Caughey, Response of a non-linear string to random loading, *J. Appl. Mech.* **26** (1959) 341–344.
4. F. Casciati and L. Faravelli, Equivalent linearization in non-linear random vibration problems, *Proc. Int. Conf. on Vibration Problems in Eng.*, Xian China, 1986, pp. 986–991.
5. F. G. Fan and G. Ahmadi, On loss of accuracy and non-uniqueness of solutions generated by equivalent linearization and cumulant-neglect methods, *J. Sound and Vibration* **137** (3) (1990) 385–401.
6. W. D. Iwan and I. M. Yang, Statistical linearization for non-linear structures, *ASCE J. Engr. Mech. Div.* **97** (6) (1971) 1609–1623.
7. R. N. Iyengar, Higher order linearization for non-linear random vibration, *Int. J. Non-linear Mech.* **23** (1988) 385–391.
8. H. J. Pradlwater, G. I. Schueller, and X. W. Chen, Consideration of non-Gaussian response properties by use of stochastic equivalent linearization, *Proc. Recent Advances in Structural Dynamics*, Southamton, Petyt, vol. II, H. F. Wolfe and C. Mei (eds.), 1988, pp. 737–752.
9. J. B. Roberts and P. D. Spanos, *Random Vibration and Stochastic Linearization*, John Wiley and Sons, 1990.

10. R. V. Roy and P. D. Spanos, Pade-type approach to non-linear random vibration analysis, *Prob. Eng. Mech.*, Part 2, **6** (3, 4) (1991) 119–128.
11. L. Socha and T. T. Soong, Linearization in analysis of non-linear stochastic systems, *ASME J. Appl. Mech. Rev.* **4** (10) (1991) 399–422.
12. X. Zhang, I. Elishakoff, R. Zhang, A stochastic linearization technique based on square deviation of potential energies, *Stochastic Structural Dynamics 1, New Theoretical Developments*, Y. K. Lin, I. Elishakoff (eds.), Springer-Verlag, 1991, pp. 327–338.