Vietnam Journal of Mathematics **25**:1 (1997) 15–24

Vietnam Journal
of
MATHEMATICS

Spline Collocation Methods for Fredholm–Volterra Integro–Differential Equations of High Order

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Received April 29, 1995
Revised January 12, 1996

Abstract. In this paper, we study a general boundary value problem for Fredholm–Volterra integro–differential equations of high order by spline collocation methods. In this general case, it is not appropriate to use diagonally dominant matrix to solve problems of existence and uniqueness of approximate solutions as in [3]. Projection methods are used instead. In many cases, a theorem on the convergence rate of high order of approximants to exact solution is established and shown to be much better than the result in [3].

1. Consider the equation

$$Lx(t) = x^{(m)}(t) + \sum_{j=0}^{m-1} a_{m-1-j}x^{(m-1-j)}(t)$$

$$+ \lambda \int_a^b K_1(t,s)x(s)\,ds + \int_a^t K_2(t,s)x(s)\,ds = f(t), \tag{1}$$

with m independent boundary conditions

$$\sum_{k=0}^{m-1} \alpha_{ik}x^{(k)}(a) + \sum_{k=0}^{m-1} \beta_{ik}x^{(k)}(b) = \gamma_i, i = 1, \dots, m, \tag{2}$$

where $a_k(t) \in C[a,b], \alpha_{ik}, \beta_{ik}, \gamma_i, \lambda \in \mathbf{R}$, $k = 0, \dots, m-1$, $i = 1, \dots, m$, $a \le t, s \le b$, $K_j(t,s) \in C(\Omega)$, $\Omega \equiv [a,b] \times [a,b]$, $j = 1, 2$. Without loss of generality, we can assume $\gamma_i = 0$, $i = 1, \dots, m$. Indeed, if $g(t) \in C^m[a,b]$, satisfying conditions (2), and $u(t)$ is a solution of the equation

$$Lu(t) = f(t) - L(g(t)), \tag{3}$$

This work was supported in part by the National Basic Research Programme in Natural Science, Vietnam.

varifying the independent conditions

$$\sum_{k=0}^{m-1} \alpha_{ik} u^{(k)}(a) + \sum_{k=0}^{m-1} \beta_{ik} u^{(k)}(b) = 0, \qquad i = 1, \ldots, m, \tag{4}$$

then it is easy to see that $x(t) = u(t) + g(t)$ is the solution of Eq. (1) satisfying conditions (2). Consequently, from now on we shall consider only Eq. (1) with conditions (4).

2. Let

$$\pi_n : a = t_0 < t_1 < \cdots < t_n = b$$

be a partition of $[a, b]$.

Set

$$h_i = t_{i+1} - t_i, \quad i = 0, \ldots, n-1, \qquad h_n = \max_{1 \le i \le n-1} h_i.$$

We always suppose that $f \in C[a, b]$ with $\|f\| = \max_{a \le t \le b} |f(t)|$, and the sequence $\{\pi_n\}$ has the property $\lim_{n \to \infty} h_n = 0$. We shall use the set

$$S_n = \{\zeta_i | \zeta_i \in [a, b], i = 1, \ldots, N_n, N_n \in \mathbb{N}\}$$

for a collocation set in $[a, b]$, where N_n is a constant dependent on n.

Denote

$$Sp(\pi_n, p, q) = \left\{ v(t) \in C^q[a, b] : v(t)|_{[t_i, t_{i+1}]} \in Q_p, i = 0, \ldots, n-1 \right\},$$

$$\overset{0}{Sp}(\pi_n, p, q) = \{v(t) \in Sp(\pi_n, p, q) : v(t) \text{ satisfying } (4)\},$$

where p, q are integers satisfying $0 \le q \le p - 1, p \ge 1$, and Q_p is the set of polynomials of order $\le p$.

Assume that the problem

$$\begin{cases} x^{(m)}(t) = 0, \\ \sum_{k=0}^{m-1} \alpha_{ik} x^{(k)}(a) + \sum_{k=0}^{m-1} \beta_{ik} x^{(k)}(b) = 0, \qquad i = 1, \ldots, m, \end{cases} \tag{5}$$

has only trivial solution and let $G(t, s)$ be its Green function. Then it is well known that the problem

$$\begin{cases} x^{(m)}(t) = v(t), \\ \sum_{k=0}^{m-1} \alpha_{ik} x^{(k)}(a) + \sum_{k=0}^{m-1} \beta_{ik} x^{(k)}(b) = 0, \qquad i = 1, \ldots, m, \end{cases}$$

has a unique solution defined by

$$x(t) = \int_a^b G(t, s) v(s)\, ds, \tag{6}$$

(see [1], p. 132). Consider the operator $U : C[a,b] \to C[a,b]$, defined by

$$Uv = \int_a^b G(t,s)v(s)\,ds.$$

By (6) we get

$$x^{(k)}(t) = \int_a^b \frac{\partial^k G(t,s)}{\partial t^k} v(s)\,ds, \qquad k = 1,\dots,m-1.$$

Hence, Eq. (1) becomes

$$v(t) + \int_a^b \sum_{j=0}^{m-1} a_j(t) \frac{\partial^j G(t,s)}{\partial t^j} v(s)\,ds + \lambda \int_a^b \left[\int_a^b K_1(t,\zeta)G(\zeta,s)v(s)\,ds \right] d\zeta$$

$$+ \int_a^t \left[\int_a^b K_2(t,\zeta)G(\zeta,s)v(s)\,ds \right] d\zeta = f(t).$$

By the continuity of $K_i(t,s), i = 1,2$ and $G(s,\zeta)$, we have

$$v(t) + \int_a^b \left[\sum_{j=0}^{m-1} a_j(t) \frac{\partial^j G(t,s)}{\partial t^j} + \lambda \int_a^b K_1(t,\zeta)G(\zeta,s)\,d\zeta \right.$$

$$\left. + \int_a^t K_2(t,\zeta)G(\zeta,s)\,d\zeta \right] v(s)\,ds = f(t). \tag{7}$$

If we denote by T the operator $T : C[a,b] \to C[a,b]$ with

$$Tv = \int_a^b \left[\sum_{j=0}^{m-1} a_j(t) \frac{\partial^j G(t,s)}{\partial t^j} + \lambda \int_a^b K(t,\zeta)G(\zeta,s)\,d\zeta \right.$$

$$\left. + \int_a^t K_2(t,\zeta)G(\zeta,s)\,d\zeta \right] v(s)\,ds,$$

then Eq. (7) can be written in the form

$$(I+T)v = f, \tag{8}$$

where I is the identity operator in $C[a,b]$.

Therefore, if $a_j(t) \in C[a,b]$, $j = 0,\dots,m-1$, $K_i(t,s) \in C(\Omega)$, $i = 1,2$ and Eq. (5) has only trivial solution, then the problem (1), (4) is equivalent to (8).

If we approximate the solution of (8) by an element $v_n \in Sp(\pi_n, p, q)$ such that

$$(I+T)v_n(\zeta_i) = f(\zeta_i), \zeta_i \in S_n,$$

then the element

$$x_n(t) = \int_a^b G(t,s)v_n(s)\,ds, \qquad x_n(t) \in \overset{0}{Sp}(\pi_n, p+m, q+m)$$

satisfies the equation

$$Lx_n(\zeta_i) = f(\zeta_i), \qquad \zeta_i \in S_n.$$

Consequently, $x_n(t)$ is the desired collocation solution. To find the above-mentioned element we take a continuous linear projection P_n,

$$P_n : C[a,b] \to Sp(\pi_n, p, q) \subset C[a,b],$$

having the property $P_n f(\zeta_i) = f(\zeta_i), \forall \zeta_i \in S_n$. Then, from (8), we obtain

$$P_n v + P_n T v = P_n f. \tag{9}$$

It is obvious that if $v_n \in Sp(\pi_n, p, q)$ satisfies (9), then we get

$$v_n + P_n T v_n = P_n f, \tag{10}$$

which means that the desired element v_n is found.

We shall make use of the following.

Lemma. *Let $a_j(t) \in C[a,b]$, $j = 0, \ldots, m-1$, $K_i(t,s) \in C(\Omega)$, $i = 1, 2$,*

$$[K_2(t_1,s) - K_2(t_2,s)] \le L|t_1 - t_2|, \qquad \forall t_1, t_2 \in [a,b],$$

(L is a positive constant), P_n are continuous linear projections,

$$P_n : C[a,b] \to Sp(\pi_n, p, q),$$

satisfying

$$\|P_n f - f\| \le M\omega(f, h_n), \qquad \forall f \in C[a,b],$$

$P_n f(\zeta_i) = f(\zeta_i)$, $\forall \zeta_i \in S_n$, $\forall f \in C[a,b]$, where M is a constant independent of n, and $\omega(f, h_n)$ is the modulus of continuity of function f with respect to h_n. Then
(i) *The sequence P_n converges pointwise to the identity operator I in $C[a,b]$.*
(ii) *The sequence $P_n T$ converges to T in the space $\mathscr{L}(C[a,b], C[a,b])$ (the space of continuous linear operators in $C[a,b]$).*

Proof. The proof of statement (i) is easy and hence it is omitted. Since $a_j(t) \in C[a,b]$, $K_i(t,s)$, $G(t,s) \in C(\Omega)$ and $K_2(t,s)$ satisfy the Lipschitz condition with repect to t, the operator T is completely continuous in $C[a,b]$. Let $S(0,\rho) = \{x(t) \in C[a,b] : \|x\| \le \rho, \rho > 0\}$. By the complete continuity of T, it follows that for arbitrary positive ε there exists a finite set $\mathscr{B} = \{y_1, \ldots, y_v\}$ such that for every $x_1 \in S(0,\rho)$ there is $y_i \in \mathscr{B}$ such that

$$\|Tx_1 - y_i\| < \varepsilon,$$

and we get

$$\begin{aligned}\|Tx_1 - P_n T x_1\| &= \|(I - P_n)Tx_1\| \\ &\le \|(I - P_n)(Tx_1 - y_i)\| + \|(I - P_n)y_i\| \\ &\le \|(I - P_n)\|\varepsilon + \|(I - P_n)y_i\|.\end{aligned}$$

Since $\{P_n\}$ is a sequence of continuous linear projections pointwise converging to

I, and $C[a,b]$ is a Banach space, it follows from the Banach Steinhauss theorem that the sequence $\{P_n\}$ is bounded: there exists a positive number M_1 such that $\|P_n\| \le M_1\|, \forall n$. Taking n sufficiently large we have

$$\|Tx_1 - P_nTx_1\| \le (1+M_1)\varepsilon + \varepsilon \le (2+M_1)\varepsilon, \forall x_1 \in S(0,\rho),$$

hence $\|P_nT - T\| \to 0$ as $n \to \infty$, and the lemma is proved.

Theorem 1. *Let* $a_j, f \in C[a,b]$, $j = 0,\ldots,m-1$, $K_i(t,s) \in C(\Omega)$, $i = 1,2$,

$$|K_2(t_1,s) - K_2(t_2,s)| \le L|t_1 - t_2|, \qquad \forall t_1,\ t_2 \in [a,b],$$

(L is a positive constant). Assume that P_n *are the projections mentioned in the above lemma and there exists an inverse operator* $(I+T)^{-1}$ *of Eq. (8) and Eq. (5) has only trivial solution. Then*
(i) *For sufficiently large* $n (n \ge N_0)$, *there exists a unique collocation solution* x_n *of the problem* (1), (4) *such that* $x_n \in \overset{0}{Sp}(\pi_n, m+p, q+m)$ *on* S_n.
(ii) *The convergence rate of the approximate solution* x_n *to the exact solution* x *is estimated by*

$$\|x - x_n\| \le \beta\omega(x^{(m)}, h_n),$$

where $\beta = M\gamma\|U\|$.

Proof. (i) Set $A = I + T$, $B = P_nT - T$.

It is clear that A and B are bounded linear operators in $C[a,b]$. On the other hand, by the above lemma, $\|P_nT - T\| \to 0$ as $n \to \infty$. Thus, there exists a natural number N_0 such that for all $n \ge N_0$ we have

$$\|P_nT - T\| \le \alpha, \qquad 0 < \alpha = \text{const}, \qquad \alpha\|(I+T)^{-1}\| < 1.$$

By using Lemma 15.2 from [2], we can assert that there exists an inverse operator $(A+B)^{-1} = (I+P_nT)^{-1}$, and

$$\|(I+P_nT)^{-1}\| \le \frac{\|(I+T)^{-1}\|}{1-\alpha\|(I+T)^{-1}\|} = \gamma.$$

So Eq. (10) has a unique solution v_n, $v_n = (I+P_nT)^{-1}P_nf$.

Since (5) has only trivial solution, there exists a unique collocation solution

$$x_n(t) = \int_a^b G(t,s)v_n(s)\,ds, \qquad x_n(t) \in \overset{0}{Sp}(\pi_n, p+m, q+m),\ n \ge N_0,$$

that proves the statement (i).

(ii) By (9) we have

$$v + P_nTv = P_nf + v - P_nv,$$

$$v - v_n + P_nT(v - v_n) = v - P_nv,$$

$$v - v_n = (I+P_nT)^{-1}(v - P_nv).$$

It follows that

$$\begin{aligned} x - x_n &= U(v - v_n) \\ &= U(I + P_nT)^{-1}(x^{(m)} - P_nx^{(m)}), \qquad (11) \\ \|x - x_n\| &\le \gamma\|U\|M\omega(x^{(m)}, h_n) \\ &\le \beta\omega(x^{(m)}, h_n), \end{aligned}$$

where $\beta = \gamma\|U\|M$. This completes the proof of the theorem.

3. Below, we shall consider some concrete projections which can be used to solve approximately the problem (1), (4). Note that, with respect to each projection, we obtain a spline collocation method to solve Eq. (1) with conditions (4).

(a) We take $d + 1$ points from the segment $[0, 1]$ as follows:

$$0 = \eta_0 < \eta_1 < \cdots < \eta_d = 1.$$

Taking into account the partition

$$\pi_n : a = t_0 < t_1 < \cdots < t_n = b,$$

we get

$$S_n = \{\zeta_{ij} = t_i + h_i\eta_j, i = 0, \ldots, n - 1, j = 0, \ldots, d\}.$$

Define the mapping $\hat{P}_n : C[a, b] \to Sp(\pi_n, d, 0)$ by setting

$$(\hat{P}_n)f(\zeta_{ij}) = f(\zeta_{ij}), \qquad \forall\zeta_{ij} \in S_n. \qquad (12)$$

It is obvious that $\hat{P}_n$ is a linear projection from $C[a, b]$ into $Sp(\pi_n, d, 0)$. On each segment $[t_i, t_{i+1}]$, $i = 0, \ldots, n - 1$, $\hat{P}_nf$ is a polynomial of order $\le d$ interpolating the function f at ζ_{ij}. If we denote by $\{l_j(t)\}_{j=0}^{d}$ the basic system of polynomials of order $\le d$ on $[t_i, t_{i+1}]$ satisfying $l_j(\zeta_{ik}) = \delta_{jk}, k = 0, \ldots, d$ (where δ_{jk} is the delta Kronecker), then for every $f \in C[a, b]$ we get

$$\begin{aligned} \|\hat{P}_nf\| &\le \max_{0 \le i \le n-1} \max_{t_i \le t \le t_{i+1}} \left|\sum_{j=0}^{d} f(\zeta_{ij})l_j(t)\right|, \\ &\le \|f\| \max_{t_i \le t \le t_{i+1}} \sum_{j=0}^{d} |l_j(t)|, \end{aligned}$$

where

$$l_j(t) = \prod_{k=0, k \ne j}^{d} \frac{t - \zeta_{ik}}{\zeta_{ij} - \zeta_{ik}}.$$

As in [6, p. 5] we have

$$l_j(t) = \prod_{r=0, r \ne j}^{d} \frac{\eta - \eta_r}{\eta_j - \eta_r} \equiv l_j^*(\eta),$$

and

$$M_2 = \max_{t_i \le t \le t_{i+1}} \sum_{j=0}^{d} |l_j(t)| = \max_{0 \le \eta \le 1} \sum_{j=0}^{d} |l_j^*(\eta)|.$$

M_2 is the Lebesgue constant independent of i and n. Consequently, $\|\hat{P}_n\| \le M_2$, so $\hat{P}_n$ is continuous. Let p^* be the best approximation of f by a polynomial of order d in $[t_i, t_{i+1}]$ (see [4], p. 43). Then

$$\begin{aligned} |p^*(t) - \hat{P}_n f(t)| &= |\sum_{j=0}^{d} (p^*(\zeta_{ij}) - f(\zeta_{ij})) l_j(t)| \\ &\le \|p^* - f\| \max_{t_i \le t \le t_{i+1}} \sum_{j=0}^{d} |l_j(t)| \\ &\le M_2 \|f - p^*\|. \end{aligned}$$

By Jackson's Theorem (see [4], p. 43) we obtain

$$\max_{t_i \le t \le t_{i+1}} |p^*(t) - \hat{P}_n f(t)| \le M_2 g_k(f, d),\ f \in C^k[a, b],$$

where

$$g_k(f, d) = \begin{cases} 6\omega\left(f, \frac{h_n}{2d}\right), & \text{when } k = 0, \\ \frac{3h_n}{d} \|f'\|, & \text{when } k = 1, \\ \frac{6^k (k-1)^{k-1}}{(k-1)! d^k} k h_n^k \|f^{(k)}\|, & \text{when } k > 1, d > k - 1 \ge 1. \end{cases}$$

So

$$\begin{aligned} \|f - \hat{P}_n f(t)\| &\le \|p^* - \hat{P}_n f\| + \|f - p^*\| \\ &\le (M_2 + 1) g_k(f, d). \end{aligned} \tag{13}$$

For $k = 0$, we have

$$\|f - \hat{P}_n f\| \le 6(M_2 + 1)\omega\left(f, \frac{h_n}{2d}\right), \qquad \forall f \in C[a, b].$$

The projections $\hat{P}_n$ have thus the properties mentioned in the above lemma. Therefore, we get the following.

Theorem 2. *Let $a_j(t), f \in C[a, b]$, $j = 0, \ldots, m-1$, $K_i(t, s) \in C(\Omega)$, $i = 1, 2$,*

$$K_2(t_1, s) - K_2(t_2, s)| \le L|t_1 - t_2|, \qquad \forall t_1, t_2 \in [a, b],$$

(L is a positive constant). $\hat{P}_n$ in the projections defined by (12). Assume that there exists an inverse operator $(I + T)^{-1}$ and Eq. (5) has only trivial solution. Then

(i) *For sufficiently large* n, *there exists a unique collocation solution* x_n *of the problem* (1), (4) *such that* $x_n \in \overset{0}{Sp}(\pi_n, m+d, m)$ *on* $\overset{*}{S}_n$.
(ii) *The convergence rate of the approximants* x_n *to the exact solution* x *is given as follows:*

$$\|x - x_n\| = \begin{cases} O(\omega(x^{(m)}, h_n)), & \text{if } x \in C^m[a,b], \\ O(h_n^k), & \text{if } x \in C^{m+k}[a,b], 1 \le k \le d+1. \end{cases}$$

Proof. By using Theorem 1, we immediately get (i). For the proof of (ii) from (11), (13) we see that

$$\begin{aligned} \|x - x_n\| &\le \gamma \|U\| \|x^{(m)} - \hat{P}_n x^{(m)}\| \\ &\le \gamma \|U\| (M_2 + 1) g_k(x^{(m)}, d). \end{aligned}$$

Hence,

$$\|x - x_n\| = \begin{cases} O(\omega(x^{(m)}, h_n)), & \text{if } x \in C^m[a,b], \\ O(h_n^k), & \text{if } x \in C^{m+k}[a,b], 1 \le k \le d+1. \end{cases}$$

This completes the proof of Theorem 2. ■

Now let π_n be a uniform partition of $[a,b]$,

$$\pi_n : a = t_0 < t_1 < \cdots < t_n = b, \qquad h = \frac{b-a}{n}.$$

Let $k \ge 1$ be a natural number and $n \ge 2k-1$, and $S_n = \{t_0, \ldots, t_n\}$.
Consider now the mapping

$$\mathbf{P}_n : C[a,b] \to Sp(\pi_n, 2k-1, 2k-2)$$

such that

$$\begin{cases} (P_n f)(t_i) = f(t_i), & i = 0, \ldots, n, \\ D^j(\mathbf{P}_n f)(a) = D^j(L_{2k-1,0} f)(a), & j = 1, \ldots, k-1, \\ D^j(\mathbf{P}_n f)(b) = D^j(L_{2k-1,1} f)(b), & j = 1, \ldots, k-1, \end{cases} \tag{14}$$

where $L_{2k-1,0} f$, $(L_{2k-1,1} f)$ are Lagrange interpolation polynomials of function f at points $t_0, t_1, \ldots, t_{2k-1}$, $(t_{n-2k+1}, t_{n-2k}, \ldots, t_n)$, respectively.

It is obvious that $\mathbf{P}_n$ are continuous linear projections from $C[a,b]$ to $Sp(\pi_n, 2k-1, 2k-2)$ and

$$\|f - \mathbf{P}_n f\| \le \theta \omega(f, h),$$

where θ is a constant independent of n (see [7], p. 347).

Clearly, $\mathbf{P}_n$ also satisfies all properties of the projections mentioned in the above lemma.

Theorem 3. *Let* $a_j(t), f \in C[a,b], K_i(t,s) \in C(\Omega), i = 1,2$

$$[K_2(t_1,s) - K_2(t_2,s)] \le L|t_1 - t_2|, \qquad \forall t_1, t_2 \in [a,b],$$

(L is a positive constant), and $\mathbf{P}_n$ *be the projections defined by* (14)*. Assume that there exists an inverse operator* $(I+T)^{-1}$ *and Eq.* (5) *has only trivial solution. Then*

(i) *For sufficiently large n, there exists a unique collocation solution* x_n *of the problem* (1), (4) *such that* $x_n \in \overset{0}{Sp}(\pi_n, m+2k-1, m+2k-2)$.

(ii) *The convergence rate of the approximants* x_n *is given by*

$$\|x - x_n\| = O(h^r \omega(x^{(m+r)}, h)),$$

where $0 \le r \le 2k-1, x \in C^{r+m}[a,b], r \in \mathbf{N}$.

Proof. The statement (i) is obvious. It remains to prove (ii). From (11) we get

$$\begin{aligned}
\|x - x_n\| &\le \|U\|\gamma\|x^{(m)} - \mathbf{P}_n x^{(m)}\| \\
&\le \|U\|\gamma\|x^{(m)} - v - (\mathbf{P}_n x^{(m)} - \mathbf{P}_n v)\|, \\
&\quad \forall v \in Sp(\pi_n, 2k-1, 2k-2) \\
&\le \|U\|\gamma\|(I - \mathbf{P}_n)(x^{(m)} - v)\| \\
&\le \|U\|\gamma(1 + \|\mathbf{P}_n\|)\|x^{(m)} - v\|, \\
&\quad \forall v \in Sp(\pi_n, 2k-1, 2k-2) \\
&\le \|U\|\gamma(1 + \|\mathbf{P}_n\|) \inf_{v \in Sp(\pi_n, 2k-1, 2k-2)} \|x^{(m)} - v\|.
\end{aligned}$$

By using Theorem 1 from [7] we obtain

$$\inf_{v \in Sp(\pi_n, 2k-1, 2k-2)} \|x^{(m)} - v\| = O(h^r \omega(x^{(m+r)}, h)),$$

where $x \in C^{m+r}[a,b], 0 \le r \le 2k-1$.

So,

$$\|x - x_n\| = O(h^r \omega(x^{(m+r)}, h)).$$

The proof of Theorem 3 is complete. ∎

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