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Recent Work on Hyperbolic Spaces

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Abstract. On the topic of hyperbolic spaces there are some excellent surveys. In [28], Lang discussed the relation between hyperbolic geometry and Diophantine geometry. The paper of Zaidenberg [48] described main results on hyperbolicity in projective spaces. Noguchi [38] gave a survey of open problems in Nevanlinna theory, theory of hyperbolic spaces and Diophantine geometry. In these papers, the reader can find a picture of the development of the theory up to the 1990-th year. This survey arose out of attemps to give a brief description of recent results in the study of hyperbolic spaces. Of course, it does not pretend to be an exhaustive study, but it does focus mainly on new results leading towards a solution of Kobayashi's conjectures.

1. Basic Notions. Problems

We first recall a few basic facts concerning the concept of hyperbolicity.

Let D be a unit disc. The *Poincaré metric*, also called the hyperbolic metric, on D is defined by the form

$$\frac{dz\, d\bar{z}}{(1-|z|^2)^2}\,.$$

The tangent space $T_z(D)$ at a given point z can be identified with $\mathbf{C}$, and if $v \in T_z(D) = \mathbf{C}_2$ under this identification, then the hyperbolic norm of v under the metric is

$$|v|_{\text{kyp},z} = \frac{|v|_{\text{Euc}}}{1-|z|^2}\,,$$

where $|v|_{\text{Euc}}$ is the ordinary absolute value on $\mathbf{C}$. Note that for $z = 0$ the Poincaré metric is the Euclidean metric.

Now let X be a complex space. Let $x, y \in X$. We consider sequences of holomorphic maps

$$f_j : D \to X\,, \qquad j = 1, \ldots, m$$

and points p_j, $q_j \in D$ such that $f_1(p_1) = x$, $f_m(q_m) = y$, and

$$f_j(q_j) = f_{j+1}(p_{j+1}).$$

In other words, we join x to y by a chain of discs. We add the hyperbolic distances between p_j and q_j, and take the infimum over all such choices of f_j, p_j, q_j to define *Kobayashi semi distance*

$$d_x(x, y) = \inf \sum_{j=1}^{m} d_{\text{hyp}}(p_j, q_j).$$

Then d_x satisfies the properties of a distance, except that $d_x(x, y)$ may be 0 if $x \neq y$, so d_x is a semi distance.

Definition 1.1. *The space X is said to be* hyperbolic *(in the sense of Kobayashi) if d_x is actually a distance, namely if $d_x(x, y) > 0$ for all pairs of distinct points (x, y) in X.*

We say that a complex space X is *complete hyperbolic* if X is hyperbolic and complete with respect to the distance d_x.

Let X be a complex subspace of a complex space Y. Then X is said to be hyperbolically embedded in Y if X is hyperbolic, and satisfies the following condition. Let $\{x_n\}$, $\{y_n\}$ be two sequences in X converging to points x, y in the closure of X in Y, respectively. If $d_x(x_n, y_n) \to 0$, then $x = y$.

One of the important properties of a hyperbolic space X is that every holomorphic map from the complex plane C into X is constant. We have the following.

Definition 1.2. *A complex space X is said to be* Brody hyperbolic *if there is no nonconstant holomorphic curve $f: \mathbf{C} \to X$.*

If is well known that *for a compact complex space, Brody hyperbolic is equivalent to Kobayashi hyperbolic.*

In the 1970s, Kobayashi posed the following problems [27].

Is a generic hypersurface $X \subset \mathbf{P}^n$ of degree d large enough (say $d \geq 2n + 1$) hyperbolically? Is $\mathbf{P}^n \setminus X$ hyperbolic (moreover, hyperbolically embedded into $\mathbf{P}^n$) ?

In fact, the problems are closely related to each other. Let $X := \{P(z_0, \ldots, z_n) = 0\}$ be a smooth hypersurface of degree d in $\mathbf{P}^n$; one may look at the hypersurface

$$\tilde{X} := \{z_{n+1}^d = P(z_0, \ldots, z_n)\} \subset \mathbf{P}^{n+1}$$

which is a cyclic covering of $\mathbf{P}^n$. Since any holomorphic map $f: \mathbf{C} \to \mathbf{P}^n \setminus X$ can be lifted to $\tilde{X}$, it is clear that the hyperbolicity of $\tilde{X}$ would imply the hyperbolicity of $\mathbf{P}^n \setminus X$.

Now let us formulate more precisely Kobayashi's questions. Let $\mathbf{P}_{n,d} = \mathbf{P}^N$, $N = \binom{n+d}{d} - 1$, be the projective space whose points parameterize hypersurfaces of degree d in $\mathbf{P}^n$ (not necessary reduced). Let $H_{n,d} \subset \mathbf{P}_{n,d}$ be the subset corresponding to hyperbolic hypersurfaces. To precise the meaning of "genericity" one could ask *whether $H_{n,d}$ contains a Zariski open subset of $\mathbf{P}_{n,d}$ for d large enough with respect to n. More generally, is the complement $\mathbf{P}_{n,d} \setminus H_{n,d}$ contained in a countable union of hypersurfaces in $\mathbf{P}_{n,d}$ for $d \gg n$?*

2. Construction of Hyperbolic Hypersurfaces

Although the set of hyperbolic hypersurfaces of degree d large enough with respect to n is conjectured to be Zariski dense in $\mathbf{P}^{N(n,d)}$, it is not easy to construct explicit examples of hyperbolic hypersurfaces.

The first example of smooth hyperbolic surfaces of even degree $d \geq 50$ was given by Brody and Green [5]. We will describe some recent examples later, after recalling a general construction of Masuda and Noguchi.

Let X be a hypersurface of degree $d\ell$ of $\mathbf{P}^n$ defined by the equation

$$X : c_1 M_1^d + \cdots + c_s M_s^d = 0, \qquad c_j \in \mathbf{C}^*,\ d \in \mathbf{Z},\ d > 0,$$

where $\{M_j = z_0^{\alpha_{j0}} \cdots z_n^{\alpha_{jn}}\}_{j=1}^{s}$ is a set of distinct monomials of degree ℓ $(\alpha_{j0} + \cdots + \alpha_{jn} = \ell)$. Masuda and Noguchi proved the following:

Theorem 2.1 [33]. *Assume $d > s(s-2)$. Then there exists an algebraic subset $\Sigma \subset (\mathbf{C}^*)^s$ such that X is hyperbolic if and only if $(c_j) \in (\mathbf{C}^*)^s \setminus \Sigma$.*

We give here a sketch of the proof. Suppose that X is not hyperbolic, and let

$$f = (f_1, \ldots, f_{n+1}) : \mathbf{C} \to X$$

be a nonconstant holomorphic curve in X. We are going to show that $\{c_j\}$ belongs to an algebraic subset of $(\mathbf{C}^*)^s$. We may assume that any $f_j \not\equiv 0$.

First of all we claim that, under the hypothesis of Theorem 2.1, *there is a decomposition of indices $\{1, \ldots, s\} = \cup I_\xi$ such that*

(i) *every I_ξ contains at least 2 indices,*

(ii) *the ratio of $M_j^d \circ f(z)$ and $M_k^d \circ f(z)$ is constant for $j, k \in I_\xi$,*

(iii) $\sum_{j \in I_\xi} c_j M_j^d \circ f(z) \equiv 0$ *for all ξ.*

The statement is proved by induction on s. The case of $s = 2$ is trivial. One can reduce to the case of $(s-1)$ monomials by showing that $\{M_j^d \circ f, j = 1, \ldots, s-1\}$ are linearly dependent over $\mathbf{C}$. If it is not the case, then the holomorphic curve

$$g : z \in \mathbf{C} \mapsto \left(M_1^d \circ f(z), \ldots, M_{s-1}^d \circ f(z)\right) \in \mathbf{P}^{s-2}$$

is non-degenerate in $\mathbf{P}^{s-2}$. Consider the set of hyperplanes in general position

$$H_1 = \{z_1 = 0\}, \ldots, H_{s-1} = \{z_{s-1} = 0\},$$

$$H_s = \{c_1 z_1 + \cdots + c_{s-1} z_{s-1} = 0\}.$$

Then g ramifies at least d on $H_j, j=1,\dots,s$, and by Cartan's theorem, we have

$$\sum_{j=1}^{s}\left(1-\frac{s-2}{d}\right)\leq s-1.$$

This is a contradiction, since $d>s(s-2)$. Now for a decomposition of $\{1,\dots,s\}$ as above, we set $b_{jk}=M_j^d\circ f(z)/M_k^d\circ f(z)$. Then the linear system of equations

$$AY=B,$$

where A is the matrix $\{\alpha_{j\ell}-\alpha_{k\ell}\}$, $Y=\begin{pmatrix} y_0\\ \vdots\\ y_n\end{pmatrix}$, $B=\{\log b_{jk}\}$, has the solution $\{\log f_0,\dots,\log f_n\}$. Thus, the matrix A satisfies certain conditions on the rank. On the other hand, by the condition (iii) there exist $(A_0,\dots,A_n)\in\mathbf{P}^n$ such that $(c_i)\in(\mathbf{C}^*)^s$ satisfies the following equations:

$$\sum_{i\in I_\xi} c_i A_0^{\alpha_{i0}}\cdots A_n^{\alpha_{in}}=0. \tag{1}$$

Hence, $(c_i)\in(\mathbf{C}^*)^s$ belongs to the projection $\Sigma\subset(\mathbf{C}^*)^s$ of an algebraic subset in $(\mathbf{C}^*)^s\times\mathbf{P}^n$. The proof of the reciprocity is not difficult.

In [33], Masuda and Noguchi gave an algorithm to construct a system $\{M_j, j=1,\dots,s\}$ such that the corresponding matrix A does not satisfy the mentioned condition on the rank. Such a system is called *admissible.*

Remark 1. In the case that $\{M_j, j=1,\dots,s\}$ is an admissible system (and $d>s(s-2)$), the hypersurface X is hyperbolic for all $(c_j)\in(\mathbf{C}^*)^s$, and $\Sigma=\emptyset$. Hence, by the algorithm of Masuda and Noguchi, we can construct explicit examples of hyperbolic hypersurfaces for every n (and $d\gg n$).

Remark 2. In some cases, the set Σ is *not proper*. For example, take $M_j=z_j$, $\ell=1$, $j=1,\dots,n+1$, $n\geq 3$. For all $(c_j)\in(\mathbf{C}^*)^s$ and for arbitrary d, the hypersurface corresponding to this case is not hyperbolic (it is the Fermat variety).

Remark 3. It is interesting to show the cases when Σ is a *nonempty proper* subset of $(\mathbf{C}^*)^s$. Here we give such a case. By Masuda–Noguchi's result, for every n there exists $d(n)$ such that for $d>d(n)$ there exists a hyperbolic hypersurface of degree d in $\mathbf{P}^n$. Take d large enough so that $H_{n,d}\neq\emptyset$. Since the set $H_{n,d}$ is open (in the usual topology) (see [47]), by a small deformation, if necessary, we can take a hypersurface X in $H_{n,d}$ with all nonzero coefficients in the defining equation. Hence, if $s=N(n,d)$ the set Σ is nonempty and proper.

By using the method based on the proof of Theorem 2.1, Masuda and Noguchi constructed explicit examples of hyperbolic hypersurfaces in $\mathbf{P}^3$, $\mathbf{P}^4$, $\mathbf{P}^5$.

For the case of surfaces in $\mathbf{P}^3$ we can use the following method. Take at first a surface $X\subset\mathbf{P}^3$ such that every holomorphic curve in X is degenerate. This means that the image of a holomorphic map from $\mathbf{C}$ into X, $f:\mathbf{C}\to X$, is contained in a proper algebraic subset of X. If one could prove that the image $f(\mathbf{C})$ is contained in a curve of genus at least two, then f is constant and X is hyperbolic.

In [34], Nadel used a method based on a rather difficult and technical theorem of Siu [42] dealing with meromorphic connections to construct a class of hyperbolic surfaces in $\mathbf{P}^n$, in which every holomorphic curve is degenerate. Then he applied this result to the case of $\mathbf{P}^3$ and calculated the genus of curves containing images of holomorphic curves and gave explicit examples of hyperbolic surfaces in $\mathbf{P}^3(\mathbf{C})$ of degree $6k+3 \geq 21$.

By using Cartan's theorem, Ha Huy Khoai [21] gave other classes of hypersurfaces in $\mathbf{P}^n$ with the property of degeneracy of holomorphic curves and constructed examples of hyperbolic surfaces in $\mathbf{P}^3$ of arbitrary $d \geq 22$ (without the restriction $3|d$). We have the following.

Theorem 2.2 [21]. *Let X be a hypersurface in $\mathbf{P}^n$ defined by the equation*

$$X : c_1 M_1 + \cdots + c_s M_s = 0\,, \qquad c_i \neq 0\,,$$

where $M_j = z_0^{\alpha_{j0}} \cdots z_n^{\alpha_{jn}}$, $\alpha_{j0} + \cdots + \alpha_{jn} = d$, $j = 1, \ldots, s$. Moreover, let X be a perturbation of the Fermat variety, i.e., $s \geq n+1$, $M_j = z_j^d$, $j = 1, \ldots, n+1$. Suppose that there is an integer $k \geq 0$ such that X satisfies the following conditions
(i) *α_{jm} is either 0, or $\geq d-k$, $j = 1, \ldots, s$, $m = 0, \ldots, n$.*
(ii) $\dfrac{(n+1)(s-2)}{d} + \dfrac{(s-2)(s-n-1)}{d-k} < 1\,.$
Then any holomorphic curve in X is degenerate.

From Theorem 2.2, one can give the following examples of hyperbolic surfaces in $\mathbf{P}^3$:

$$X : z_0^d + z_1^d + z_2^d + z_3^d + c\, z_1^{\alpha_1} z_2^{\alpha_2} z_3^{\alpha_3} = 0\,,$$

where $c \neq 0$, $\alpha_1 + \alpha_2 + \alpha_3 = d$, $\alpha_i \geq 7$ $(i = 1, 2, 3)$, $d \geq 22$.

Notice that all explicit examples of hyperbolic hypersurfaces constructed before by Brody and Green, Nadel, and Masuda and Noguchi are of degree divided by some integer > 1.

Very recently, Goul [13] used a technique similar to that of Nadel to prove the following.

Theorem 2.3 [13]. *The surface $X \subset \mathbf{P}^3$ defined by the equation*

$$X : z_0^d + z_1^d + z_2^d + z_3^d + \varepsilon_1 z_0^2 z_1^{d-2} + \varepsilon_2 z_0^2 z_2^{d-2} = 0$$

is hyperbolic for $d \geq 14$ and for all but a finite number of $\varepsilon_1, \varepsilon_2 \in \mathbf{C}^*$.

3. Hyperbolicity of Complements of Hypersurfaces

As mentioned in Sec. 1, the problem of constructing hyperbolic hypersurfaces is closely related to the problem of finding hypersurfaces with hyperbolic complements. Recall that by Kobayashi's conjecture, the set of all hypersurfaces of degree d in $\mathbf{P}^n$ with hyperbolic complements would be Zariski dense in $\mathbf{P}^{N(n,d)}$ if $d \geq 2n+1$. However, it seems difficult to construct explicit examples of such hypersurfaces. The reducible case is much easier, and we will consider it first.

By the classical results of Borel, Bloch, and Cartan, the complement of d hyperplanes in general position is hyperbolic if $d \geq 2n+1$. In [40], Ru gave a necessary and sufficient condition for the set $\mathscr{H}$ of hyperplanes in $\mathbf{P}^n(\mathbf{C})$ (not necessarily in a general position) such that $\mathbf{P}^n \setminus |\mathscr{H}|$ is Brody hyperbolic.

Theorem 3.1 [40]. *Let $\mathscr{H}$ be a set of hyperplanes in $\mathbf{P}^n(\mathbf{C})$ and let $\mathscr{L}$ be the corresponding set of linear forms in $n+1$ variables defining the hyperplanes in $\mathscr{H}$. Then the complement of the union of hyperplanes in $\mathscr{H}$ is Brody hyperbolic in $\mathbf{P}^n$ if and only if* $\dim \mathscr{L} = n+1$, *and for each proper nonempty subset $\mathscr{L}_1$ of $\mathscr{L}$,*

$$\mathscr{L}_1 \cap (\mathscr{L} - \mathscr{L}_1) \cap \mathscr{L} = \emptyset .$$

Theorem 3.1 follows the mentioned result of Bloch, Borel, and Cartan, and that the complement of $2n$ hyperplanes is not hyperbolic (Kiernan [26], Snurnitsyn [44]).

Babets [2] and Eremenko and Sodin [14] generalized Cartan's theorem on value distribution of holomorphic curves to the case of hypersurfaces, and proved the following.

Theorem 3.2 [2], [14]. *Every holomorphic map $f : \mathbf{C} \to \mathbf{P}^n(\mathbf{C})$, omitting $2n+1$ hypersurfaces in general position is constant.*

Recall that hypersurfaces $X_1, \ldots, X_q$ $(q \geq n+2)$ is said to be in general position if their defining homogeneous polynomials are multiplicatively independent (pairwise linearly independent), and any $n+1$ among the equations $\mathscr{P}_j(z_0, \ldots, z_n) = 0$, $1 \leq j \leq q$ do not have common nonzero solutions in $\mathbf{C}^{n+1}$. Ru gave another proof of the Babets–Eremenko–Sodin theorem, and also discussed the case of the complement of $2n$ hypersurfaces.

Definition 3.3. *The hypersurfaces $X_1, \ldots, X_q$ are said to be geometrically in general position if they are in general position and they intersect transversally, i.e., the components have no common tangent at the points of intersection. We have the following theorem.*

Theorem 3.4 [39]. *Let X be the union of q (irreducible) hypersurfaces in $\mathbf{P}^n(\mathbf{C})$ geometrically in general position. Let $f : \mathbf{C} \to \mathbf{P}^n(\mathbf{C}) \setminus X$ be holomorphic. Then the image of f is contained in a subvariety of $\mathbf{P}^n(\mathbf{C})$ of dimension $2n - q + 1$. In particular, if $q = 2n$, then $\mathbf{P}^n(\mathbf{C}) \setminus X$ is Brody hyperbolic (in fact, it is Kobayashi hyperbolic and hyperbolically embedded) with three exceptional cases:*
(i) X is the union of $2n$ hyperplanes, have n hyperplanes intersect at a point p and the other n hyperplanes intersect at a point s, where $f(\mathbf{C})$ is contained in the line joining p and s.

(ii) *X consists of $2n-1$ hyperplanes and one smooth quadric (Q) such that n hyperplanes intersect at a point p and the rest of the hyperplanes intersect with Q at a point s, where $f(\mathbf{C})$ is contained in the line joining p and s.*
(iii) *X consists of $2n-2$ hyperplanes and two smooth quadrics (Q_1, Q_2) such that $n-1$ hyperplanes intersect with Q_1 at a point p and the rest of the hyperplanes intersect with Q_2 at a point s, where $f(\mathbf{C})$ is contained in the line joining p and s and is bitangent to these two quadrics.*

Dethloff *et al.* [11] determined the case of reducible curves ($n=2$) and proved the following theorem by using the method of the Nevanlinna theory.

Theorem 3.5 [11]. *Let S be the space of 3-tuples of quadrics. Then there exists a proper algebraic variety $V \subset S$ such that for $s \in S \setminus V$ the quasiprojective variety $\mathbf{P}^2 \setminus \bigcup_{j=1}^{3} \Gamma_j(s)$ (where $(\Gamma_1(s), \Gamma_2(s), \Gamma_3(s))$ is the 3-tuple corresponding to s) is complete hyperbolic and hyperbolically embedded.*

The algebraic subset V in Theorem 3.5 is defined by the following conditions:
(i) All $\Gamma_j(s)$ are smooth (and of multiplicity one).
(ii) The $\Gamma_j(s)$, $j=1, 2, 3$ intersect transversally.
(iii) For any common tangent line of two of quadrics $\Gamma_j(s)$ which is tangential to these in points p and s resp., the third quadric does not intersect the tangent in both point p and s.

In [46], similar results were also obtained for the case of curves with at least four irreducible components.

Now consider the case of irreducibles hypersurfaces. In [47], Zaidenberg proved the existence of smooth curves (resp. surfaces) of arbitrary degree ≥ 5 (resp. even degree ≥ 350) of $\mathbf{P}^2(\mathbf{C})$ (resp. $\mathbf{P}^3(\mathbf{C})$) of which complements are hyperbolic and hyperbolically embedded into $\mathbf{P}^2(\mathbf{C})$ (resp. $\mathbf{P}^3(\mathbf{C})$); however, their equations are not explicit.

The first example of a smooth hyperbolic curve of even degree ≥ 30 of $\mathbf{P}^2(\mathbf{C})$ of which the complement is hyperbolic and hyperbolically embedded into $\mathbf{P}^2(\mathbf{C})$ was given by Azukawa and Suzuki, based on the example of Brody and Green. In [34], Nadel constructed such examples for $d \geq 18$ which is divisible by 6. Ha Huy Khoai [21] gave explicit examples for arbitrary degree $d \geq 19$.

In the paper of Masuda and Noguchi [33] the following was proved.

Theorem 3.6 [33]. *There exists a smooth hyperbolic hypersurface of every degree $d \geq d(n)$ of $\mathbf{P}^n(\mathbf{C})$ such that its complement is complete hyperbolic and hyperbolically embedded into $\mathbf{P}^n(\mathbf{C})$, where $d(n)$ is a positive integer depending only on n.*

Theorem 3.6 follows from the existence of hyperbolic hypersurfaces (Theorem 2.1). Indeed, take a hyperbolic hypersurface in $\mathbf{P}^{n+1}(\mathbf{C})$ and consider $Y = X \cap \{z_{n+1} = 0\}$. Identify $\mathbf{P}^n(\mathbf{C})$ with the hypersurface defined by $\{z_{n+1} = 0\}$; then $\mathbf{P}^n(\mathbf{C}) \setminus Y$ is complete hyperbolic and hyperbolically embedded into $\mathbf{P}^n(\mathbf{C})$. By this method, Masuda and Noguchi gave first explicit examples of smooth hyperbolic hypersurfaces, of which complements are hyperbolic and hyperbolically embedded.

Recently, Siu and Yeung [43] made remarkable progress by proving the analog of Kobayashi's conjecture for plane curves of degree large enough.

Theorem 3.7 [43]. *There exists a positive integer δ_0 satisfying the following. Let C be a generic smooth complex curve in $\mathbf{P}^2$ such that the degree δ of C is at least δ_0. Then, there is no nonconstant holomorphic map from C to $\mathbf{P}^2\backslash C$. Here, generic means that C belongs to a Zariski open subset in the space of all complex curves of degree δ in $\mathbf{P}^2$.*

Moreover, Y. J. Siu and S. K. Yeung gave a condition to compute possible values of δ_0. For example, one can take $\delta_0 = 5 \times 10^{13}$.

The proof of Theorem 3.7 is based on the construction of independent holomorphic 2-jet differentials. The authors said that their method should be applicable also to the case of dimension n with n-jets being used instead of 2-jets; however, the technical details are far more involved.

Very recently, Dethloff and Zaidenberg [12] have given examples of families of irreducible curves for any even $d \geq 6$ of which complements are hyperbolic and hyperbolically embedded into $\mathbf{P}^2$. Notice that while all examples which are known before are hyperbolic curves, Dethloff and Zaidenberg get examples of elliptic or rational curves.

4. Diophantine Geometry

Recent studies suggest that the hyperbolicity of complete space X is related to the finiteness of the number of rational on integral points of X. In particular, Lang made the following conjecture.

Conjecture 4.1 [29]. *A projective variety X has only finitely many k-rational points for every number field k if and only if the associated complex space is Brody hyperbolic.*

For affine varieties, Lang [29] also conjectured the following.

For an affine variety X, if X is hyperbolically embedded in its projective closure, then the number of integral points of X is finite. Conversely, this finiteness conjecturely implies that X is Brody hyperbolic.

Lang's conjecture is true for smooth projective curves: curves of genus ≥ 2 are Brody hyperbolic, which is an easy consequence of the Uniformization Theorem and Liouville's theorem; such curves have only finitely many rational points by Falting's theorem (the Mordell–Weil conjecture). Faltings [15] also proved the conjecture for hyperbolic subspaces contained in an Abelian variety.

Vojta [45] observed the similarity between the Second Main Theorem (SMF) of the Value Distribution Theory and the Roth theorem in the theory of Diophantine approximations. Notice that most of the results on hyperbolic spaces mentioned in previous sections are obtained by using the Value Distribution Theory. Then, due to Vojta's correspondence, one could translate a proof of hyperbolicity of a variety into a proof of finiteness of rational points on the corresponding variety over a number field. Let us recall some recent results in this direction.

Ru and Wong [41] proved the following arithmetic analog of the Bloch–Cartan result:

Theorem 4.2 [41]. *Let k be a number field. Then $\mathbf{P}^n(k)\backslash\{2n+1\}$ hyperplanes in general position (with algebraic coefficients) have only finitely many integral points.*

This theorem was generalized to the case of the complement of a set of hyperplanes in $\mathbf{P}^n(k)$. We can formulate a theorem exactly as Theorem 3.1.

Theorem 4.3 [40]. *Let $\mathscr{H}$ be a set of hyperplanes in $\mathbf{P}^n$ with algebraic coefficients. Let $\mathscr{L}$ be the corresponding set of linear forms in $n+1$ variables defining the hyperplanes in $\mathscr{H}$. Given a number field k, let $M(k)$ be the set of all nonequivalent valuations on k. Then $\mathbf{P}^n(k)\backslash|\mathscr{H}|$ has only finitely many $(S, |\mathscr{H}|)$-integral points, for every finite set $S \subset M(k)$ containing the Archimedean valuations, if and only if,* $\dim \mathscr{L} = n+1$ *and for each proper nonempty subset $\mathscr{L}_1$ of $\mathscr{L}$,*

$$\mathscr{L}_1 \cap (\mathscr{L} - \mathscr{L}_1) \cap \mathscr{L} = \emptyset .$$

Combining Theorem 3.1 and Theorem 4.3 one can say that $\mathbf{P}^n(\mathbf{C})\backslash|\mathscr{H}|$ is Brody hyperbolic if and only if $\mathbf{P}^n(k)\backslash|\mathscr{H}|$ has only finitely many $(S, |\mathscr{H}|)$-integral points, for every number field k, every finite set $S \subset M(k)$ containing the Archimedean valuations. In other words, Lang's conjecture is true for the case of the complement of hyperplanes in $\mathbf{P}^n$.

Theorem 3.2 and Theorem 3.4 also have arithmetic analogs.

Theorem 4.4 [39]. *Let k be a number field, $M(k)$, S as above. Let $X_1, \ldots, X_q$ be hypersurfaces in $\mathbf{P}^n(k)$ in general position (any $n+1$ of them have empty intersections) with algebraic coefficients. If $q \geq 2n+1$, then $\mathbf{P}^n(k)\backslash X$ has only finitely many (S, X)-integral points, where $X = \bigcup_{i=1}^{q} X_i$.*

Theorem 4.5 [39]. *Let k be a number field $M(k)$, S as above. Let X be the union of q (irreducible) hypersurfaces in $\mathbf{P}^n(k)$ geometrically in general position. Then the set of (S, X)-integral points of $\mathbf{P}^n(k)\backslash X$ is contained in a finite union of subvarieties of $\mathbf{P}^n(k)$ of dimension $2n-q+1$.*

In particular, if $q = 2n$, then $\mathbf{P}^n(k)\backslash X$ has only finitely many (S, X)-integral points with the following three exceptional cases:

(i) X is the union of $2n$ hyperplanes, where n hyperplanes intersect at a point p and the other n hyperplanes intersect at a point s.

(ii) X consists of $2n-1$ hyperplanes and one smooth quadric (Q) such that n hyperplanes intersect at a point p and the rest of the hyperplanes intersect with Q at a point s;

(iii) *X consists of $2n-2$ hyperplanes and two smooth quadrics (Q_1, Q_2) such that $n-1$ hyperplanes intersect with Q_1 at a point p and the rest of the hyperplanes intersect with Q_2 at a point s.*

For $n = 2$ (the case of curves), similar results were also obtained by Wong [46].

From the proof of Theorem 2.1, one can see that the construction of Masuda and Noguchi would have an arithmetic analog if one could prove the following.

Conjecture 4.6 [33]. *Consider a homogeneous equation of Fermat type*

$$c_1 z_1^d + \cdots + c_s z_s^d = 0\,,$$

where $c_j \in k^* = k\setminus\{0\}$. Let S denote the set k-integral solutions. Assume that

$$d > s(s-2)\,.$$

Then there is a finite decomposition $S = \bigcup S_\nu$, and for each S_ν there is a decomposition of indices, $\{1, \ldots, s\} = \bigcup I_\xi$, satisfying the following:
(i) *Every I_ξ contains at least 2 indices.*
(ii) *Let $j, k \in I_\xi$ be arbitrary indices. Then the ratio of z_j^d and z_k^d are the same for all but finitely many $z \in S_\nu$.*
(iii) $\sum_{j \in I_\xi} c_j z_j^d = 0$ *for all but finitely many $z \in S_\nu$ and for each ν.*

Notice that the proof of Theorem 2.1 used as an essential tool the so-called Cartan's Second Main Theorem with *n-truncated counting functions.* Up to now we have only arithmetic Cartan's Second Main Theorem with *nontruncated counting functions* (Schmidt–Schlikewei; see [45]).

Bombieri and Mueller [3] proved the function field analog of Conjecture 4.6 for $d > s!(s!-2)$.

5. Final Remarks

While Kobayashi conjectured that the sets of hyperbolic hypersurfaces and hypersurfaces with hyperbolic complements contain a Zariski open subset, Zaidenberg [47] proved that these sets are open in the usual topology. Hence, the Kobayashi conjectures would be proved if one could prove that the property of being hyperbolic is an algebraic property. There have been several attempts to find an algebraic characterization of complex hyperbolicity. We refer the reader to the papers of Lang [28]–[30] for detailed discussions on this topic. Here, we describe some new ideas of Demailly and ones arising from the work of Masuda and Noguchi [33].

Let X be a projective algebraic variety. If X is hyperbolic, then X has no rational and elliptic curves, and more generally, every holomorphic map $f: Z \to X$ from an Abelian variety (or complex torus) to X must be constant. Conversely, it has been suggested by Kobayashi and Lang that these algebraic properties are equivalent to hyperbolicity. To prove this, one would have to construct a torus Z and a nontrivial holomorphic map $f : Z \to X$ whenever X is nonhyperbolic. A hint that it should be true is given by the following observation. If X is hyperbolic, there is an absolute constant $\varepsilon > 0$ such that the genus of any compact curve of X is bounded below by ε times the degree; conversely, this property fails to be true in many examples of nonhyperbolic projective varieties. Our belief, supported by some heuristic arguments, is that any sequence of compact curves (C_t) with genus (C_t)/degree $(C_t) \to 0$ should have a cluster set swept out by the image of a map $f : Z \to X$ from a complex torus Z, such that the limit of some subsequence of the sequence of universal covering map $\Delta \to C_t \to X$ (suitably reparametrized) coincides with the image of a (not necessarily compact) straight line of Z into X. A related conjecture of Lang [2] states that a projective variety is hyperbolic if all its irreducible algebraic varieties are of general type. The most elementary step would be to exclude the case of manifolds with $c_1 \equiv 0$, by showing for instance that it does admit a sequence of compact curves (C_t) with genus (C_t)/degree $(C_t) \to 0$.

Another approach to Kobayashi's conjectures is suggested by the proof of Theorem 2.1. In [22], Ha Huy Khoai introduced the notion of the *Borel curve.* By definition, a holomorphic curve in a hypersurface X is called a *Borel curve* if the conditions (i), (ii), (iii) in the proof of Theorem 2.1 hold. A hypersurface X is said to be *Borel hyperbolic* if every Borel curve in X is constant. Let $B(n, d)$ denote the set of Borel hyperbolic hypersurfaces of degree d in $\mathbf{P}^n$. It is proved [22] that $B(n, d)$ is a Zariski dense subset in $\mathbf{P}^N$, $N = N(n, d) = \binom{n+d}{d} - 1$ for d large enough with respect to n. As a consequence we have the following.

Theorem 5.1 [22]. *For every $n \in \mathbf{Z}$, $n > 0$, there exists an integer $d(n)$ satisfying the following. For any $d > d(n)$ there exists a nonempty Zariski open subset Z of $\mathbf{P}^N$, $N = N(n, d)$, such that for any X in Z and for $d' > N(N-2)$, the hypersurface $X_{d'}$ is hyperbolic.*

Here, we use the following notation. If X is defined by

$$X : c_1 M_1 + \cdots + c_s M_s = 0 ,$$

then $X_{d'}$ is defined by

$$X_{d'} : c_1 M_1^{d'} + \cdots + c_s M_s^{d'} = 0 .$$

To conclude this survey, we would like to mention that in the recent few years, the hyperbolicity in the non-Archimedean case has been investigated. We refer the reader to the papers [8], [9], [19], [20], [23], [31], [32], [36], [37] for details.

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