Vietnam Journal of Mathematics **40**:2&3(2012) 319-330

Vietnam Journal
of
MATHEMATICS

An Iteration Method for Finding the Unique Solution of a Variational Inequality Problem Constrained by the Intersection of the Sets of Fixed Points of a Countable Family of Nonexpansive Mappings in Hilbert Spaces

Lai-Jiu Lin and Chih-Sheng Chuang

Department of Mathematics, National Changhua University of Education, Changhua, 50058, Taiwan

Dedicated to Professor Phan Quoc Khanh on the occasion of his 65th-birthday

Received October 14, 2011
Revised January 18, 2012

Abstract. In this paper, we consider an iteration process which converges strongly to a common fixed point of a countable family of nonexpansive mappings, and it is the unique solution of a variational inequality constrained by this set of common fixed points. Note that our results generalize the corresponding results by Ceng et al. (Comput. Math. Appl. **61** (2011), 2447-2455).

2000 Mathematics Subject Classification. 47H10, 47H05, 47H17.

Key words. Variational inequality, common fixed point, nonexpansive mapping.

1. Introduction

Throughout this paper, let $\mathbb{N}$ be the set of positive integers and let $\mathbb{R}$ be the set of real numbers.

Let C be a nonempty closed convex subset of a real Hilbert space H. A mapping $T : C \to H$ is nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. Let $\text{Fix}(T) := \{x \in C : Tx = x\}$ denote the fixed point set of T. It is well-known

This research was supported by the National Science Council of Republic of China.

that $\mathrm{Fix}(T)$ is a closed convex subset of C if T is a nonexpansive mapping with $\mathrm{Fix}(T) \neq \emptyset$. An operator $B \subseteq H \times H$ is said to be accretive if $\langle x_1 - x_2, y_1 - y_2 \rangle \geq 0$ for each $(x_1, y_1) \in B$ and $(x_2, y_2) \in B$. A mapping $A : H \to H$ is a strongly positive operator with coefficient $\bar{\gamma} > 0$ if $\langle Ax, x \rangle \geq \bar{\gamma} \|x\|^2$ for all $x \in H$. A nonlinear mapping S whose domain $D(S) \subseteq H$ and range $R(S) \subseteq H$ is said to be strongly monotone with coefficient $\eta > 0$ if $\langle x - y, Sx - Sy \rangle \geq \eta \|x - y\|^2$ for all $x, y \in D(S)$.

Let $f : C \to C$ be a contraction mapping, and let $T : C \to C$ be a nonexpansive mapping, and let $A : H \to H$ be a strongly positive operator.

In 2004, Xu [13] proved that under certain appropriate conditions on $\{\alpha_n\}$, the sequence $\{x_n\}$ generated by

$$x_{n+1} := \alpha_n f(x_n) + (1 - \alpha_n) T x_n, \quad n \in \mathbb{N},$$

converges strongly to the unique solution $\bar{x} \in \mathrm{Fix}(T)$ of the variational inequality

$$\langle (I - f)\bar{x}, x - \bar{x} \rangle \geq 0 \quad \text{ for all } \ x \in \mathrm{Fix}(T).$$

In 2006, Marino and Xu [7] considered the following general iterative process:

$$x_{n+1} := \alpha_n \gamma f(x_n) + (I - \alpha_n A) T x_n, \quad n \in \mathbb{N}. \tag{1}$$

They proved that if the sequence $\{\alpha_n\}$ of parameters satisfies appropriate conditions, then the sequence $\{x_n\}$ generated by (1) converges strongly to a point $\bar{x} \in \mathrm{Fix}(T)$, and it is the unique solution of the following variational inequality

$$\langle (\gamma f - A)\bar{x}, x - \bar{x} \rangle \leq 0 \quad \text{for all } x \in C.$$

In 2010, Tian [9] introduced the following general iterative process:

$$x_{n+1} := \alpha_n \gamma f(x_n) + (I - \mu \alpha_n F) T x_n, \quad n \in \mathbb{N}. \tag{2}$$

The author proved that if $\{\alpha_n\}$ satisfies appropriate conditions, then the sequence $\{x_n\}$ generated by (2) converges strongly to a point $\bar{x} \in \mathrm{Fix}(T)$, and it is the unique solution of the variational inequality

$$\langle (\gamma f - \mu F)\bar{x}, x - \bar{x} \rangle \leq 0 \quad \text{for all } x \in \mathrm{Fix}(T).$$

This scheme improves and extends the corresponding ones given by Marino and Xu [7], and Yamada [14].

In 2011, Ceng et al. [5] consider the following general composite iterative process:

$$\begin{cases} x_1 \in C \text{ chosen arbitrary}, \\ y_n := (I - \alpha_n \mu F) T x_n + \alpha_n \gamma f(x_n), \\ x_{n+1} := (I - \beta_n A) T x_n + \beta_n y_n, \end{cases} \tag{3}$$

where A is a strongly positive bounded linear operator on H with coefficient

$\bar{\gamma} \in (1,2)$, and $\{\alpha_n\} \subseteq [0,1]$ and $\{\beta_n\} \subseteq (0,1]$ satisfy appropriate conditions. They proved that the sequence $\{x_n\}$ generated by (3) converges strongly to a point $\bar{x} \in \mathrm{Fix}(T)$, and it is the unique solution of the variational inequality

$$\langle (I-A)\bar{x}, x-\bar{x}\rangle \leq 0 \quad \text{for all } x \in \mathrm{Fix}(T).$$

Recall that the metric projection from a real Hilbert space H onto a nonempty closed convex subset C of H is the mapping $P_C : H \to C$ which assigns to each point $x \in H$ the unique point $P_C x$ satisfying the property

$$\|x - P_C x\| = \inf_{y \in C} \|x-y\|.$$

Motivated by the above works, we consider the following iterative process. Let C be a nonempty closed convex subset of a real Hilbert space H. For each $n \in \mathbb{N}$, let $T_n : C \to C$ be a nonexpansive mapping. Let $f : H \to H$ be a contraction mapping with coefficient $\alpha \in (0,1)$, $F : H \to H$ be Lipschitz with coefficient $k > 0$ and a strongly monotone mapping with coefficient $\eta > 0$, $A : H \to H$ be a strongly positive bounded linear operator with coefficient $\bar{\gamma} \in (1,2)$. Let P_C be the metric projection. Let $\{\alpha_n\} \subseteq [0,1]$ and $\{\beta_n\} \subseteq (0,1]$. Let $0 < \mu < 2\eta/k^2$, $0 < \gamma < \mu(\eta - \frac{\mu k^2}{2})/\alpha = \tau/\alpha$. Let $\{x_n\}$ be generated by

$$\begin{cases} x_1 \in C \text{ chosen arbitrary}, \\ y_n := P_C[(I - \alpha_n \mu F)T_n x_n + \alpha_n \gamma f(x_n)], \\ x_{n+1} := P_C[(I-\beta_n A)T_n x_n + \beta_n y_n]. \end{cases} \tag{4}$$

We prove that if $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy suitable conditions, then the sequence $\{x_n\}$ generated by (4) converges strongly to a point $\bar{x} \in \cap_{n=1}^{\infty} \mathrm{Fix}(T_n)$, and it is the unique solution of the variational inequality

$$\langle (A-I)\bar{x}, \bar{x} - x\rangle \leq 0 \quad \text{for all } x \in \cap_{n=1}^{\infty} \mathrm{Fix}(T_n).$$

Note that our results are different from the corresponding ones in [4, 5, 7, 9, 10, 14] since these iteration processes are different.

Finally, we consider the problem of finding a zero of an accretive operator. Let C be a nonempty closed convex subset of a real Hilbert space H. Let $f : H \to H$ be a contraction mapping with coefficient $\alpha \in (0,1)$, $F : H \to H$ be a Lipschitz mapping with coefficient $k > 0$ and strongly monotone with coefficient $\eta > 0$, $A : H \to H$ be a strongly positive bounded linear self-adjoint operator with coefficient $\bar{\gamma} \in (1,2)$. Let B be an accretive operator such that $B^{-1}(0) \neq \emptyset$. Let $\{\alpha_n\} \subseteq [0,1]$, $\{\beta_n\} \subseteq (0,1]$, and let $\{\lambda_n\}$ be a sequence in $(0,\infty)$. Let $0 < \mu < 2\eta/k^2$, $0 < \gamma < \mu(\eta - \frac{\mu k^2}{2})/\alpha = \tau/\alpha$. Let $\{x_n\}$ be generated by

$$\begin{cases} x_1 \in C \text{ chosen arbitrary}, \\ y_n := P_C[(I - \alpha_n \mu F)J_{\lambda_n} x_n + \alpha_n \gamma f(x_n)], \\ x_{n+1} := P_C[(I-\beta_n A)J_{\lambda_n} x_n + \beta_n y_n], \end{cases} \tag{5}$$

where J_λ is the resolvent of B. (For details, see Section 4.) We prove that if $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\lambda_n\}$ satisfy suitable conditions, then the sequence $\{x_n\}$ generated by (5) converges strongly to a point $\bar{x} \in B^{-1}(0)$, and it is the unique solution of the variational inequality

$$\langle (A - I)\bar{x}, \bar{x} - z \rangle \leq 0 \quad \text{for all } z \in B^{-1}(0).$$

Note that our result is different from Theorems 4.2 and 4.3 in [2] since these iteration processes are different.

2. Preliminaries

Let H be a (real) Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. We denote the strongly convergence and the weak convergence of $\{x_n\}$ to $x \in H$ by $x_n \to x$ and $x_n \rightharpoonup x$, respectively. It is easy to see that for each $x, y \in H$ and $\lambda \in [0, 1]$, we have

$$\|\lambda x + (1 - \lambda)y\|^2 = \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)\|x - y\|^2.$$

Furthermore, we have

$$2\langle x - y, u - v \rangle = \|x - v\|^2 + \|y - u\|^2 - \|x - u\|^2 - \|y - v\|^2.$$

Lemma 2.1. [11] *Assume that $\{a_n\}_{n\in\mathbb{N}}$ is a sequence of nonnegative real numbers such that $a_{n+1} \leq (1 - \gamma_n)a_n + \delta_n$, $n \in \mathbb{N}$, where $\{\gamma_n\} \subseteq (0, 1)$ and $\{\delta_n\}$ is a sequence in $\mathbb{R}$ such that* (i) $\sum_{n=1}^{\infty} \gamma_n = \infty$*;* (ii) $\limsup_{n\to\infty} \frac{\delta_n}{\gamma_n} \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$*. Then* $\lim_{n\to\infty} a_n = 0$.

Lemma 2.2. [3] *Let C be a nonempty closed convex subset of a real Hilbert space H. Let T be a nonexpansive mapping of C into itself, and let $\{x_n\}$ be a sequence in C. If $x_n \rightharpoonup w$ and $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$, then $Tw = w$.*

Lemma 2.3. [8] *Let C be a nonempty closed convex subset of a Hilbert space H. Let P_C be the metric projection from H onto C. Then we have*

(i) *$y = P_C x$ if and only if $\langle x - y, y - z \rangle \geq 0$ for all $z \in C$;*
(ii) *$\|x - P_C x\|^2 + \|P_C x - z\|^2 \leq \|x - z\|^2$ for all $z \in C$;*
(iii) *$\|P_C x - P_C y\|^2 \leq \langle x - y, P_C x - P_C y \rangle$ for all $x, y \in H$. Consequently, P_C is a nonexpansive and monotone mapping.*

In 2007, Aoyama, Kimura, Takahashi, and Toyoda [2] gave the following definition and lemma.

Definition 2.4. [2] Let C be a nonempty subset of a real Hilbert space H. Let $\{T_n\}$ be a countable family of mappings from C into itself. We say that a family

$\{T_n\}$ satisfies AKTT-condition if

$$\sum_{n=1}^{\infty} \sup_{x\in B} \|T_{n+1}x - T_n x\| < \infty$$

for each nonempty bounded subset B of C.

Lemma 2.5. [2] *Let C be a nonempty closed subset of a real Hilbert space H, and let $\{T_n\}$ be a sequence of mappings from C into itself. Suppose that $\{T_n\}$ satisfies AKTT-condition. Then, for each $x \in C$, $\{T_n x\}$ converges strongly to a point in C. Furthermore, let $T : C \to C$ be defined by*

$$Tx := \lim_{n\to\infty} T_n x, \ x \in C.$$

Then, for each bounded subset B of C,

$$\lim_{n\to\infty} \sup\{\|Tz - T_n z\| : z \in B\} = 0.$$

In the sequel, we say that $\{T_n, T\}$ satisfies AKTT-condition if T is defined as above and $\{T_n\}$ satisfies AKTT-condition.

3. Main results

Theorem 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. For each $n \in \mathbb{N}$, let $T_n : C \to C$ be a nonexpansive mapping. Let $f : H \to H$ be a contraction mapping with coefficient $\alpha \in (0, 1)$, $F : H \to H$ be a Lipschitz mapping with coefficient $k > 0$ and strongly monotone with coefficient $\eta > 0$, $A : H \to H$ be a strongly positive bounded linear self-adjoint operator with coefficient $\bar{\gamma} \in (1, 2)$. Let $\{\alpha_n\} \subseteq [0, 1]$ and $\{\beta_n\} \subseteq (0, 1]$. Let $0 < \mu < 2\eta/k^2$, $0 < \gamma < \mu(\eta - \frac{\mu k^2}{2})/\alpha = \tau/\alpha$. Suppose that $\cap_{n=1}^{\infty}\mathrm{Fix}(T_n) \neq \emptyset$. Let $\{x_n\}$ be generated by*

$$\begin{cases} x_1 \in C \ \text{chosen arbitrary}, \\ y_n := P_C[(I - \alpha_n \mu F)T_n x_n + \alpha_n \gamma f(x_n)], \\ x_{n+1} := P_C[(I - \beta_n A)T_n x_n + \beta_n y_n]. \end{cases}$$

Assume that:

(i) $\lim_{n\to\infty} \alpha_n = 0$; $\lim_{n\to\infty} \beta_n = 0$; $\sum_{n=1}^{\infty} \beta_n = \infty$;

(ii) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$; $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$;

(iii) $\{T_n, T\}$ *satisfies AKTT-condition, and* $\mathrm{Fix}(T) = \cap_{n=1}^{\infty}\mathrm{Fix}(T_n)$.

Then $\lim_{n\to\infty} x_n = \bar{x}$, where $\bar{x} \in \cap_{n=1}^{\infty}\mathrm{Fix}(T_n)$ and $\langle (A-I)\bar{x}, \bar{x}-z\rangle \leq 0$ for all $z \in \cap_{n=1}^{\infty}\mathrm{Fix}(T_n)$, that is, $P_{\cap_{n=1}^{\infty}\mathrm{Fix}(T_n)}(2I - A)\bar{x} = \bar{x}$.

Proof. Since $\|A\| \geq \bar{\gamma} > 1$, without loss of generality, we may assume that $0 < \beta_n \leq \|A\|^{-1}$ for all $n \in \mathbb{N}$. Since $\lim_{n\to\infty} \alpha_n = 0$, we may assume that $1 > \alpha_n(\tau - \gamma\alpha)$ and $1 > 2\alpha_n\tau$ for each $n \in \mathbb{N}$. For each $x, y \in X$ and $n \in \mathbb{N}$, we have

$$\begin{aligned}
&\|(I-\alpha_n\mu F)x-(I-\alpha_n\mu F)y\|^2\\
&=\|(x-y)-(\alpha_n\mu Fx-\alpha_n\mu Fy)\|^2\\
&=\|x-y\|^2-2\langle x-y,\alpha_n\mu Fx-\alpha_n\mu Fy\rangle+\|\alpha_n\mu Fx-\alpha_n\mu Fy\|^2\\
&\leq\|x-y\|^2-2\alpha_n\mu\eta\|x-y\|^2+\alpha_n^2\mu^2k^2\|x-y\|^2\\
&=(1-2\alpha_n\mu\eta+\alpha_n^2\mu^2k^2)\|x-y\|^2\\
&\leq(1-\alpha_n\mu(2\eta-\mu k^2))\|x-y\|^2\\
&=(1-2\alpha_n\tau)\|x-y\|^2\\
&\leq(1-\alpha_n\tau)\|x-y\|^2.
\end{aligned}\tag{6}$$

Take any $w \in \cap_{n=1}^{\infty}\mathrm{Fix}(T_n)$ and let w be fixed. Hence

$$\begin{aligned}
&\|y_n-w\|\\
&\leq\|(I-\alpha_n\mu F)T_nx_n+\alpha_n\gamma f(x_n)-w\|\\
&=\|\alpha_n(\gamma f(x_n)-\mu F(w))+(I-\alpha_n\mu F)T_nx_n-(I-\alpha_n\mu F)T_nw\|\\
&\leq\|\alpha_n\gamma f(x_n)-\alpha_n\gamma f(w)\|+\|\alpha_n\gamma f(w)-\alpha_n\mu F(w))\|\\
&\quad+\|(I-\alpha_n\mu F)T_nx_n-(I-\alpha_n\mu F)T_nw\|\\
&\leq\alpha_n\gamma\alpha\|x_n-w\|+\alpha_n\|\gamma f(w)-\mu F(w))\|+(1-\alpha_n\tau)\|x_n-w\|\\
&=(1-\alpha_n(\tau-\gamma\alpha))\|x_n-w\|+\alpha_n\|\gamma f(w)-\mu F(w))\|\\
&\leq\|x_n-w\|+\alpha_n\|\gamma f(w)-\mu F(w))\|,
\end{aligned}$$

and

$$\begin{aligned}
&\|x_{n+1}-w\|\\
&\leq\|(I-\beta_nA)T_nx_n+\beta_ny_n-w\|\\
&=\|(I-\beta_nA)T_nx_n-(I-\beta_nA)T_nw+\beta_n(y_n-w)+\beta_n(I-A)w\|\\
&\leq\|(I-\beta_nA)T_nx_n-(I-\beta_nA)T_nw\|+\beta_n\|y_n-w\|+\beta_n\|I-A\|\cdot\|w\|\\
&\leq(1-\beta_n\bar{\gamma})\|T_nx_n-T_nw\|+\beta_n\|I-A\|\cdot\|w\|\\
&\quad+\beta_n[\|x_n-w\|+\alpha_n\|\gamma f(w)-\mu F(w))\|]\\
&\leq(1-\beta_n(\bar{\gamma}-1))\|x_n-w\|+\beta_n\|I-A\|\cdot\|w\|+\beta_n\|\gamma f(w)-\mu F(w))\|\\
&=(1-\beta_n(\bar{\gamma}-1))\|x_n-w\|+\beta_n(\bar{\gamma}-1)\frac{\|I-A\|\cdot\|w\|+\|\gamma f(w)-\mu F(w))\|}{\bar{\gamma}-1}.
\end{aligned}$$

By induction, we get

$$\|x_n-w\|\leq\max\left\{\|x_1-w\|,\frac{\|I-A\|\cdot\|w\|+\|\gamma f(w)-\mu F(w))\|}{\bar{\gamma}-1}\right\},\ n\in\mathbb{N}.$$

This implies that $\{x_n\}$ is a bounded sequence. So, $\{T_nx_n\}$, $\{f(x_n)\}$, $\{FT_nx_n\}$, $\{AT_nx_n\}$, and $\{y_n\}$ are bounded sequences. Let

$M:= \sup\{\|x_n\|, \|f(x_n)\|, \|\gamma f(x_n)\|, \|T_nx_n\|, \|y_n\|, \|\mu FT_nx_n\|, \|AT_nx_n\| : n \in \mathbb{N}\}.$

Besides, we also have

$$\lim_{n\to\infty} \|y_n - T_nx_n\| \lim_{n\to\infty} \|\alpha_n\gamma f(x_n) - \alpha_n\mu FT_nx_n\| = 0,$$

and

$$\lim_{n\to\infty} \|x_{n+1} - T_nx_n\| \lim_{n\to\infty} \|\beta_nAT_nx_n - \beta_ny_n\| = 0. \tag{7}$$

By (7), $\lim_{n\to\infty} ||x_{n+1} - y_n|| = 0$. Furthermore, by (6), we have

$$\begin{aligned}
&\|y_n - y_{n-1}\| \\
\le\ &\|(I - \alpha_n\mu F)T_nx_n + \alpha_n\gamma f(x_n) - (I - \alpha_{n-1}\mu F)T_{n-1}x_{n-1} - \alpha_{n-1}\gamma f(x_{n-1})\| \\
\le\ &\|\alpha_n\gamma f(x_n) - \alpha_n\gamma f(x_{n-1})\| + \|\alpha_n\gamma f(x_{n-1}) - \alpha_{n-1}\gamma f(x_{n-1})\| \\
&+ \|(I - \alpha_n\mu F)T_nx_n - (I - \alpha_n\mu F)T_{n-1}x_{n-1}\| \\
&+ \|(I - \alpha_n\mu F)T_{n-1}x_{n-1} - (I - \alpha_{n-1}\mu F)T_{n-1}x_{n-1}\| \\
\le\ &\alpha_n\gamma\alpha\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}| \cdot \|\gamma f(x_{n-1})\| \\
&+ (1 - \alpha_n\tau)\|T_nx_n - T_{n-1}x_{n-1}\| + |\alpha_n - \alpha_{n-1}| \cdot \mu\|FT_{n-1}x_{n-1}\| \\
\le\ &\alpha_n\gamma\alpha\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}| \cdot \|\gamma f(x_{n-1})\| + (1 - \alpha_n\tau)\|T_nx_n - T_nx_{n-1}\| \\
&+ \|T_nx_{n-1} - T_{n-1}x_{n-1}\| + |\alpha_n - \alpha_{n-1}| \cdot \mu\|FT_{n-1}x_{n-1}\|. \\
\le\ &(1 - \alpha_n(\tau - \gamma\alpha)\|x_n - x_{n-1}\| + 2M|\alpha_n - \alpha_{n-1}| + \|T_nx_{n-1} - T_{n-1}x_{n-1}\|.
\end{aligned}$$

Next, we have

$$\begin{aligned}
&\|x_{n+1} - x_n\| \\
\le\ &\|(I - \beta_nA)T_nx_n + \beta_ny_n - (I - \beta_{n-1}A)T_{n-1}x_{n-1} - \beta_{n-1}y_{n-1}\| \\
\le\ &\|(I - \beta_nA)T_nx_n - (I - \beta_nA)T_{n-1}x_{n-1}\| + \|(I - \beta_nA)T_{n-1}x_{n-1} \\
&- (I - \beta_{n-1}A)T_{n-1}x_{n-1}\| + \beta_n\|y_n - y_{n-1}\| + |\beta_n - \beta_{n-1}| \cdot \|y_{n-1}\| \\
\le\ &(1 - \beta_n\bar{\gamma})\|T_nx_n - T_{n-1}x_{n-1}\| + |\beta_n - \beta_{n-1}| \cdot \|AT_{n-1}x_{n-1}\| \\
&+ \beta_n\|y_n - y_{n-1}\| + |\beta_n - \beta_{n-1}| \cdot \|y_{n-1}\| \\
\le\ &(1 - \beta_n\bar{\gamma})\|x_n - x_{n-1}\| + \|T_nx_{n-1} - T_{n-1}x_{n-1}\| + 2M|\beta_n - \beta_{n-1}| \\
&+\beta_n\left[(1 - \alpha_n(\tau - \gamma\alpha)\|x_n - x_{n-1}\| + 2M|\alpha_n - \alpha_{n-1}| + \|T_nx_{n-1} - T_{n-1}x_{n-1}\|\right] \\
\le\ &(1 - \beta_n(\bar{\gamma} - 1))\|x_n - x_{n-1}\| + \|T_nx_{n-1} - T_{n-1}x_{n-1}\| + 2M|\beta_n - \beta_{n-1}| \\
&+ 2M\beta_n|\alpha_n - \alpha_{n-1}| + \beta_n\|T_nx_{n-1} - T_{n-1}x_{n-1}\| \\
\le\ &(1 - \beta_n(\bar{\gamma} - 1))\|x_n - x_{n-1}\| + 2\|T_nx_{n-1} - T_{n-1}x_{n-1}\| \\
&+ 2M\beta_n|\alpha_n - \alpha_{n-1}| + 2M|\beta_n - \beta_{n-1}| \\
\le\ &(1 - \beta_n(\bar{\gamma} - 1))\|x_n - x_{n-1}\| + 2\sup\{\|T_nx - T_{n-1}x\| : x \in \{x_n\}\} \\
&+ 2M|\alpha_n - \alpha_{n-1}| + 2M|\beta_n - \beta_{n-1}|.
\end{aligned} \tag{8}$$

By (i), (ii), (iii), (8), and Lemma 2.1, we know that $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$. This implies that

$$\lim_{n\to\infty} \|T_n x_n - x_n\| = 0. \tag{9}$$

Furthermore,

$$\begin{aligned}\|x_n - Tx_n\| &\le \|x_n - T_n x_n\| + \|T_n x_n - Tx_n\| \\ &\le \|x_n - T_n x_n\| + \sup\{\|T_n z - Tz\| : z \in \{x_n\}\} \\ &\le \|x_n - T_n x_n\| + \sum_{k=n}^{\infty} \sup\{\|T_n z - T_{n+1} z\| : z \in \{x_n\}\}. \end{aligned} \tag{10}$$

By (iii), (9), and (10), we get $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$.

Clearly, T is a nonexpansive mapping. For this nonexpansive mapping T, by Theorem 3.1 in [5], there exists $\bar{x} \in \mathrm{Fix}(T)$ such that $\bar{x}$ is the unique solution of the problem:

$$x^* \in \mathrm{Fix}(T) : \langle (A - I)x^*, x^* - z\rangle \le 0 \ \forall z \in \mathrm{Fix}(T). \tag{11}$$

Since $\{x_n\}$ is a bounded sequence, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty} \langle x_n - \bar{x}, (I - A)\bar{x}\rangle = \lim_{k\to\infty} \langle x_{n_k} - \bar{x}, (I - A)\bar{x}\rangle. \tag{12}$$

Without loss of generality, we may assume that $x_{n_k} \rightharpoonup z$. By Lemma 2.2, $z \in \mathrm{Fix}(T)$. Hence, by (11) and (12), we get

$$\limsup_{n\to\infty} \langle x_n - \bar{x}, (I - A)\bar{x}\rangle = \langle z - \bar{x}, (I - A)\bar{x}\rangle \le 0. \tag{13}$$

Let $u_n := (I - \alpha_n \mu F)T_n x_n + \alpha_n \gamma f(x_n)$. By Lemma 2.3,

$$\begin{aligned} &\|y_n - \bar{x}\|^2 \\ =\ & \|P_C(u_n) - \bar{x}\|^2 \\ =\ & \langle P_C(u_n) - u_n, y_n - \bar{x}\rangle + \langle u_n - \bar{x}, y_n - \bar{x}\rangle \\ \le\ & \langle u_n - \bar{x}, y_n - \bar{x}\rangle \\ \le\ & \langle (I - \alpha_n \mu F)T_n x_n + \alpha_n \gamma f(x_n) - \bar{x}, y_n - \bar{x}\rangle \\ =\ & \langle (I - \alpha_n \mu F)T_n x_n + \alpha_n \gamma f(x_n) - \bar{x}, y_n - \bar{x}\rangle \\ =\ & \langle (I - \alpha_n \mu F)T_n x_n - (I - \alpha_n \mu F)T_n \bar{x} + \alpha_n(\gamma f(x_n) - \mu F\bar{x}), y_n - \bar{x}\rangle \\ \le\ & \|(I - \alpha_n \mu F)T_n x_n - (I - \alpha_n \mu F)T_n \bar{x}\| \cdot \|y_n - \bar{x}\| + \alpha_n \langle \gamma f(x_n) - \mu F\bar{x}, y_n - \bar{x}\rangle \\ \le\ & (1 - \alpha_n \tau)\|x_n - \bar{x}\| \cdot \|y_n - \bar{x}\| + \alpha_n \|\gamma f(x_n) - \mu F\bar{x}\| \cdot \|y_n - \bar{x}\| \\ \le\ & \|x_n - \bar{x}\| \cdot \|y_n - \bar{x}\| + \alpha_n \|\gamma f(x_n) - \mu F\bar{x}\| \cdot \|y_n - \bar{x}\|. \end{aligned}$$

Without loss of generality, we may assume that $\|y_n - \bar{x}\| \neq 0$ for each $n \in \mathbb{N}$. So,

$$\|y_n - \bar{x}\| \le \|x_n - \bar{x}\| + \alpha_n \|\gamma f(x_n) - \mu F\bar{x}\|. \tag{14}$$

Let $w_n := (I - \beta_n A)T_n x_n + \beta_n y_n$. By (14) and Lemma 2.3 again,

$$\begin{aligned}
&\|x_{n+1} - \bar{x}\|^2 \\
&\le \langle w_n - \bar{x}, x_{n+1} - \bar{x}\rangle \\
&= \langle (I - \beta_n A)T_n x_n + \beta_n y_n - \bar{x}, x_{n+1} - \bar{x}\rangle \\
&= \langle (I - \beta_n A)T_n x_n - (I - \beta_n A)T_n \bar{x}, x_{n+1} - \bar{x}\rangle + \beta_n\langle y_n - \bar{x}, x_{n+1} - \bar{x}\rangle \\
&\quad + \beta_n\langle (I - A)\bar{x}, x_{n+1} - \bar{x}\rangle \\
&\le \|(I - \beta_n A)(T_n x_n - T_n\bar{x})\| \cdot \|x_{n+1} - \bar{x}\| + \beta_n\langle y_n - \bar{x}, x_{n+1} - \bar{x}\rangle \\
&\quad + \beta_n\langle (I - A)\bar{x}, x_{n+1} - \bar{x}\rangle \\
&\le (1 - \beta_n\bar{\gamma})\|x_n - \bar{x}\| \cdot \|x_{n+1} - \bar{x}\| + \beta_n\|y_n - \bar{x}\| \cdot \|x_{n+1} - \bar{x}\| \\
&\quad + \beta_n\langle (I - A)\bar{x}, x_{n+1} - \bar{x}\rangle \\
&\le (1 - \beta_n\bar{\gamma})\|x_n - \bar{x}\| \cdot \|x_{n+1} - \bar{x}\| + \beta_n\|x_n - \bar{x}\| \cdot \|x_{n+1} - \bar{x}\| \\
&\quad + \alpha_n\beta_n\|\gamma f(x_n) - \mu F\bar{x}\| \cdot \|x_{n+1} - \bar{x}\| + \beta_n\langle (I - A)\bar{x}, x_{n+1} - \bar{x}\rangle \\
&\le (1 - \beta_n(\bar{\gamma} - 1))\|x_n - \bar{x}\| \cdot \|x_{n+1} - \bar{x}\| + \alpha_n\beta_n\|\gamma f(x_n) - \mu F\bar{x}\| \cdot \|x_{n+1} - \bar{x}\| \\
&\quad + \beta_n\langle (I - A)\bar{x}, x_{n+1} - \bar{x}\rangle \\
&\le \frac{1 - \beta_n(\bar{\gamma} - 1)}{2}\left[\|x_n - \bar{x}\|^2 + \|x_{n+1} - \bar{x}\|^2\right] \\
&\quad + \alpha_n\beta_n\|\gamma f(x_n) - \mu F\bar{x}\| \cdot \|x_{n+1} - \bar{x}\| + \beta_n\langle (I - A)\bar{x}, x_{n+1} - \bar{x}\rangle.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\|x_{n+1} - \bar{x}\|^2 \\
&\le \frac{1 - \beta_n(\bar{\gamma} - 1)}{1 + \beta_n(\bar{\gamma} - 1)}\|x_n - \bar{x}\|^2 + \frac{\beta_n}{1 + \beta_n(\bar{\gamma} - 1)}\big[\alpha_n\|\gamma f(x_n) - \mu F\bar{x}\| \cdot \|x_{n+1} - \bar{x}\| \\
&\quad + \langle (I - A)\bar{x}, x_{n+1} - \bar{x}\rangle\big] \\
&= \left(1 - \frac{2\beta_n(\bar{\gamma} - 1)}{1 + \beta_n(\bar{\gamma} - 1)}\right)\|x_n - \bar{x}\|^2 \\
&\quad + \frac{2\beta_n(\bar{\gamma} - 1)}{1 + \beta_n(\bar{\gamma} - 1)} \cdot \frac{1}{2(\bar{\gamma} - 1)}\big[\alpha_n\|\gamma f(x_n) - \mu F\bar{x}\| \cdot \|y_n - \bar{x}\| \\
&\quad + 2\langle (I - A)\bar{x}, x_{n+1} - \bar{x}\rangle\big].
\end{aligned}$$

By the integral test for series, we know that

$$\sum_{n=1}^{\infty} \frac{2\beta_n(\bar{\gamma} - 1)}{1 + \beta_n(\bar{\gamma} - 1)} = \infty.$$

Hence, by (i), (13), and Lemma 2.1, we know that $\lim_{n\to\infty} x_n = \bar{x}$. Therefore, the proof is complete. ■

The following result is a special case of Theorem 3.1. Note that our results are different from the corresponding ones given by Marino and Xu [7], Yamada [14] and Tian [9] since these iteration processes are different and A is not an identity mapping.

Corollary 3.2. [5] *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to C$ be a nonexpansive mapping, $f : H \to H$ be a contraction mapping with coefficient $\alpha \in (0,1)$, $F : H \to H$ be a Lipschitz mapping with coefficient $k > 0$ and strongly monotone with coefficient $\eta > 0$, $A : H \to H$ be a strongly positive bounded linear operator with coefficient $\bar{\gamma} \in (1,2)$. Let $\{\alpha_n\} \subseteq [0,1]$ and $\{\beta_n\} \subseteq (0,1]$. Let $0 < \mu < 2\eta/k^2$, $0 < \gamma < \mu(\eta - \frac{\mu k^2}{2})/\alpha = \tau/\alpha$. Suppose that $\mathrm{Fix}(T) \neq \emptyset$. Let $\{x_n\}$ be generated by*

$$\begin{cases} x_1 \in C \text{ chosen arbitrary}, \\ y_n := P_C[(I - \alpha_n \mu F)Tx_n + \alpha_n \gamma f(x_n)], \\ x_{n+1} := P_C[(I - \beta_n A)Tx_n + \beta_n y_n]. \end{cases}$$

Assume that:

(i) $\lim_{n\to\infty} \alpha_n = 0$; $\lim_{n\to\infty} \beta_n = 0$; $\sum_{n=1}^{\infty} \beta_n = \infty$;

(ii) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$; $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$.

Then $\lim_{n\to\infty} x_n = \bar{x}$, where $\bar{x} \in \mathrm{Fix}(T)$ and $\langle (A-I)\bar{x}, \bar{x}-z\rangle \leq 0$ for all $z \in \mathrm{Fix}(T)$, that is, $P_{\mathrm{Fix}(T)}(2I - A)\bar{x} = \bar{x}$.

4. Applications

Let H be a real Hilbert space, and let C be a nonempty closed convex subset of H. An accretive operator B is said to satisfy the range condition if $\mathrm{cl}(D(B)) \subseteq R(I + \lambda B)$ for all $\lambda > 0$, where $D(B)$ is the domain of B, I is the identity mapping on H, $R(I + \lambda B)$ is the range of $I + \lambda B$, and $\mathrm{cl}(D(B))$ is the closure of $D(B)$.

If B is an accretive operator which satisfies the range condition, then we can define, for each $\lambda > 0$, a mapping $J_\lambda : R(I + \lambda B) \to D(B)$ by $J_\lambda := (I + \lambda B)^{-1}$, which is called the resolvent of B. We know that $\mathrm{Fix}(J_\lambda) = B^{-1}(0)$ for all $\lambda > 0$, and

$$\|J_\lambda x - J_\lambda y\|^2 \leq \|x - y\|^2 - \|(I - J_\lambda)x - (I - J_\lambda)y\|^2$$

for all $x, y \in R(I + \lambda B)$. Hence, J_λ is a nonexpansive mapping. Furthermore, we know that [6]: for each $\lambda_1, \lambda_2 > 0$ and $x \in R(I + \lambda_1 B) \cap R(I + \lambda_2 B)$, we have

$$\|J_{\lambda_1} x - J_{\lambda_2} x\| \leq \frac{|\lambda_1 - \lambda_2|}{\lambda_1} \|x - J_{\lambda_1} x\|.$$

Theorem 4.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $f : H \to H$ be a contraction mapping with coefficient $\alpha \in (0,1)$, F :*

$H \to H$ be a Lipschitz mapping with coefficient $k > 0$ and strongly monotone with coefficient $\eta > 0$, $A : H \to H$ be a strongly positive bounded linear self-adjoint operator with coefficient $\bar{\gamma} \in (1, 2)$. Let B be an accretive operator such that $B^{-1}(0) \neq \emptyset$ and $\mathrm{cl}(D(B)) \subseteq C \subseteq \cap_{\lambda>0} R(I + \lambda B)$. Let $\{\alpha_n\} \subseteq [0, 1]$, $\{\beta_n\} \subseteq (0, 1]$, and let $\{\lambda_n\}$ be a sequence in $(0, \infty)$. Let $0 < \mu < 2\eta/k^2$, $0 < \gamma < \mu(\eta - \frac{\mu k^2}{2})/\alpha = \tau/\alpha$. Let $\{x_n\}$ be generated by

$$\begin{cases} x_1 \in C \text{ chosen arbitrary}, \\ y_n := P_C[(I - \alpha_n \mu F) J_{\lambda_n} x_n + \alpha_n \gamma f(x_n)], \\ x_{n+1} := P_C[(I - \beta_n A) J_{\lambda_n} x_n + \beta_n y_n]. \end{cases}$$

Assume that:

(i) $\lim_{n\to\infty} \alpha_n = 0$; $\lim_{n\to\infty} \beta_n = 0$; $\sum_{n=1}^{\infty} \beta_n = \infty$;

(ii) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$; $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$.

Then $\lim_{n\to\infty} x_n = \bar{x}$, where $\bar{x} \in B^{-1}(0)$ and $\langle (A - I)\bar{x}, \bar{x} - z\rangle \leq 0$ for all $z \in B^{-1}(0)$, that is, $P_{B^{-1}(0)}\bar{x} = \bar{x}$.

Proof. For each $n \in \mathbb{N}$, let $T_n : C \to C$ be defined by $T_n x := J_{\lambda_n} x$ for each $x \in C$. By following the same argument in the proof of Theorem 4.3 [2], we know that the condition (iii) of Theorem 3.1 holds. By Theorem 3.1, we get the conclusion of Theorem 4.1. ∎

Acknowledgments. The authors wish to express their gratitude to the referees for their valuable suggestions during the preparation of this paper. This research was supported by the National Science Council of Republic of China.

References

1. P. N. Anh and D. X. Son, A new method for a finite family of pseudocontractions and equilibrium problems, *J. Appl. Math. Inform.* **29** (2011), 1179–1191.
2. K. Aoyama, Y. Kimura, W. Takahashi and M. Toyoda, Approximation of common fixed points of a countable family of nonexpansive mappings in a Banach space, *Nonlinear Anal.* **67** (2007), 2350–2360.
3. F. E. Browder, Fixed point theorems for noncompact mappings in Hilbert spaces, *Proc. Natl. Acad. Sci. USA* **53** (1965), 1272–1276.
4. A. Bunyawat and S. Suantai, Strong convergence theorems for variational inequalities and fixed points of a countable family of nonexpansive mappings, *Fixed Point Theory Appl.* (2011), 2011:47, 13 pages, Doi:10.1186/1687-1812-2011-47.
5. L. C. Ceng, S. Y. Guu and J. C. Yao, A general composite iterative algorithm for nonexpansive mappings in Hilbert spaces, *Comput. Math. Appl.* **61** (2011), 2447–2455.
6. K. Eshita and W. Takahashi, Approximating zero points of accretive operators in general Banach spaces, *JP J. Fixed Point Theory Appl.* **2** (2007), 105–116.

7. G. Marino and H. K. Xu, A general iterative method for nonexpansive mappings in Hilbert spaces, *J. Math. Anal. Appl.* **318** (2006), 43–52.
8. W. Takahashi, *Nonlinear Functional Analysis-Fixed Point Theory and its Applications*, Yokohama Publishers, Yokohama, 2000.
9. M. Tian, A general iterative algorithm for nonexpansive mappings in Hilbert spaces, *Nonlinear Anal.* **73** (2010), 689–694.
10. R. Wangkeeree and U. Kamraksa, A general iterative method for solving the variational inequality problem and fixed point problem of an infinite family of nonexpansive mappings in Hilbert spaces, *Fixed Point Theory Appl.* (2009), Article ID 369215, 23 pages.
11. H. K. Xu, Iterative algorithm for nonlinear operators, *J. Lond. Math. Soc.* **2** (2002), 240–256.
12. H. K. Xu, An iterative approach to quadratic optimization, *J. Optim. Theory Appl.* **116** (2003), 659–678.
13. H. K. Xu, Viscosity approximation methods for nonexpansive mappings, *J. Math. Anal. Appl.* **298** (2004), 279–291.
14. I. Yamada, The hybrid steepest-descent method for variational inequality problems over the intersection of the fixed-point sets of nonexpansive mappings, In: *Inherently Parallel Algorithms in Feasibility and Optimization and their Applications*, D. Butnariu, Y. Censor, S. Reich (eds.), North-Holland, Amsterdam, Holland, 2001, pp. 473–504.