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# Lifts of Calibrations and Minimal Surfaces on the Tangent Bundle of a Riemannian Manifold

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**Abstract.** Let $M$ be a Riemannian manifold. The tangent bundle $TM$ has naturally an induced metric. This paper is devoted to properties of the horizontal lifts of calibrations and $\omega$-manifolds on $TM$. We then describe some minimal surfaces on it.

## 1. Introduction

The method of calibrations is one of the effective methods to show some surface to be homologically minimal in a Riemannian manifold. This method based on the principle of calibrations which has been presented by Dao Trong Thi [1] and later by Harvey and Lawson [7]. By using the method of calibrations, several authors obtained interesting results on minimal surfaces in Grassman manifolds, complex manifolds (see for example, [4, 7, 9]) and in homogenous spaces, Lie groups (see [1, 8]).

For a Riemannian manifold $M$, we denote by $TM$ the tangent bundle of $M$. It is well known that $TM$ can also be considered as a Riemannian manifold. A problem in which several authors are interested is stated as follows (see [4]). Given $M$, find the unit vector field on $M$ having least volume, that is, find a section $V$ on $M$ such that $V$ has least volume. In particular, the problem of determination of optimal vector fields on spheres is of interest and has attracted much attention in recent years. The answer in the case of $S^3$ is given by Gluck, Ziller [5]. In the general case, the problem is still open.

In this paper, by using the method of calibrations and horizontal lift, we discover minimal surfaces in the tangent bundle $TM$. The main results are Theorems 2, 3 and 4.

We need the following theorem which is called the principle of calibrations (see [1, 7]).

**Theorem 1.** *Let $V$ be an oriented compact connected k-surface in the Riemannian manifold $M$. Let $\omega$ be a closed k-form on $M$ such that*

$$\omega(\xi) \leq 1 \tag{1}$$

*for every unit simple k-vector* $\xi$, *and*

$$\omega(\vec{V}_x) = 1 \tag{2}$$

*for almost every* $x \in M$, *where* $\vec{V}_x$ *is the oriented tangent space of* $V$ *at* $x$. *Then* $V$ *is homologically minimal (i.e.,* $V$ *has least volume in its class of homology).*

The form $\omega$ satisfying conditions (1) and (2) is called a calibration and $V$ is called an $\omega$-manifold.

One can show that a calibration is a form of comass one and an $\omega$-manifold is a submanifold whose tangent directions are maximal directions of $\omega$.

## 2. Vertical and Horizontal Spaces on *TM*

Let $M$ be a differential manifold of dimension $n$, $\nabla$ a linear connection, and $K$ the map of connection. Let $\Pi\colon TM \to M$ be the natural projection. For each $u \in TM$, the tangent space $T_uTM$ is the direct sum of $H_u$ and $V_u$, where $H_u$ is the horizontal subspace defined by

$$H_u = \{a \in T_uTM;\ K(a) = 0\},$$

and $V_u$ is the vertical subspace defined by

$$V_u = \{a \in T_uTM;\ \Pi_{*u}(a) = 0\}.$$

For a Riemannian manifold $M$, one can construct different metrics on $TM$. However, we are interested only in the following metric (see, for example, [3]). Denote by $\mathfrak{X}(TM)$ the set of differentiable vector fields on $TM$, and by $\mathscr{F}(TM)$ the set of differentiable functions. Let $g$ be a Riemannian metric of $M$.

Consider the map

$$\tilde{g}\colon \mathfrak{X}(TM) \times \mathfrak{X}(TM) \to \mathscr{F}(TM)$$

given by

$$\tilde{g}(A, B) = g(\Pi_* A, \Pi_* B) + g(KA, KB), \tag{3}$$

where $K$ is the map of Levi–Civita connection.

It is easy to check that $\tilde{g}$ is a metric on $TM$ and $TM$ becomes a Reimannian manifold.

Let us consider $TM$ with the above indicated metric. For each $u \in TM$, $H_u$ is perpendicular to $V_u$. Actually, for $a \in H_u$, $b \in V_u$, we have

$$\tilde{g}(a, b) = g(\Pi_* a, \Pi_* b) + g(Ka, Kb). \tag{4}$$

Since $\Pi_* b = 0$, $Ka = 0$,

$$g(\Pi_* a, \Pi_* b) = g(Ka, Kb) = 0.$$

Thus, $a$ is perpendicular to $b$.

From now, assume $TM$ is equipped with the metric as indicated above.

**Definition 1.** *Let $a \in T_xM$. A vector $\tilde{a} \in T_uTM$ is said to be the horizontal lift of a if $\Pi(u) = x$, $K(\tilde{a}) = 0$, and $\Pi_{*u}(\tilde{a}) = a$. Denote $\tilde{a} = a^H$. A vector $a' \in T_uTM$ is said to be the vertical lift of a if $\Pi(u) = x$, $\Pi_{*u}(a') = 0$, and $K(a') = a$. Denote $a' = a^V$.*

**Proposition 1.** *Let $\Pi(u) = x$. The map $H$: $T_xM \to T_uTM$ defined by $H(a) = a^H$ is a linear isomorphism.*

*Proof.* Let $c = \lambda a + \mu b$, where $a, b \in T_xM$, $\lambda, \mu \in \mathbb{R}$. Assume $\tilde{a} = H(a)$ and $\tilde{b} = H(b)$. Then $\lambda\tilde{a} + \mu\tilde{b}$ is horizontal and

$$\Pi_{*u}(\lambda\tilde{a} + \mu\tilde{b}) = \lambda a + \mu b.$$

Thus, $\lambda\tilde{a} + \mu\tilde{b}$ is the horizontal lift of $\lambda a + \mu b$. Hence, $H(\lambda a + \mu b) = \lambda H(a) + \mu H(b)$ and $H$ is linear.

Now assume $a \in T_xM$ and $H(a) = 0$. Then $\Pi_*(0) = a = 0$. So $H$ is monomorphic. It is easy to check that $H$ is epimorphic.

Thus, $H$ is linearly isomorphic and the proof is complete.

**Definition 2.** *Let $H_u$ be the horizontal subspace at $u \in M$. A subspace $\tilde{V}$ of $H_u$ is called the horizontal lift of subspace $V \subset T_xM$ if $H(V) = \tilde{V}$.*

### 3. Lifts of $K$-form and Their Properties

Let $\Pi$: $TM \to M$ be the natural projection. Then $\Pi$ induces the map $\Pi_*$: $TTM \to TM$. Denote by $E^kM$ the space of differential $k$-forms on $M$. The map also induces the map $\Pi^*$: $E^kM \to E^kTM$ defined by the formula

$$\Pi^*\omega(X_1, \dots, X_k) = \omega(\Pi_*(X_1), \dots, \Pi_*(X_k)) \tag{5}$$

for $\omega \in E^kM$, where $X_1, \dots, X_k \in \mathfrak{X}(TM)$.

Because the operator $\Pi^*$ commutes with the operator $d$, $\Pi^*\omega$ is closed (resp., exact) if $\omega$ is closed (resp., exact).

**Definition 3.** *Let $N$ be a connected submanifold of $M$. A connected submanifold $\tilde{N} \subset TM$ is called a horizontal lift of $N$ if $\Pi(\tilde{N}) = N$ and $\tilde{N}_u$ is the horizontal lift of $N_{\Pi(u)}$.*

**Proposition 2.** *Let $\tilde{N}$ be a horizontal lift of a connected oriented k-surface $N$. Then $\tilde{N}$ is an oriented k-surface.*

*Proof.* Because $N$ is oriented, there exists a $k$-form $\omega$ on $N$ such that $\omega_x \neq 0$ everywhere. Note that $(\Pi^*\omega)|_{\tilde{N}}$ is a $k$-form on $\tilde{N}$. Actually, by definition,

$$(\Pi^*\omega)_u(X_1, \dots, X_k) = \omega_{\Pi(u)}(\Pi_*X_1, \dots, \Pi_*X_k)$$

for $X_1, \dots, X_k \in T_uTM$.

On the other hand, $\Pi_*$ is isomorphic from $H_u$ to $T_{\Pi(u)}M$. Therefore, $(\Pi^*\omega)|_{\tilde{N}}$ is a $k$-form on $\tilde{N}$. Denote $(\Pi^*\omega)|_{\tilde{N}} = \tilde{\omega}$. It is easy to see $\tilde{\omega}_u \neq 0$ everywhere. Thus, $\tilde{N}$ is oriented and the proof is complete. ■

For a $k$-form $\omega$ on $M$, $\Pi^*\omega$ is called the horizontal lift of $\omega$. Denote by $\|\omega\|^*$ the comass of $\omega$ (see [2]). We have the following.

**Theorem 2.** *For an arbitrary k-form $\omega$ on $M$, the following formulas hold:*
(a) $\|\Pi^*\omega\|^* = \|\omega\|^*$,
(b) $G((\Pi^*\omega)_u) = \{\xi^H;\ \xi \in G(\omega_{\Pi(u)})\}$,
*where $G(\omega_{\Pi(u)})$ is the set of maximal directions of $\omega$ at $\Pi(u)$.*

*Proof.* By definition, we have

$$\|(\Pi^*\omega)_u\|^* = \sup\{(\Pi^*\omega)_u(\tilde{\xi});\ \tilde{\xi} \in \Lambda_k T_u TM, \tilde{\xi} \text{ is simple}, |\tilde{\xi}| \le 1\}. \tag{5}$$

Express $\tilde{\xi} = \tilde{v}_1 \wedge \cdots \wedge \tilde{v}_k$ such that $\tilde{v}_i \in T_u TM$, $|\tilde{v}_i| \le 1$. Then

$$\begin{aligned}\|(\Pi^*\omega)_u\|^* &= \sup\{(\Pi^*\omega)_u(\tilde{v}_1 \wedge \cdots \wedge \tilde{v}_k);\ \tilde{v}_i \in T_u TM, |\tilde{v}_i \le 1\}\\ &= \sup\{\omega_{\Pi(u)}(\Pi_*\tilde{v}_1, \ldots, \Pi_*\tilde{v}_k);\ \tilde{v}_i \in T_u TM, |\tilde{v}_i| \le 1\}.\end{aligned} \tag{6}$$

Put $v_i = \Pi_*\tilde{v}_i$. Let $\tilde{X}^i \in \mathfrak{X}(TM)$ such that $\tilde{X}^i_u = \tilde{v}_i$. Then $|\tilde{v}_i|^2 = \tilde{g}(\tilde{X}^i|_u, \tilde{X}^i_u) = g(\Pi_{*u}\tilde{X}^i_u, \Pi_{*u}\tilde{X}^i_u) + g(K\tilde{X}^i_u, K\tilde{X}^i_u) \ge g(\Pi_{*u}\tilde{v}_i, \Pi_{*u}\tilde{v}_i) = g(v_i, v_i)$.

It follows that $|\tilde{v}_i|^2 \ge |v_i|^2$. The equality holds if and only if $\tilde{v}_i$ is horizontal.

Put $\xi = v_1 \wedge \cdots \wedge v_k$. If the system $\{v_1, \ldots, v_k\}$ is linearly dependent, then $\xi = 0$, and it follows that

$$(\Pi^*\omega)_u(\tilde{\xi}) = \omega_{\Pi(u)}(\xi) = 0 \le \|\omega_{\Pi(u)}\|^*.$$

Assume the system $\{v_1, \ldots, v_k\}$ is linearly independent. From (6), we have

$$\begin{aligned}\|(\Pi^*\omega)_u\|^* &= \sup\{\omega_{\Pi(u)}(v_1, \ldots, v_k);\ v_i = \Pi_{*u}\tilde{v}_i\}\\ &\le \sup\{\omega_{\Pi(u)}(\eta);\ \eta \text{ is simple}, |\eta| \le 1\}\\ &= \|\omega_{\Pi(u)}\|^*.\end{aligned}$$

Thus,

$$\|(\Pi^*\omega)_u\|^* \le \|\omega_{\Pi(u)}\|^*. \tag{7}$$

On the other hand, because of the compactness of the set $\{\xi \in \Lambda_k(T_{\Pi(u)}M);\ \xi$ is simple, $|\xi| \le 1\}$, there exists a simple $k$-vector $\xi_0 \in \Lambda_k T_{\Pi(u)}M$, $|\xi_0| \le 1$ such that $\omega_{\Pi(u)}(\xi_0) = \|\omega_{\Pi(u)}\|^*$.

By the previous remark, $\xi_0$ can be expressed in the form $\xi_0 = v_1^0 \wedge \cdots \wedge v_k^0$, where $v_i^0 \in T_{\Pi(u)}M$, $|v_i^0| \le 1$ and $\{v_1^0, \ldots, v_k^0\}$ is linearly independent.

Denote by $\tilde{v}_1^0, \ldots, \tilde{v}_k^0$ the horizontal lifts of $v_1^0, \ldots, v_k^0$. Then the system $\{\tilde{v}_1^0, \ldots, \tilde{v}_k^0\}$ is linearly independent and $|\tilde{v}_i^0| = |v_i^0| \le 1$, $i = 1, \ldots, k$.

Put $\tilde{\xi}_0 = \tilde{v}_1^0 \wedge \cdots \wedge \tilde{v}_k^0$. We can see $|\tilde{\xi}_0| \le 1$. Therefore,

$$\|(\Pi^*\omega)_u\|^* \ge (\Pi^*\omega)_u(\tilde{\xi}_0) = \omega_{\Pi(u)}(\xi_0) = \|\omega_{\Pi(u)}\|^*.$$

Thus,

$$\|\omega_{\Pi(u)}\|^* \le \|(\Pi^*\omega)_u\|^*. \tag{8}$$

From (7) and (8), it follows that

$$\|\omega_{\Pi(u)}\|^* = \|(\Pi^*\omega)_u\|^*. \tag{9}$$

Hence, the first conclusion of the theorem is proved.

To obtain the conclusion (b) we first show that, for every $\xi \in G(\omega_{\Pi(u)})$, the horizontal lift $\tilde{\xi}$ of $\xi$ belongs to $G(\Pi^*\omega)_u$. The simple $k$-vector $\xi$ can be expressed by $\xi = v_1 \wedge \cdots \wedge v_k$, where $\{v_1, \ldots, v_k\}$ is linearly independent and $v_i$ is perpendicular to $v_j$, $|v_i| = 1$.

Denote by $\tilde{v}_i$ the horizontal lift of $v_i$. Then the system $\{\tilde{v}_i\}$ is linearly independent and $\tilde{v}_i$ is perpendicular to $\tilde{v}_j$. Moreover, $|\tilde{v}_i| = 1$.

Let $\tilde{\xi} = \tilde{v}_1 \wedge \cdots \wedge \tilde{v}_k$. It is easy to see that $\tilde{\xi} = \xi^H$ and $|\tilde{\xi}| = 1$. Hence,

$$(\Pi^*\omega)_u(\tilde{\xi}) = \omega_{\Pi(u)}(\xi) = \|\omega_{\Pi(u)}\|^*.$$

Thus, $\tilde{\xi} \in G((\Pi^*\omega)_u)$.

Conversely, for each $\tilde{\xi} \in G((\Pi^*\omega)_u)$, we shall show that $\tilde{\xi}$ is the horizontal lift of some $k$-vector $\xi \in G(\omega_{\Pi(u)})$. Actually, $\tilde{\xi} = \tilde{w}_1 \wedge \cdots \wedge \tilde{w}_k$, where $\{\tilde{w}_1, \ldots, \tilde{w}_k\}$ is an orthonormal system. Let $w_i = \Pi_{*u}(\tilde{w}_i)$. Then $|w_i| \leq |\tilde{w}_i| = 1$. Moreover, if $\{\tilde{w}_1, \ldots, \tilde{w}_k\}$ is linearly dependent, then $\xi = w_1 \wedge \cdots \wedge w_k = 0$. It implies

$$(\Pi^*\omega)_u(\tilde{\xi}) = \omega_{\Pi(u)}(\xi) = 0,$$

a contradiction. Thus, $\{w_i\}$ must be linearly independent. From $|w_i| \leq 1$, we have $|\xi| \leq 1$. From (a) of the theorem, it follows that

$$\|\omega_{\Pi(u)}\|^* = \|(\Pi^*\omega)_u\|^* = (\Pi^*\omega)_u(\tilde{\xi}) = \omega_{\Pi(u)}(\xi).$$

Therefore, $\xi \in G(\omega_{\Pi(u)})$ and $|\xi| = 1$. Then $|w_i| = 1 = |\tilde{w}_i|$. This shows that $\tilde{w}_i$ is horizontal and $\tilde{w}_i = w_i^H$. Thus, the conclusion (b) of the theorem is proved. ■

**Definition 4.** *Let $TM$ be the tangent bundle, $V_u$ the vertical space at $u \in TM$, and $K$ the map of connection. A covector $\bar{\varphi}$ in $T_uTM$ is called the vertical lift of a covector $\varphi$ in $T_pM$ if $\bar{\varphi}(v) = \varphi(K(v))$ for every $v \in V_u$. A differential 1-form $\bar{\xi}$ on $TM$ is called the vertical lift of a 1-form $\xi$ on $M$ if $\bar{\xi}_u$ is the vertical lift of $\xi_{\Pi(u)}$.*

**Lemma 1.** *On a parallelizable Riemannian manifold $M$ of dimension $n$, there exist differential 1-forms $\xi^1, \ldots, \xi^n$ such that $\{\xi^1_p, \ldots, \xi^n_p\}$ is linearly independent for every $p \in M$.*

*Proof.* Let $\{X_1, \ldots, X_n\}$ be a parallelization of $M$. For each $i = 1, \ldots, n$, define a 1-form $\xi^i$ as follows

$$\xi^i_p((X_j)_p) = \delta_{ij},\ j = 1, \ldots, n. \tag{10}$$

We shall show that the functions $\xi^i$, $i = 1, \ldots, n$, are differentiable. For $X \in \mathfrak{X}(M)$, $X$ is expressed by

$$X = \sum_{j=1}^{n} f_j X_j,\ f_j \in \mathscr{F}(M). \tag{11}$$

Then

$$(\xi^i(X))_p = \xi^i_p(X_p) = \xi^i_p(\sum f_j(p)(X_j)_p) = f_i(p). \tag{12}$$

It follows that $\xi^i(X) = f_i$ is differentiable.

It remains to prove that, for $p \in M$, the system $\{\xi^i_p\}$ is linearly independent. Indeed, if $\sum_{i=1}^{n} \lambda_i \xi^i_p = 0$ for $\lambda_i \in R$, then, for every $v \in T_pM$, $\sum_{i=1}^{n} (\lambda_i \xi^i_p)(v) = 0$. Therefore,

$$\sum \lambda_i \xi^i_p(v) = 0. \tag{13}$$

In (13), if $v = X_j$, then $\lambda_j = 0$. This holds for every $j = 1, \ldots, n$. Thus, the proof is complete. ■

**Lemma 2.** *Let $\xi^1_p, \ldots, \xi^n_p$ be linearly independent on TM. Then vertical lifts of them are linearly independent.*

*Proof.* Denote by $\bar{\xi}^1, \ldots, \bar{\xi}^n$ the vertical lifts of covectors $\xi^1_p, \ldots, \xi^n_p$. Assume

$$\sum_{i=1}^{n} \lambda_i \bar{\xi}^i = 0.$$

It follows that $\sum \lambda_i \xi^i(K\bar{v}) = 0$ for $\bar{v} \in TTM$. Since $K$ is epimorphic, $\sum_{i=1}^{n} \lambda_i \xi^i = 0$. The independence of $\{\xi^i\}$ implies $\lambda_i = 0$ for $i = 1, \ldots, n$. Thus, the system $\{\bar{\xi}^1, \ldots, \bar{\xi}^n\}$ is linearly independent. The proof of the lemma is complete. ■

Now, let us construct some calibrations on $TM$. This is based on the following theorem and some properties in [6].

**Theorem 3.** *Let M be a parallelizable Riemannian manifold and ω a k-form on M. Then there exists an (n + k)-form Ω on TM such that*

$$\Omega_u = e^*_{V_u} \wedge (\Pi^* \omega)_u, \; u \in TM. \tag{14}$$

*Moreover,*

$$\|\Omega\|^* = \|\omega\|^*, \tag{15}$$

*where $e^*_{V_u}$ is the unit simple n-covector associated with $V_u$.*

*Proof.* By Lemma 1, there is a system of linearly independent 1-forms $\{\xi^1, \ldots, \xi^n\}$. Denote by $\bar{\xi}^i$ the vertical lift of $\xi^i$ for each $i = 1, \ldots, n$. Then

$$\Omega_1 = \bar{\xi}^1 \wedge \cdots \wedge \bar{\xi}^n$$

is an $n$-form on $TM$. We can identify $\lambda\Omega_1$ with $e^*_{V_u}$, where $\lambda$ is some real value.

Consider an $(n + k)$-form $\Omega$ defined by

$$\Omega = (\lambda\Omega_1) \wedge (\Pi^* \omega). \tag{16}$$

For

$$u \in TM, \; \Omega_u = e^*_{V_u} \wedge (\Pi^* \omega)_u, \tag{17}$$

we shall prove that

$$\|\Omega_u\|^* = \|(\Pi^*\omega)_u\|^*. \tag{18}$$

Since (17), by [2], we have

$$\|\Omega_u\|^* = \|e^*_{V_u} \wedge (\Pi^*\omega)_u\|^* \leq \|e^*_{V_u}\|^*.\|(\Pi^*\omega)_u\|^*. \tag{19}$$

The comass of $e^*_{V_u}$ is equal to 1 because $e^*_{V_u}$ is a unit simple $n$-covector. It follows that

$$\|\Omega_u\|^* \leq \|(\Pi^*\omega)_u\|^*. \tag{20}$$

Let $\eta = v_1 \wedge \cdots \wedge v_p$ be a unit simple $p$-vector on $T_uTM$ such that $(\Pi^*\omega)_u(\eta) = \|(\Pi^*\omega)_u\|^*$, where $v_1, \ldots, v_p$ are linearly independent in $T_uTM$. We can choose $v_1, \ldots, v_p$ in the horizontal space $H_u$. Then $e_{V_u} \wedge \eta$ is simple. Hence,

$$\|\Omega_u\|^* \geq \Omega_u(e_{V_u} \wedge \eta) = (e^*_{V_u} \wedge (\Pi^*\omega)_u)(e_{V_u} \wedge \eta)$$
$$= (\Pi^*\omega)_u(\eta) = \|(\Pi^*\omega)_u\|^*.$$

Thus,

$$\|\Omega_u\|^* \geq \|(\Pi^*\omega)_u\|^*. \tag{21}$$

From (20) and (21), we have

$$\|\Omega_u\|^* = \|(\Pi^*\omega)_u\|^*.$$

This shows that $\|\Omega_u\|^* = \|\Pi^*\omega\|^*$ and the proof of the theorem is complete. ■

A direct corollary of this theorem can be stated as follows. If $\omega$ is of comass one, then $\Omega$ is of comass one. This helps us to find some $(n+k)$-surfaces which are $\Omega$ submanifolds in $TM$ and therefore, will be homologically minimal.

**Theorem 4.** *Let $\omega$ be a calibration of an oriented connected compact k-surface $N$ of the Riemannian manifold $M$. If the horizontal lift $\tilde{N}$ is compact, then $\Pi^*\omega$ is a calibration of $\tilde{N}$ and $\tilde{N}$ is homologically minimal.*

*Proof.* By Proposition 2, $\tilde{N}$ is oriented. Because $\omega$ is a calibration, $\omega$ is closed and of comass one. By Theorem 2, $\Pi^*\omega$ is also closed and its comass is equal to one. For $u \in \tilde{N}$, $\Pi(u) = p \in N$, $\tilde{N}_u$ is the horizontal lift of $N_p$.

Let $\{e_1, \ldots, e_k\}$ be an orthonormal basis of $N_p$. Then the set of horizontal lifts $\bar{e}_1, \ldots, \bar{e}_k$ forms an orthonormal basis of $\tilde{N}_u$. Hence, $\tilde{N}_u$ is the horizontal lift of $N_p$. We have $\omega_p(\tilde{N}_p) = 1$ for every $p \in N$. It follows that $N_p \in G(\omega_p)$. According to Theorem 2, we obtain $\tilde{N}_u \in G((\Pi^*\omega)_u)$. Thus, $\Pi^*\omega$ is a calibration of $\tilde{N}$ and therefore, $\tilde{N}$ is homologically minimal. This completes the proof. ■

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