Vietnam Journal of Mathematics **40**:2&3(2012) 285-304

Vietnam Journal of MATHEMATICS

Generalized Dubovitskii-Milyutin Approach in Set-Valued Optimization*

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Dedicated to Professor Phan Quoc Khanh on the occasion of his 65th birthday

Received December 3, 2011
Revised July 3, 2012

Abstract. The primary objective of this paper is to employ the Dubovitskii-Milyutin approach to give first and second-order optimality conditions for set-valued optimization problems with more general set-valued constraints. In the particular case when some constraints are single-valued, we recover the case of set-valued optimization problems with many set-valued inequality constraints and many single-valued equality constraints. It is known that the classical Dubovitskii-Milyutin approach is not suitable for optimization problems with multi-equality constraints. The main reason for this deficiency is the fact that the separation arguments used in the classical Dubovitskii-Milyutin approach are applicable to an empty intersection of cones in which at most one cone can be closed. However, a proper formulation of multi-equality constraints leads to an empty intersection with more than one closed cones. To study optimization problems with multi-equality constraints, a generalized Dubovitskii-Milyutin theory has been developed. In this work we present an extension of the generalized Dubovitskii-Milyutin theory to the set-valued optimization problems. In this process, we also obtain new applications of this theory to nonsmooth optimization and to more general vector optimization problems. New second-order asymptotic derivatives of set-valued maps are introduced and used to give the optimality conditions.

* This work was supported by a grant from the Simons Foundation (#210443 to Akhtar Khan).

2000 Mathematics Subject Classification. 90C26, 90C29, 90C30.

Key words. Set-valued optimization, lower Dini derivative, second-order lower Dini asymptotic derivative, asymptotic contingent cones, interiorly tangent cones, contingent cones, optimality conditions.

1. Introduction

Let $I := \{1, 2, \dots, m\}$ and $J := \{1, 2, \dots, n\}$ be two index sets. Let X and Y be normed spaces, where the space Y is partially ordered by a pointed, closed, and convex cone K with a nonempty interior, let $Q \subset X$ be nonempty, and let $F : X \rightrightarrows Y$ be a given set-valued map. For $i \in I$, let Z_i be a normed space, let $C_i \subset Z_i$ be a pointed, closed, and convex cone with nonempty interior, and let $G_i : X \rightrightarrows Z_i$ be a given set-valued map. Analogously, for $j \in J$, let W_j be a normed space, let $D_j \subset W_j$ be a closed and convex set, and let $H_j : X \rightrightarrows W_j$ be a given set-valued map.

In this work, we focus on the following set-valued optimization problem (P):

$$\text{minimize}\, F(x)$$
$$\text{subject to}$$
$$x \in S := \{x \in Q \mid G_i(x) \cap -C_i \neq \emptyset \ \forall\ i \in I;\ H_j(x) \cap -D_j \neq \emptyset \ \forall\ j \in J\}.$$

Our objective is to give optimality conditions for a local weak-minimizer. However, our approach can readily be extended to other kinds of optimality as well. We recall that a point $(\bar{x}, \bar{y}) \in X \times Y$ is a local weak-minimizer of (P), if there is a neighborhood U of $\bar{x}$ such that

$$F(S \cap U) \cap (\{\bar{y}\} - \operatorname{int}(K)) = \emptyset,$$

where $\bar{y} \in F(\bar{x})$, the notion $\operatorname{int}(K)$ designates the interior of the cone K, and

$$F(S \cap U) := \cup_{x \in S \cap U} F(x).$$

Since most available results in set-valued optimization deal only with inequality constraints, we will also focus on the following optimization problem (P_1):

$$\text{minimize}\, F(x)$$
$$\text{subject to } x \in S := \{x \in Q \mid G_i(x) \cap -C_i \neq \emptyset \ \forall\ i \in I\}.$$

Another important case when there are no explicit constraints is considered in the following optimization problem (P_0):

$$\text{minimize}\, F(x)$$
$$\text{subject to } x \in Q.$$

By appropriately adjusting the data and by choosing the maps G_i and H_j to be single-valued, from (P) we recover the following set-valued optimization problem with single-valued constraints:

$$\begin{aligned}&\text{minimize}\, F(x)\\ &\text{subject to } x \in S := \{x \in Q \mid G_i(x) \geq 0 \ \forall\ i \in I;\ H_j(x) = 0 \forall\ j \in J\}.\end{aligned}$$

We remark that the above optimization problem with multi-equality constraints can not be tackled by the classical Dubovitskii-Milyutin approach (see [10]). This is mainly due to the fact that the separation arguments used in the classical Dubovitskii-Milyutin approach are applicable to an empty intersection of cones in which at most one cone can be closed. On the other hand, a proper formulation of multi-equality constraints leads to an empty intersection with several closed cones. To handle optimization problems with multi-equality constraints, a generalized Dubovitskii-Milyutin theory has been developed (see [19]). This theory is enriched by the notion of the cones of the same sense and the cones of the opposite sense introduced by Walczak [28]. Although many interesting contributions have appeared in this direction, the generalized Dubovitskii-Milyutin theory so far has only been used to study vector optimization problems with differentiable data. Motivated by this, the primary objective of this paper is to present an extension of the generalized Dubovitskii-Milyutin theory to the set-valued optimization problems. In this process, we also obtain new extensions of this theory to nonsmooth optimization and to more general vector optimization problems. Some of the recent developments in set-valued optimization can be found in [1, 3, 8, 9, 12, 17, 18, 27] and the cited references therein.

The contents of this paper are organized into four sections. In Section 2, we collect some concepts and results to be used in the rest of the paper. In Section 3, we give first-order optimality conditions for set-valued optimization problems. Section 4 presents some second-order optimality conditions by using a new notion of second-order lower Dini asymptotic derivative.

2. Preliminaries

We begin by recalling the notions of some tangent cones. We set $\mathbb{P} := \{t \in \mathbb{R} \mid t > 0\}$.

Definition 2.1. Let Ξ be a normed space, let $\Omega \subset \Xi$ and let $\bar{z} \in \mathrm{cl}(\Omega)$ (the closure of Ω).

(a) The contingent cone $T(\Omega, \bar{z})$ of Ω at $\bar{z}$ is the set of all $z \in \Xi$ such that there are sequences $(\lambda_n) \subset \mathbb{P}$ and $(z_n) \subset \Xi$ with $\lambda_n \downarrow 0$ and $z_n \to z$ such that $\bar{z} + \lambda_n z_n \in \Omega$ for every $n \in \mathbb{N}$.

(b) The interiorly contingent cone $IT(\Omega, \bar{z})$ of Ω at $\bar{z}$ is the set of all $z \in \Xi$ such that for any sequences $(\lambda_n) \subset \mathbb{P}$ and $(z_n) \subset \Xi$ with $\lambda_n \downarrow 0$ and $z_n \to z$, there exists an integer $m \in \mathbb{N}$ such that $\bar{z} + \lambda_n z_n \in \Omega$ for every $n \geq m$.

Remark 2.2. It is known that $T(\Omega,\bar{z})$ is a closed cone possessing the isotony property, that is, for subsets Ω_1 and Ω_2 such that $\Omega_1 \subset \Omega_2$, the inclusion $T(\Omega_1,\bar{z}) \subset T(\Omega_2,\bar{z})$ holds for every $\bar{z} \in \mathrm{cl}(\Omega_1) \cap \mathrm{cl}(\Omega_2)$. On the other hand the interiorly contingent cone $IT(\Omega,\bar{z})$ is an isotone open cone. As concern the relationship between $T(\Omega,\bar{z})$ and $IT(\Omega,\bar{z})$, we have $IT(\Omega,\bar{z}) = \Xi \setminus T(\Xi \setminus \Omega,\bar{z})$. As a useful implication of this relationship, the cones $T(\Omega,\bar{z})$ and $IT(\Omega,\bar{z})$ form an admissible pair, that is, for every pair of sets $\Omega_1,\Omega_2 \subset \Xi$ with $\Omega_1 \cap \Omega_2 = \emptyset$, we have $T(\Omega_1,\bar{z}) \cap IT(\Omega_2,\bar{z}) = \emptyset$ for every $\bar{z} \in \Xi$. Also for arbitrary sets $\Omega_1,\Omega_2 \subset \Xi$ we have $IT(\Omega_1 \cap \Omega_2,\bar{z}) = IT(\Omega_1,\bar{z}) \cap IT(\Omega_2,\bar{z})$ for every $\bar{z} \in \Omega_1 \cap \Omega_2$. In general, this important property is not shared by the contingent cones. However, for arbitrary sets $\Omega_1,\Omega_2 \subset \Xi$, we have $T(\Omega_1 \cap \Omega_2,\bar{z}) \subset T(\Omega_1,\bar{z}) \cap T(\Omega_2,\bar{z})$ for every $\bar{z} \in \Omega_1 \cap \Omega_2$. For any $\Omega \subset \Xi$, the identities $T(\Omega,\bar{z}) = T(\mathrm{cl}(\Omega),\bar{z})$ and $IT(\Omega,\bar{z}) = IT(\mathrm{int}(\Omega),\bar{z})$ hold. Moreover, for a convex solid set Ω, we have $\mathrm{cl}(IT(\Omega,\bar{z})) = T(\Omega,\bar{z})$ and $\mathrm{int}(T(\Omega,\bar{z})) = IT(\Omega,\bar{z})$ (see [2, 11] for details).

Next we collect some notions for set-valued maps. Given the normed spaces X and Y, let $F : X \rightrightarrows Y$ be a set-valued map. The (effective) domain and the graph of F are defined by

$$\mathrm{dom}(F) := \{x \in X| \quad F(x) \neq \emptyset\},$$
$$\mathrm{graph}(F) := \{(x,y) \in X \times Y| \quad y \in F(x)\}.$$

We shall say that F is *strict* if $\mathrm{dom}(F) = X$. Given a convex cone $C \subset Y$, which induces a partial ordering in Y, the profile map $F_+ : X \rightrightarrows Y$ is given by: $F_+(x) := F(x) + C$ for every $x \in \mathrm{dom}(F)$. Now the epigraph of F can be defined as the graph of F_+, that is, $\mathrm{epi}(F) = \mathrm{graph}(F_+)$. The map F is called convex, if $\mathrm{graph}(F)$ is a convex set and C-convex, if $\mathrm{epi}(F)$ is a convex set. Finally, we define the *weak-inverse image* $F[\Theta]^-$ of F with respect to a set $\Theta \subset Y$ as

$$F[\Theta]^- := \{x \in X| \quad F(x) \cap \Theta \neq \emptyset\}.$$

Now, let X^* be the dual of X and let $L \subset X$ be arbitrary. The negative dual of L, denoted by $L^\diamond$, is a subset of X^* defined by:

$$L^\diamond = \{\ell \in X^* : \ \ell(x) \leq 0 \text{ for every } x \in L\}.$$

It is known that if $L_1 \subseteq L_2$ then $L_2^\diamond \subseteq L_1^\diamond$. Additionally, $L \subset (L^\diamond)^\diamond$ with equality if and only if L is a closed and convex cone. The positive dual then is the set defined by $L^* = -L^\diamond$. Both the positive dual and the negative dual are closed and convex cones. Moreover, the properties just mentioned for the negative dual hold for the positive dual as well.

Given a set-valued map $F : X \rightrightarrows Y$ and a point $(\bar{x},\bar{y}) \in \mathrm{graph}(F)$, the contingent derivative of F at $(\bar{x},\bar{y})$ is the set-valued map $D_cF(\bar{x},\bar{y}) : X \rightrightarrows Y$ defined by:

$$D_cF(\bar{x},\bar{y})(x) := \{y \in Y \,| \quad (x,y) \in T(\mathrm{graph}(F),(\bar{x},\bar{y}))\}.$$

The contingent derivative and related contingent epiderivative and generalized contingent epiderivative are commonly used derivative notions in set-valued optimization (see [8, 9, 14, 15]).

The derivative for set-valued maps that turns out to be of great importance in the present approach is the so-called lower Dini derivative introduced by Penot [24].

We recall that given a set-valued map $F : X \rightrightarrows Y$ and a point $(\bar{x}, \bar{y}) \in \text{graph}(F)$, the lower Dini derivative of F at $(\bar{x}, \bar{y})$ is the set-valued map $D_l F(\bar{x}, \bar{y}) : X \rightrightarrows Y$ defined by:

$$D_l F(\bar{x}, \bar{y})(x) := \liminf_{(t,z)\to(0_+,x)} \frac{F(\bar{x}+tz)-\bar{y}}{t}.$$

Equivalently $y \in D_l F(\bar{x}, \bar{y})(x)$ if and only if for every $(\lambda_n) \subset \mathbb{P}$ and for every $(x_n) \subset X$ with $\lambda_n \downarrow 0$ and $x_n \to x$ there are a sequence $(y_n) \subset Y$ with $y_n \to y$ and an integer $m \in \mathbb{N}$ such that $\bar{y} + \lambda_n y_n \in F(\bar{x} + \lambda_n x_n)$ for every $n \geq m$.

We also need to recall the following important notion of certain cones:

Definition 2.3. [28] Let Ξ be a normed space, let $\{K_i\}_{i=1}^k$ be a system of cones in Ξ, and let B_ε be a ball with center 0 and radius $\varepsilon > 0$ in the space Ξ.

(a) The cones $\{K_i\}_{i=1}^k$ are of the same sense, if for any $\varepsilon > 0$ there exist $\varepsilon_1, \ldots, \varepsilon_k > 0$ such that for any $x \in B_\varepsilon \cap (K_1 + \cdots + K_k)$, where $x = x_1 + \cdots + x_k$ with $x_i \in K_i$, we have $x_i \in B_{\varepsilon_i} \cap K_i$ for $i = 1, \ldots, k$.

(b) The cones $\{K_i\}_{i=1}^k$, are of the opposite sense, if there exist nontrivial vectors $\{x_i\}_i^n$, that is $(x_1, \ldots, x_k) \neq (0, \ldots, 0)$, $x_i \in K_i$, such that $x_1 + x_2 + \cdots + x_k = 0$.

Remark 2.4. It follows from the above definitions that the cones of the same sense and the cones of the opposite sense do not intersect. Furthermore, any subsystem of a system of the cones of the same sense is of the same sense, and if a subsystem is of the opposite sense, then the whole system is also of the opposite sense. Moreover, if cones K_1 and K_2 are subspaces, and if $K_1 \cap K_2$ contains a nonzero element, then these cones are of the opposite sense.

We also need to formulate the following notion:

Definition 2.5. A system of sets $\{\Omega_1, \Omega_2, \ldots, \Omega_n\}$ is called optimally positioned, if for any $z \in \Omega_1 \cap \Omega_2 \cap \cdots \cap \Omega_n$, we have

$$T(\Omega_1 \cap \Omega_2 \cap \cdots \cap \Omega_n, z) = T(\Omega_1, z) \cap T(\Omega_2, z) \cap \cdots \cap T(\Omega_n, z). \tag{1}$$

Conditions ensuring (1) are available in [20] (see also [2]).

Finally, we conclude this section by recalling the following important result.

Theorem 2.6. [19] *Let Ξ be a normed space, let $\{K_i\}_{i=1}^k$, be a system of cones in Ξ. Assume that the following conditions hold:*

(a) *The cones* $K_1, \ldots, K_p \subset \Xi$ *are open and convex with vertices at* 0.

(b) *The cones* $K_{p+1}, \ldots, K_k \subset \Xi$ *are closed and convex with vertices at* 0.

(c) *The (positive) dual cones* $K^*_{p+1}, \ldots, K^*_k$ *to* $K_{p+1}, \ldots, K_k$ *are either of the same sense or the opposite sense.*

Then the following two statements are equivalent:

(i) $\bigcap_{i=1}^{k} K_i = \emptyset$.

(ii) *There exist linear, continuous functionals* $f_i \in K_i^*$ *for* $i = 1, \ldots, k$, *not all simultaneously zero, such that*

$$f_1 + \cdots + f_k = 0.$$

3. First-Order generalized Dubovitskii-Milyutin approach

We begin by introducing the following notions.

Definition 3.1. Let X and Y be normed spaces and let $R : X \rightrightarrows Y$ be a set-valued map. The map R is called locally convex at $(\bar{x}, \bar{y}) \in \operatorname{graph}(R)$, if the lower Dini derivative $D_l R(\bar{x}, \bar{y})$ of R at $(\bar{x}, \bar{y})$ is a convex set-valued map. The map R is called regular at $(\bar{x}, \bar{y}) \in \operatorname{graph}(R)$, if additionally R is strict and the weak-inverse image $D_l R(\bar{x}, \bar{y})$ with respect to an open cone is an open cone.

Given $A \subset \mathbb{R}$ and $b \in \mathbb{R}$, by the inequality $A \geq b$ we understand that $a \geq b$ for every $a \in A$. With this convention in mind we are ready to give the promised multiplier rule.

Theorem 3.2. *Let* $(\bar{x}, \bar{y}) \in \operatorname{graph}(F)$ *be a local weak-minimizer of* (P) *and let* $\bar{z}_i \in G_i(\bar{x}) \cap (-C_i)$, *where* $i \in I := \{1, 2, \ldots, m\}$. *Let there exist an open convex cone* $L \subseteq IT(Q, \bar{x})$. *Assume that the sets* $\{H_j[-D_j]^-\}_{j=1}^n$ *are optimally positioned. Assume that* $M_j := T(H_j[-D_j]^-, \bar{x})$ *for* $j \in J$ *are nontrivial and convex and* $\{M_j^*\}_{j=1}^n$ *are either of the same sense or the opposite sense. Let* F *be regular at* $(\bar{x}, \bar{y})$ *and let* G_i *be regular at* $(\bar{x}, \bar{z}_i)$ *for* $i \in I$. *Then there exist functionals* $s \in L^*$, $t \in K^*$, $u_i \in (T(C_i, -\bar{z}_i))^*$, $v_j \in M_j^*$, *not all zero, such that* $u_i(\bar{z}_i) = 0$. *Moreover, the following inequality holds for every* $x \in X$:

$$t \circ D_l F(\bar{x}, \bar{y})(x) + u_1 \circ D_l G_1(\bar{x}, \bar{z}_1)(x) + \cdots + u_m \circ D_l G_m(\bar{x}, \bar{z}_m)(x) \geq s(x) + v_1(x) + \cdots + v_n(x). \quad (2)$$

We shall divide the proof in several parts. We begin with the following:

Proposition 3.3. *Let* $(\bar{x}, \bar{y}) \in \operatorname{graph}(F)$ *be a local weak-minimizer to* (P). *Then*

$$U \bigcap Q \bigcap F[\bar{y} - \operatorname{int}(K)]^- \bigcap_{i=1}^{m} G_i[-C_i]^- \bigcap_{j=1}^{n} H_j[-D_j]^- = \emptyset,$$

where U is a neighborhood of $\bar{x}$ corresponding to the definition of the local weak-minimality.

Proof. Assume that there exists

$$x \in U \bigcap Q \bigcap F[\bar{y} - \mathrm{int}(K)]^- \bigcap_{i=1}^{m} G_i[-C_i]^- \bigcap_{j=1}^{n} H_j[-D_j]^-.$$

From the containment

$$x \in Q \bigcap_{i=1}^{m} G_i[-C_i]^- \bigcap_{j=1}^{n} H_j[-D_j]^-,$$

we notice that $x \in Q$ and for all $i \in I$ and all $j \in J$, we have

$$G_i(x) \cap -C_i \neq \emptyset,$$
$$H_j(x) \cap -D_j \neq \emptyset,$$

which ensures that x is a feasible point. Furthermore, from the containment

$$x \in U \cap F[\bar{y} - \mathrm{int}(K)]^-,$$

we obtain that there exists $x \in U$, neighborhood of $\bar{x}$, such that

$$F(x) \cap (\bar{y} - \mathrm{int}(K)) \neq \emptyset.$$

This, however, in view of the feasibility of x, contradicts the local weak-minimality of $(\bar{x}, \bar{y})$. ■

The above result then leads to the following new optimality condition:

Proposition 3.4. *Let* $(\bar{x}, \bar{y}) \in \mathrm{graph}(F)$ *be a local minimizer to* (P). *Assume that the sets* $H_j[-D_j]^-$ *for* $j \in J$*, are optimally positioned. Then*

$$IT(Q, \bar{x}) \bigcap IT(F[\bar{y} - \mathrm{int}(K)]^-, \bar{x}) \bigcap_{i=1}^{m} IT(G_i[-C_i]^-, \bar{x}) \bigcap_{j=1}^{n} T\left(H_j[-D_j]^-, \bar{x}\right) = \emptyset.$$

Proof. In view of Proposition 3.3 and the properties of the interiorly contingent cones and the contingent cones stated in Remark 2.2, we obtain

$$IT(Q, \bar{x}) \bigcap IT\Big(F[\bar{y} - \mathrm{int}(K)]^-, \bar{x}\Big) \bigcap_{i=1}^{m} IT(G_i[-C_i]^-, \bar{x}) \bigcap T\Big(\bigcap_{j=1}^{n} H_j[-D_j]^-, \bar{x}\Big) = \emptyset.$$

The assertion then follows by the assumption that the sets $H_j[-D_j]^-$ for $j \in J$ are optimally positioned and hence

$$T\Big(\bigcap_{j=1}^{n} H_j[-D_j]^-, \bar{x}\Big) = \bigcap_{j=1}^{n} T(H_j[-D_j]^-, \bar{x}).$$

■

The following results are based on our previous work (see [12, 16]). However, due to the important nature of these results and for the sake of completeness we give shorter proofs here.

Lemma 3.5. [12, 16] *Let X and Y be normed spaces, let $F : X \rightrightarrows Y$ be a set valued map and let $(\bar{x}, \bar{y}) \in \mathrm{graph}(F)$. Let K be a proper, open and convex cone. Then*

$$D_l F(\bar{x}, \bar{y})[-\mathrm{int}(K)]^- \subseteq IT(F[\bar{y} - \mathrm{int}(K)]^-, \bar{x}).$$

Proof. Let $x \in D_l F(\bar{x}, \bar{y})[-\mathrm{int}(K)]^-$ be arbitrary. Therefore, there exists $y \in D_l F(\bar{x}, \bar{y})(x) \cap -\mathrm{int}(K)$. Let $(x_n) \subset X$ and $(\lambda_n) \subset \mathbb{P}$ be arbitrary sequences such that $x_n \to x$ and $\lambda_n \downarrow 0$. It suffices to show that there exists $m \in \mathbb{N}$ such that $\bar{x}+\lambda_n x_n \in F[\bar{y}-\mathrm{int}(K)]^-$ for every $n \geq m$. By the definition of $D_l F(\bar{x}, \bar{y})(\cdot)$, there exist $(y_n) \subset Y$ with $y_n \to y$ and $n_1 \in \mathbb{N}$ such that $\bar{y} + \lambda_n y_n \in F(\bar{x} + \lambda_n x_n)$ for every $n \geq n_1$. Since $y \in -\mathrm{int}(K)$ and $y_n \to y$, there exists $n_2 \in \mathbb{N}$ such that $\lambda_n y_n \in -\mathrm{int}(K)$ for every $n \geq n_2$. This implies that $\bar{y} + \lambda_n y_n \in F(\bar{x} + \lambda_n x_n) \cap (\bar{y} - \mathrm{int}(K))$ for $n \geq m := \max\{n_1, n_2\}$. Hence for the sequences (x_n) and (λ_n) we have $\bar{x} + \lambda_n x_n \in F[\bar{y} - \mathrm{int}(K)]^-$ for $n \geq m$. This is equivalent to saying that $x \in IT(F[\bar{y} - \mathrm{int}(K)]^-, \bar{x})$. The proof is complete. ■

Lemma 3.6. [12, 16] *Let X and Z be normed spaces, let $G : X \rightrightarrows Z$ be a set valued map and let $(\bar{x}, \bar{z}) \in \mathrm{graph}(G)$. Let $A \subset Z$ with $\mathrm{int}(A) \neq \emptyset$. Then the following holds:*

$$D_l G(\bar{x}, \bar{z})[IT(-A, \bar{z})]^- \subseteq IT(G[-A]^-, \bar{x}).$$

Proof. Let $u \in D_l G(\bar{x}, \bar{z})[IT(-A, \bar{z})]^-$ be arbitrary. Let $(u_n) \subset X$ and $(\lambda_n) \subset \mathbb{P}$ be arbitrary sequences such that $u_n \to u$ and $\lambda_n \downarrow 0$. It suffices to show that there exists $m \in \mathbb{N}$ such that $\bar{x} + \lambda_n u_n \in G[-A]^-$ for every $n \geq m$. Since $u \in D_l G(\bar{x}, \bar{z})[IT(-A, \bar{z})]^-$, there exists $v \in D_l G(\bar{x}, \bar{z})(u) \cap IT(-A, \bar{z})$. Therefore, there are a sequence $(v_n) \subset Z$ and an integer $n_1 \in \mathbb{N}$ such that $v_n \to v$ and $\bar{z} + \lambda_n v_n \in G(\bar{x}+\lambda_n u_n)$ for every $n \geq n_1$. Because of the containment $v \in IT(-A, \bar{z})$ there exists $n_2 \in \mathbb{N}$ such that $\bar{z}+\lambda_n v_n \in -A$ for every $n \geq n_2$. Therefore we have $\bar{z} + \lambda_n v_n \in G(\bar{x} + \lambda_n u_n) \cap (-A)$ for every $n \geq m := \max\{n_1, n_2\}$. Consequently $u \in IT(G[-A]^-, \bar{x})$. ■

Lemma 3.7. [12, 16] *Let X and Y be normed spaces, let $D \subseteq X$ be convex and let $A \subset Y$ be a solid closed convex cone. Let $T : D \rightrightarrows Y$ be an A-convex set-valued map. If $T[-\mathrm{int}(A)]^- \neq \emptyset$, then for every $p \in P^\diamond$ where $P := T[-A]^-$, there exists $t \in A^*$ such that*

$$t \circ T(x) \geq p(x) \quad \text{for every} \quad x \in D.$$

If $T[-\mathrm{int}(A)]^- = \emptyset$, *then there exists* $t \in A^*\backslash\{0_{Y^*}\}$ *such that*

$$t \circ T(x) \geq 0 \quad \textit{for every} \quad x \in D.$$

Proof. We begin with the case when the set $T[-\mathrm{int}(A)]^-$ is nonempty. Then the (negative) dual $P^\diamond$ of $P := T[-A]^-$ is also nonempty. We choose $p \in P^\diamond$ arbitrarily and define a set $E := \{(y, p(x)) \in Y \times \mathbb{R} \mid y \in T(x) + A,\ x \in D\}$. In view of the assumptions that D is convex, T is A-convex and $p \in Y^*$, we deduce that E is a convex set. Indeed, let $(y_1, z_1), (y_2, z_2) \in E$ be arbitrary. Then by the definition of E, for $i = 1, 2$, there exists $x_i \in X$ with $z_i = p(x_i)$ and $y_i \in T(x_i) + A$. For $\lambda \in (0, 1]$, we have $\lambda z_1 + (1 - \lambda)z_2 = p(\lambda x_1 + (1 - \lambda)x_2)$. Further, in view of the A-convexity of T, we have $\lambda y_1 + (1 - \lambda)y_2 \in \lambda T(x_1) + (1 - \lambda)T(x_2) + A \subseteq T(\lambda x_1 + (1 - \lambda)x_2) + A$. This, in view of the convexity of the set D, implies that $\lambda(y_1, z_1) + (1 - \lambda)(y_2, z_2) \in E$.

Next, we claim that $E \cap (-\mathrm{int}(A) \times \mathbb{P}) = \emptyset$. In fact, if this is not the case, then there exists $(x, y) \in X \times Y$ such that $y \in (T(x) + A) \cap (-\mathrm{int}(A))$ and $p(x) > 0$. Let $w \in T(x)$ be such that $y \in w + A$. Then $w \in y - A \subset -\mathrm{int}(A) - A = -\mathrm{int}(A)$. This however contradicts that $p \in P^\diamond$. Therefore $E \cap (-\mathrm{int}(A) \times \mathbb{P}) = \emptyset$ and hence by a separation theorem, we get the existence of $(f, g) \in Y^* \times \mathbb{R}\backslash\{0_{Y^*}, 0\}$ and a real number α such that we have

$$f(u) + g(v) \geq \alpha \quad \text{for every} \quad (u, v) \in E, \tag{3}$$

$$f(c) + g(d) < \alpha \quad \text{for every} \quad (c, d) \in -\mathrm{int}(A) \times \mathbb{P}. \tag{4}$$

Since A is a cone, we can set $\alpha = 0$ in (3) and (4). By taking $d \in \mathbb{P}$ arbitrary close to 0 and $c \in -\mathrm{int}(A)$ arbitrary close to 0_Y, we obtain $f \in A^*$ and $g \leq 0$, respectively. We claim that $g < 0$. Indeed, if $g = 0$, we get $f(c) < 0$ for every $c \in -\mathrm{int}(A)$ and $f(u) \geq 0$ for every $u \in T(D) + A$. This, however is impossible because we have $(T(D) + A) \cap (-\mathrm{int}(A)) \neq \emptyset$. Therefore $g < 0$. Moreover, from (3), for every $x \in D$ we have $f \circ (T + A)(x) \geq -(g \cdot p)(x)$. By setting $t = (-f/g) \in A^*$ and noticing that $0_Y \in A$, we finish the proof of the first part.

For the second part, we notice that if $T(-\mathrm{int}(A)) = \emptyset$, we have $T(D) \cap -\mathrm{int}(A) = \emptyset$ and hence by the arguments similar to those given above we can prove the existence of $t \in A^*\backslash\{0_{Y^*}\}$ such that $t \circ T(x) \geq 0$ for every $x \in D$. ■

The above result and its proof are motivated by a similar observation made by Rigby [26] for single-valued maps.

Proof of Theorem 3.2. Notice that the equation

$$IT(Q, \bar{x}) \bigcap IT(F[\bar{y} - \mathrm{int}(K)]^-, \bar{x}) \bigcap_{i=1}^{m} IT(G_i[-C_i]^-, \bar{x}) \bigcap_{j=1}^{n} T(H_j[-D_j]^-, \bar{x}) = \emptyset,$$

in view of the inclusions

$$D_l F(\bar{x}, \bar{y})[-\mathrm{int}(K)]^- \subseteq IT(F[\bar{y} - \mathrm{int}(K)]^-, \bar{x}),$$

$$D_lG_i(\bar{x},\bar{z})[IT(-C_i,\bar{z}_i)]^- \subseteq IT(G[-C_i]^-,\bar{x}),$$

that follow from Lemma 3.5 and Lemma 3.6 (by choosing $A = C_i$ and $G = G_i$) ensure that

$$IT(Q,\bar{x})\bigcap D_lF(\bar{x},\bar{y})[-\mathrm{int}(K)]^- \bigcap_{i=1}^{m} D_lG_i(\bar{x},\bar{z}_i)[IT(-C_i,\bar{z}_i)]^- \bigcap_{j=1}^{n} T(H_j[-D_j]^-,\bar{x}) = \emptyset.$$

We define

$$\Phi := D_lF(\bar{x},\bar{y})[-\mathrm{int}(K)]^-,$$
$$\Psi_i := D_lG_i(\bar{x},\bar{z}_i)[IT(-C_i,\bar{z}_i)]^- \quad (i \in I).$$

We shall prove the theorem by analyzing the three possibilities, namely:

(i) $\Phi = \emptyset$;

(ii) $\Psi_i = \emptyset$ for some $i \in I$;

(iii) $\Phi \neq \emptyset$ and $\Psi_i \neq \emptyset$ for every $i \in I$.

We begin with the case (i). Since $\Phi = \emptyset$, we can apply Lemma 3.7 for

$$T := D_lF(\bar{x},\bar{y}) : X \rightrightarrows Y,\ D := X,\ A := K$$

such that there exists $t \in K^*\backslash\{0_{Y^*}\}$ with

$$t \circ D_lF(\bar{x},\bar{y})(x) \geq 0 \quad \text{for every } x \in X.$$

By choosing $s = 0_{X^*}$, $u_i = 0_{Z_i^*}$ for every $i \in I$, and $v_j = 0_{W_j^*}$ for every $j \in J$, we obtain the desired result.

For the case (ii), let there exist $i \in I$ such that $\Psi_i = \emptyset$. Then again by invoking Lemma 3.7 with

$$T := D_lG_i(\bar{x},\bar{z}_i) : X \rightrightarrows Z_i,\ D := X,\ A := T(C_i,-\bar{z}_i),\ (i \in I)$$

we obtain $u_i \in (T(C_i,-\bar{z}_i))^*\backslash\{0_{Z_i^*}\}$ such that

$$(u_i \circ DG_i(\bar{x},\bar{z}_i))(x) \geq 0 \quad \text{for every } x \in X.$$

By setting $s = 0_{X^*}$, $v_j = 0_{W^*}$ for every $j \in J$, and $u_j = 0_{Z_j^*}$, $i \neq j \in I$, we obtain (2). For $u_i(C_i + \bar{z}_i) \geq 0$, it suffices to notice that in view of the convexity of C_i, we have $T(C_i,-\bar{z}_i) \supseteq C_i + \bar{z}_i$ and hence $u_i(z + \bar{z}_i) \geq 0$ for every $z \in C_i$.

Finally, we consider the case (iii). Since $(\bar{x},\bar{y})$ is a local-minimizer of (P), it follows from Proposition 3.4 and the imposed conditions that we have

$$L \bigcap \Phi \bigcap_{i=1}^{m} \Psi_i \bigcap_{j=1}^{n} M_j = \emptyset.$$

Since L, Φ, Ψ_i, $(i \in I)$ and M_j, $(j \in J)$ are all nonempty, we can apply Theorem 2.6 to assure the existence of

$$\begin{aligned} \ell &\in L^*; \\ \ell_0 &\in (D_l F(\bar{x}, \bar{y})[-\mathrm{int}(K)]^-)^*; \\ \ell_i &\in (D_l G_i(\bar{x}, \bar{z}_i)[IT(-C_i, \bar{z}_i)]^-)^*, \quad i \in I; \\ \ell_{m+j} &\in M_j^*, \quad j \in J \end{aligned}$$

such that

$$-\ell - \ell_0 - \ell_1 - \ell_2 - \cdots - \ell_m - \ell_{m+1} - \cdots - \ell_{m+n} = 0. \tag{5}$$

Now, in the case $\Phi \neq \emptyset$ in view of Lemma 3.7 for

$$T := D_l F(\bar{x}, \bar{y}) : X \rightrightarrows Y, \; D := X, \; A := K$$

we get the existence of functionals $t \in K^*$ such that for all $x \in X$ and $-\ell_0 \in (D_l F(\bar{x}, \bar{y})[-Q]^-)^\diamond$ the following inequality holds

$$(t \circ D_l F(\bar{x}, \bar{y}))(x) \geq -\ell_0(x). \tag{6}$$

Analogously, in the case $\Psi \neq \emptyset$ with Lemma 3.7 for

$$T := D_l G_i(\bar{x}, \bar{y}) : X \rightrightarrows Z_i, \; D := X, \; A := T(C_i, -\bar{z}_i), \; (i \in I)$$

we get the existence of functionals $u_i \in (T(C_i, -\bar{z}_i))^*$ such that for all $x \in X$ and $-l_i \in (D_l G_i(\bar{x}, \bar{z}_i)[T(-C_i, \bar{z}_i)]^-)^\diamond$ the inequality

$$(u_i \circ D_l G_i(\bar{x}, \bar{z}_i))(x) \geq -\ell_i(x), \quad i \in I \tag{7}$$

holds.

Combining of the above inequalities (6) and (7) with (5) and setting $s = \ell$ and $v_j = \ell_{m+j}$ with $j \in J$ yield (2). This completes the proof. ■

Remark 3.8. It is of interest to obtain optimality conditions that involve derivatives of the set-valued map H_j with $j \in J$. In this regard, when H_j are suitable single-valued maps, then some variants of the well-known Lyusternik theorem can be used. We plan to address this issue in a forthcoming work.

4. Second-Order generalized Dubovitskii-Milyutin approach

We now discuss the second-order analogues of the results obtained in the previous section. We begin by recalling some second-order tangent cones.

Definition 4.1. Let Ξ be a normed space, let $\Omega \subset \Xi$ be nonempty and let $w \in \Xi$.

1. The second order asymptotic tangent cone $\widetilde{T}^2(\Omega, \bar{z}, w)$ of Ω at $\bar{z} \in \mathrm{cl}(\Omega)$ in the direction $w \in \Omega$ is the set of all $z \in \Omega$ such that there are a sequence $(z_n) \subset \Xi$ with $z_n \to z$ and a sequence $(s_n, t_n) \subset P \times P$ with $(s_n, t_n) \downarrow (0, 0)$ and $s_n/t_n \to 0$ so that $\bar{z} + s_n w + s_n t_n z_n \in \Omega$.
2. The second order asymptotic adjacent cone $\widetilde{K}^2(\Omega, \bar{z}, w)$ of Ω at $\bar{z} \in \mathrm{cl}(\Omega)$ in the direction $w \in \Omega$ is the set of all $z \in \Omega$ such that for every sequence $(s_n, t_n) \subset P \times P$ with $(s_n, t_n) \downarrow (0, 0)$ and $s_n/t_n \to 0$ there exists a sequence $(z_n) \subset \Xi$ with $z_n \to z$ and $\bar{z} + s_n w + s_n t_n z_n \in \Omega$.
3. The interior second order adjacent cone $\widetilde{IT}^2(\Omega, \bar{z}, w)$ of Ω at $\bar{z} \in \mathrm{cl}(\Omega)$ in the direction $w \in \Xi$ is the set of all $z \in \Xi$ such that for every sequence $(z_n) \subset \Xi$ with $z_n \to z$ and for every sequence $(s_n, t_n) \subset P \times P$ with $(s_n, t_n) \downarrow (0, 0)$ and $s_n/t_n \to 0$ we have $\bar{z} + s_n w + s_n t_n z_n \in \Omega$, for sufficiently large n.

Remark 4.2. We notice that $\widetilde{T}^2(S, \bar{z}, w)$ and $\widetilde{K}^2(S, \bar{z}, w)$ are closed cones, whereas $\widetilde{IT}^2(S, \bar{z}, w)$ is an open cone. We refer the reader to an interesting and timely survey by Giorgi, Jimenez, and Novo [11] that contains significant details of the asymptotic cones mentioned above (see also [25]). We also remark that second-order contingent sets and second-order adjacent sets (see [11]) are more commonly used objects in set-valued and nonsmooth optimization. However, their asymptotic analogues, being cones, are more suitable for our approach.

In the following definition, we recall the notion of the second-order asymptotic derivative, and being inspired by the lower Dini derivative and its usefulness in the preset approach, we also introduce the second-order lower Dini asymptotic derivative.

Definition 4.3. Let $F : X \rightrightarrows Y$ be set-valued, let $(\bar{x}, \bar{y}) \in \mathrm{graph}(F)$, and let $(\bar{u}, \bar{v}) \in X \times Y$.

(i) The second order asymptotic derivative of F at $(\bar{x}, \bar{y})$ in the direction $(\bar{u}, \bar{v})$ is the set-valued map $D^2F(\bar{x}, \bar{y}, \bar{u}, \bar{v}) : X \rightrightarrows Y$ defined by

$$D^2F(\bar{x}, \bar{y}, \bar{u}, \bar{v})(x) := \left\{ y \in Y \mid (x, y) \in \widetilde{T}^2(\mathrm{graph}(F), (\bar{x}, \bar{y}), (\bar{u}, \bar{v})) \right\}.$$

(ii) The second-order lower Dini asymptotic derivative of F at $(\bar{x}, \bar{y})$ in the direction $(\bar{u}, \bar{v})$ is the set-valued map $D_l^2F(\bar{x}, \bar{y}, \bar{u}, \bar{v}) : X \rightrightarrows Y$ such that $(x, y) \in \mathrm{graph}(D_l^2F(\bar{x}, \bar{y}, \bar{u}, \bar{v}))$ if and only if for every $(x_n) \subset X$ with $x_n \to x$ and for every sequence $(s_n, t_n) \in \mathbb{P} \times \mathbb{P}$ with $(s_n, t_n) \to 0$, and $s_n/t_n \to 0$, there exist a sequence $(y_n) \subset Y$ with $y_n \to y$ and an integer $m \in \mathbb{N}$ such that $\bar{y} + s_n \bar{v} + s_n t_n y_n \in F(\bar{x} + s_n \bar{u} + s_n t_n x_n)$ for every $n \geq m$.

We begin with the following necessary optimality condition for Problem (P):

Theorem 4.4. *Let* $(\bar{x},\bar{y}) \in \mathrm{graph}(F)$ *be a local weak minimizer of* (P). *Let* $(\bar{u},\bar{v}) \in \mathrm{graph}(D_l F(\bar{x},\bar{y}))$. *Then*

$$\widetilde{IT}^2(Q,\bar{x},\bar{u}) \bigcap D_l^2 F(\bar{x},\bar{y},\bar{u},\bar{v})[IT(-K,\bar{v})]^- \bigcap_{i=1}^{m} \widetilde{IT}^2(G_i[-C_i]^-,\bar{x},\bar{u}) \bigcap \widetilde{T}^2\Big(\bigcap_{j=1}^{n} H_j[-D_j]^-,\bar{x},\bar{u}\Big) = \emptyset. \tag{8}$$

Proof. Since $(\bar{x},\bar{y})$ is a local weak-minimizer of (P), there exists a neighborhood U of $\bar{x}$ such that

$$F(S\cap U)\cap(\{\bar{y}\}-\mathrm{int}(K))=\emptyset.$$

We will show that if (8) fails then the above criteria for the weak-minimality will be violated.

For the sake of argument, we assume that there exists an $x\in X$ such that

$$x \in \widetilde{IT}^2(Q,\bar{x},\bar{u}) \bigcap D_l^2 F(\bar{x},\bar{y},\bar{u},\bar{v})[IT(-K,\bar{v})]^- \bigcap_{i=1}^{m} \widetilde{IT}^2(G_i[-C_i]^-,\bar{x},\bar{u}) \bigcap \widetilde{T}^2\Big(\bigcap_{j=1}^{n} H_j[-D_j]^-,\bar{x},\bar{u}\Big).$$

In view of the containment $x\in \widetilde{T}^2(\bigcap_{j=1}^n H_i[-D_i]^-,\bar{x},\bar{u})$, and the definition of the second order asymptotic tangent cone, we ensure that there are a sequence $(x_n)\subset X$ with $x_n\to x$ and a sequence $(s_n,t_n)\subset \mathbb{P}\times\mathbb{P}$ with $(s_n,t_n)\downarrow(0,0)$ and $s_n/t_n\to 0$ so that for every $n\in\mathbb{N}$, we have

$$\bar{x}+s_n\bar{u}+s_nt_nx_n \in \bigcap_{j=1}^{n} H_j[-D_j]^-,$$

implying that

$$H_j(\bar{x}+s_n\bar{u}+s_nt_nx_n)\cap -D_j\neq\emptyset \quad \forall\, j\in J.$$

Furthermore, due to the containment $x\in\widetilde{IT}^2(G_i[C_i]^-,\bar{x},\bar{u})$, and the facts that $x_n\to x$, $(s_n,t_n)\downarrow(0,0)$ and $s_n/t_n\to 0$, we ensure the existence of $n_1\in\mathbb{N}$ such that

$$\bar{x}+s_n\bar{u}+s_nt_nx_n\in G_i[-C_i]^- \quad \text{for every } n\geq n_1,\ i\in I,$$

or equivalently

$$G_i(\bar{x}+s_n\bar{u}+s_nt_nx_n)\cap -C_i\neq\emptyset \quad \text{for every } n\geq n_1,\ i\in I.$$

Moreover, since $x\in D_l^2F(\bar{x},\bar{y},\bar{u},\bar{v})[IT(-K,\bar{v})]^-$, there exists $y\in IT(-K,\bar{v})$ such that

$$(x,y)\in \mathrm{graph}(D_l^2F(\bar{x},\bar{y},\bar{u},\bar{v})).$$

This, in view of the definition of the derivative $D_l^2 F(\bar{x}, \bar{y}, \bar{u}, \bar{v})$ and the sequences (x_n) and (s_n, t_n), ensure that there are a sequence $(y_n) \subset Y$ and an integer $n_2 \in \mathbb{N}$ such that for $n \geq n_2$

$$\bar{y} + s_n \bar{v} + s_n t_n y_n \in F(\bar{x} + s_n \bar{u} + s_n t_n x_n).$$

Since $y \in IT(-\mathrm{int}(K), \bar{v})$, $t_n \downarrow 0$ and $y_n \to y$, it follows from the definition of the interior tangent cones that there exists $n_3 \in \mathbb{N}$ such that for $n \geq n_3$, we have

$$\bar{v} + t_n y_n \in -\mathrm{int}(K).$$

Using the fact that $s_n > 0$, we obtain that

$$\bar{y} + s_n \bar{v} + s_n t_n y_n \in \bar{y} - \mathrm{int}(K).$$

Finally, from the containment $x \in \widetilde{IT}^2(Q, \bar{x}, \bar{u})$ and the definition of the sequences (x_n) and (s_n, t_n), we infer that there exists an integer $n_4 \in \mathbb{N}$ such that for every $n > n_4$, we have

$$\bar{x} + s_n \bar{u} + s_n t_n x_n \in U \cap Q.$$

Therefore, we have shown that for every $n \geq \max\{n_1, n_2, n_3, n_4\}$ there are

$$u_n := \bar{x} + s_n \bar{u} + s_n t_n x_n \in Q \cap U,$$

satisfying the constraints such that

$$F(u_n) \cap (\bar{y} - \mathrm{int}(K)) \neq \emptyset.$$

This however contradicts the weak optimality of $(\bar{x}, \bar{y})$. The proof is complete. ■

To obtain a variant of the above optimality condition that involves the derivative of the maps G_i for $i \in I$, as well, we need the following:

Theorem 4.5. *Let X and Z be normed spaces, let $G : X \to Z$ be a set-valued map, let $(\bar{x}, \bar{z}) \in \mathrm{graph}(G)$ and let $(\bar{u}, \bar{w}) \in \mathrm{graph}(D_l G(\bar{x}, \bar{z}))$. Let $A \subset Z$ with $\mathrm{int}(A) \neq \emptyset$. Then the following inclusion holds:*

$$D_l^2 G(\bar{x}, \bar{z}, \bar{u}, \bar{w})[\widetilde{IT}^2(-A, \bar{z}, \bar{w})]^- \subset \widetilde{IT}^2(G[-A]^-, \bar{x}, \bar{u}).$$

Proof. Let $x \in D_l^2 G(\bar{x}, \bar{z}, \bar{u}, \bar{w})[\widetilde{IT}^2(-A, \bar{z}, \bar{w})]^-$ be arbitrary. Choose arbitrary sequences $(x_n) \subset X$ with $x_n \to x$, and $(s_n, t_n) \subset P \times P$ with $(s_n, t_n) \downarrow (0, 0)$ and $s_n/t_n \to 0$. It suffices to show that there exists $m \in \mathbb{N}$ such that

$$\bar{x} + s_n \bar{u} + s_n t_n x_n \in G[-A]^-$$

for every $n \geq m$. Since $x \in D_l^2 G(\bar{x}, \bar{z}, \bar{u}, \bar{w})[\widetilde{IT}^2(-A, \bar{z}, \bar{w})]^-$, there exists

$$z \in D_l^2 G(\bar{x}, \bar{z}, \bar{u}, \bar{w})(x) \cap \widetilde{IT}^2(-A, \bar{z}, \bar{w}).$$

Therefore, there are a sequence $(z_n) \subset Z$ and an integer $n_1 \in \mathbb{N}$ such that $z_n \to z$ and

$$\bar{z} + s_n \bar{w} + s_n t_n z_n \in G(\bar{x} + s_n \bar{u} + s_n t_n x_n)$$

for every $n \geq n_1$. Because of the containment $z \in \widetilde{IT}^2(-A, \bar{z}, \bar{w})$ there exists $n_2 \in \mathbb{N}$ such that $\bar{z} + s_n \bar{w} + s_n t_n z_n \in -A$ for every $n \geq n_2$. Therefore we have

$$\bar{z} + s_n \bar{w} + s_n t_n z_n \in G(\bar{x} + s_n \bar{u} + s_n t_n x_n) \cap (-A)$$

for every $n \geq m := \max\{n_1, n_2\}$. Consequently $x \in \widetilde{IT}^2(G[-A]^-, \bar{x}, \bar{u})$. ■

Combining the above two results, we obtain the following optimality condition:

Theorem 4.6. *Let* $(\bar{x}, \bar{y}) \in \operatorname{graph}(F)$ *be a local weak minimizer of* (P). *Let* $(\bar{u}, \bar{v}) \in \operatorname{graph}(D_l F(\bar{x}, \bar{y}))$. *Let* $(\bar{x}, \bar{z}_i) \in \operatorname{graph}(G_i)$, *let* $\bar{u} \in \cap_{i=1}^m \operatorname{dom}(D_l G_i(\bar{x}, \bar{z}_i))$, *and let* $\bar{w}_i \in D_l G_i(\bar{x}, \bar{z}_i)(\bar{u})$. *Then*

$$\widetilde{IT}^2(Q, \bar{x}, \bar{u}) \bigcap D_l^2 F(\bar{x}, \bar{y}, \bar{u}, \bar{v})[IT(-K, \bar{v})]^- \bigcap_{i=1}^m D_l^2 G_i(\bar{x}, \bar{z}_i, \bar{u}, \bar{w}_i)[\widetilde{IT}^2(-C_i, \bar{z}, \bar{w}_i)]^- \bigcap \widetilde{T}^2(\bigcap_{j=1}^n H_j[-D_j]^-, \bar{x}, \bar{u}) = \emptyset.$$

Remark 4.7. Using the structure of the asymptotic derivatives, the above theorem can be combined with the techniques of the previous section to obtain second-order multiplier rules. A high-order generalization of the Lyusternik theorem can be used to include the derivatives of H_j for some specific cases (see [21]). We plan to address this issue in a forthcoming paper.

In view of the proof of Theorem 4.4, it is evident that the arguments used allow only one closed cone. Therefore, in the absence of the generalized multi-equality constraints, either a bigger cone $\widetilde{T}^2(Q, \bar{x}, \bar{u})$ or a larger second-order asymptotic derivative can be employed. Therefore, the following two variants are easy to prove.

Theorem 4.8. *Let* $(\bar{x}, \bar{y}) \in \operatorname{graph}(F)$ *be a local weak minimizer of* (P_1). *Let* $(\bar{u}, \bar{v}) \in \operatorname{graph}(D_l F(\bar{x}, \bar{y}))$. *Let* $(\bar{x}, \bar{z}_i) \in \operatorname{graph}(G_i)$, *let* $\bar{u} \in \cap_{i=1}^m \operatorname{dom}(D_l G_i(\bar{x}, \bar{z}_i))$, *and let* $\bar{w}_i \in D_l G_i(\bar{x}, \bar{z}_i)(\bar{u})$. *Then*

$$\widetilde{T}^2(Q, \bar{x}, \bar{u}) \bigcap D_l^2 F(\bar{x}, \bar{y}, \bar{u}, \bar{v})[IT(-K, \bar{v})]^- \bigcap_{i=1}^m D_l^2 G_i(\bar{x}, \bar{z}_i, \bar{u}, \bar{w}_i)[\widetilde{IT}^2(-C_i, \bar{z}, \bar{w}_i)]^- = \emptyset.$$

Theorem 4.9. *Let* $(\bar{x},\bar{y}) \in \mathrm{graph}(F)$ *be a local weak minimizer of* (P_1). *Let* $(\bar{u},\bar{v}) \in \mathrm{graph}(D_l F(\bar{x},\bar{y}))$. *Let* $(\bar{x},\bar{z}_i) \in \mathrm{graph}(G_i)$, *let* $\bar{u} \in \cap_{i=1}^m \mathrm{dom}(D_l G_i(\bar{x},\bar{z}_i))$, *and let* $\bar{w}_i \in D_l G_i(\bar{x},\bar{z}_i)(\bar{u})$. *Then*

$$\widetilde{IT}^2(Q,\bar{x},\bar{u}) \bigcap D^2F(\bar{x},\bar{y},\bar{u},\bar{v})[IT(-K,\bar{v})]^- \bigcap_{i=1}^m D_l^2 G_i(\bar{x},\bar{z}_i,\bar{u},\bar{w}_i)[\widetilde{IT}^2(-C_i,\bar{z},\bar{w}_i)]^- = \emptyset.$$

Another variant, a sort of compromise between the above two results is the following result in which F_Q represents the restriction of the map F on the set constraint Q.

Theorem 4.10. *Let* $(\bar{x},\bar{y}) \in \mathrm{graph}(F)$ *be a local weak minimizer of* (P_1). *Let* $(\bar{u},\bar{v}) \in \mathrm{graph}(D_l F(\bar{x},\bar{y}))$. *Let* $(\bar{x},\bar{z}_i) \in \mathrm{graph}(G_i)$, *let* $\bar{u} \in \cap_{i=1}^m \mathrm{dom}(D_l G_i(\bar{x},\bar{z}_i))$, *and let* $\bar{w}_i \in D_l G_i(\bar{x},\bar{z}_i)(\bar{u})$. *Then*

$$D^2F_Q(\bar{x},\bar{y},\bar{u},\bar{v})[IT(-K,\bar{v})]^- \bigcap_{i=1}^m D_l^2 G_i(\bar{x},\bar{z}_i,\bar{u},\bar{w}_i)[\widetilde{IT}^2(-C_i,\bar{z},\bar{w}_i)]^- = \emptyset.$$

The above result holds in terms of the second-order contingent derivatives or the second-order contingent epiderivatives (see [13]). Therefore, several known results either through second-order asymptotic derivatives or through second-order contingent derivatives or epiderivatives can be recovered as particular cases. In the following, we collect a few of such results.

The following result which is a generalization [7, Theorem 5.3] can be recovered from Theorem 4.10.

Corollary 4.11. *Consider problem* (P_0) *with* $Y = \mathbb{R}$ *and* $K = \mathbb{R}_+ := \{t \in \mathbb{R} \mid t \geq 0\}$. *Set* $\widetilde{IT}^2(Q,\bar{x},\bar{u}) = X$. *Let* $(\bar{x},\bar{y}) \in \mathrm{graph}(F)$ *be a local weak minimizer of* (P_0). *Then*

$$D_l F(\bar{x},\bar{y})(x) \subseteq \mathbb{R}_+ \quad \textit{for every } x \in S_0 := \mathrm{dom}(D_l F(\bar{x},\bar{y})).$$

Furthermore, for $\bar{u} \in S_0$ *with* $0 \in D_l F(\bar{x},\bar{y})(\bar{u})$, *we have*

$$D_l^2 F(\bar{x},\bar{y},\bar{u},0)(x) \subseteq \mathbb{R}_+ \quad \textit{for every } x \in \mathrm{dom}(D_l^2 F(\bar{x},\bar{y},\bar{u},0)).$$

Proof. In this particular case, we have

$$D_l^2 F(\bar{x},\bar{y},\bar{u},0)[IT(-K,0)]^- = \emptyset,$$

or

$$D_l^2 F(\bar{x},\bar{y},\bar{u},0)(x) \cap IT(-K,0) = \emptyset,$$

which gives the desired estimate involving the second-order derivative. The inclusion involving the first-order derivative then follows by taking $\bar{u} = 0$. ■

We recall that if $F : X \to Y$ is a single-valued map which is twice continuously Fréchet differentiable around $\bar{x} \in Q \subset X$ (notations: $F^{'}(\bar{x})$ and $F^{''}(\bar{x})$), then the second order contingent derivative of the restriction F_Q of F to Q at $\bar{x}$ in a direction $\bar{u}$ is given by the formula (see [2, p. 215]):

$$D_c^2 F_Q \Big(\bar{x}, F(\bar{x}), \bar{u}, F^{'}(\bar{x})(\bar{u})\Big)\,(x) = F^{'}(\bar{x})(x) + F^{''}(\bar{x})(\bar{u}, \bar{u}) \text{ whenever } x \in T^2(K, \bar{x}, \bar{u}).$$

It is empty when $x \notin T^2(Q, \bar{x}, \bar{u})$.

Therefore, for the case when the asymptotic derivative coincides with the second-order contingent derivative, the second-order asymptotic cone $\widetilde{T}^2(Q, \bar{x}, \bar{u})$ coincides with the second-order contingent set $T^2(Q, \bar{x}, \bar{u})$ and the map F is single-valued, we have the following conlusion from Theorem 4.8:

Corollary 4.12. *Consider problem* (P_0) *with* $Y = \mathbb{R}$ *and* $K = \mathbb{R}_+ := \{t \in \mathbb{R} |\ t \geq 0\}$. *Let* $F : X \to Y$ *be a single-valued map being twice continuously Fréchet differentiable around a point* $\bar{x} \in Q$. $(\bar{x}, F(\bar{x}))$ *is assumed to be a local weak minimizer of* (P_0). *Then*

$$F_Q^{'}(\bar{x})(x) \geq 0 \quad \textit{for every } x \in T(Q, \bar{x}).$$

Furthermore, for every $\bar{u} \in T(Q, \bar{x})$ *such that* $\bar{v} := F_K^{'}(\bar{x})(\bar{u})$ *we have*

$$F_Q^{'}(\bar{x})(x) + F_Q^{''}(\bar{x})(\bar{u}, \bar{u}) \geq -\bar{v} \quad \textit{for every } x \in T^2(Q, \bar{x}, \bar{u}).$$

Proof. In this particular case, we have

$$\widetilde{T}^2(Q, \bar{x}, \bar{u}) \bigcap D_l^2 F(\bar{x}, \bar{y}, \bar{u}, \bar{v})[IT(-K, \bar{v})]^- = \emptyset$$

or

$$D_l^2 F(\bar{x}, \bar{y}, \bar{u}, \bar{v})[IT(-K, \bar{v})]^- = \emptyset \quad \text{for every } x \in T^2(Q, \bar{x}, \bar{v}).$$

Therefore,

$$D_l^2 F(\bar{x}, \bar{y}, \bar{u}, \bar{v})(x) \cap IT(-K, \bar{v}) = \emptyset \quad \text{for every } x \in T^2(Q, \bar{x}, \bar{v}).$$

Under the assumptions, we have

$$D_l^2 F_Q \Big(\bar{x}, F(\bar{x}), \bar{u}, F^{'}(\bar{x})(\bar{u})\Big)(x) = F^{'}(\bar{x})(x) + F^{''}(\bar{x})(\bar{u}, \bar{u}) \text{ whenever } x \in T^2(K, \bar{x}, \bar{u}).$$

This immediately gives the desired estimate for the second-order derivative. The first-order derivative then follows by taking $(\bar{u}, \bar{v}) = (0, 0)$. ■

As a further specialization of Theorem 4.8, we recover the following necessary optimality condition in finite dimensional mathematical programming. This result is well comparable to the similar results obtained in [4].

Corollary 4.13. *Consider problem* (P_0) *with* $Y = \mathbb{R}^n$ *and* $K = \mathbb{R}^n_+ := \{x \in \mathbb{R}^n |\ \ x_i \geq 0$ *for every* $i \in \hat{I} := \{1, 2, \ldots, n\}\}$. *Let* $F : X \to Y$ *be a single-*

valued map being twice continuously Fréchet differentiable around a point $\bar{x} \in Q$. $(\bar{x}, F(\bar{x}))$ *is assumed to be a local weak minimizer of* (P_0). *For simplicity set* $F_Q = F$ *and define* $\hat{I}(x) := \{i \in \hat{I} \mid x_i = 0\}$. *Then*

$$F^{'}(\bar{x})(x) \notin -\mathrm{int}(\mathbb{R}^n_+) \quad \textit{for every } x \in T(Q, \bar{x}). \tag{9}$$

Furthermore, for every $\bar{u} \in T(K, \bar{x})$ *such that* $\bar{v} := F^{'}(\bar{x})(\bar{u}) \in (-\partial \mathbb{R}^n_+)$, *we have*

$$F^{'}(\bar{x})(x) + F^{''}(\bar{x})(\bar{u}, \bar{u}) \notin -\mathrm{int}(\mathbb{R}^n_+) - \{\bar{v}\} \quad \textit{for every } x \in T^2(Q, \bar{x}, \bar{u}). \tag{10}$$

Remark 4.14. Notice that (9) implies that there is no $x \in T(Q, \bar{x})$ with $F_i^{'}(\bar{x})(x) < 0$ for every $i \in \hat{I}$. Moreover, if for every $\bar{u} \in T(Q, \bar{x})$ such that $F_i(\bar{x})(\bar{u}) \leq 0$ for all $i \in \hat{I}$ and $\hat{I}(\bar{v}) \neq \emptyset$, we have the incompatibility of the system

(i) $x \in T^2(K, \bar{x}, \bar{u})$;
(ii) $F_i^{'}(\bar{x})(x) + F_i^{''}(\bar{x})(\bar{u}, \bar{u}) < 0 \quad \text{whenever} \quad i \in \hat{I}(\bar{v})$,

then this implies the condition (10).

The following corollary, under the same assumptions as above, also extends results given in [6], [5] and in [29] for special cases.

Corollary 4.15. *Let* $F : X \to \mathbb{R}$ *be a single-valued map being twice continuously Fréchet differentiable around a point* $\bar{x} \in Q$. $(\bar{x}, F(\bar{x}))$ *is assumed to be a local weak minimizer of* (P_0). *Then*

$$F_Q^{'}(\bar{x})(x) \geq 0 \quad \textit{for every } x \in T(Q, \bar{x}).$$

Furthermore, for every $\bar{u} \in T(Q, \bar{x})$ *such that* $F_Q^{'}(\bar{x})(\bar{u}) = 0$ *we have*

$$F_Q^{'}(\bar{x})(x) + F_Q^{''}(\bar{x})(\bar{u}, \bar{u}) \geq 0 \quad \textit{for every} \ \ x \in T^2(Q, \bar{x}, \bar{u}).$$

We conclude by giving the following simple illustrating example:

Example 4.16. Consider the following set-valued optimization problem

$$\min_{x \in \mathbb{R}} F(x)$$

with $F : \mathbb{R} \rightrightarrows \mathbb{R}$ and

$$F(x) := \{y \in \mathbb{R} \mid y \geq x^2\} \quad \text{for all } x \in \mathbb{R}.$$

Clearly, $(\bar{x}, \bar{y}) := (0, 0)$ is a local weak minimizer of this problem. For every $x \in \mathbb{R}$, for every $\bar{u} \in \mathbb{R}$, we determine the second-order asymptotic tangent cone

$$\tilde{T}^2(\mathrm{epi}(F), (0,0), (\bar{u}, 0)) = \{(x, y) \in \mathbb{R} \times \mathbb{R} : \ y \geq 0\}.$$

Therefore,

$$D^2F(0,0,\bar{u},0)(x) \cap -\mathrm{int}(R_+) = \emptyset \quad \text{for every } x \in \mathbb{R}$$

which is in compromise with the necessary optimality conditions given above.

5. Concluding Remarks

We have given new optimality conditions for the general set-valued optimization problems with multi-equality and multi-inequality constraints. The given conditions do not include the derivatives of the map defining the equality constraints and it is our future goal to deal with deficiency. This generalization seems necessary to compare our results with more complete results where Dubovitskii-Milyutin approach has been used to obtain first and second-order optimality conditions (see [22, 23]). The generalized Dubovitskii-Milyutin approach for the smooth data was introduced to study control problems with several equations corresponding to the equality constraints (see [19]). The solution maps in this setting were assumed to be smooth. Clearly, our results can be used to study similar control problems without assuming that the solution maps are smooth. However, we plan to give detailed applications of our results in a forthcoming paper.

Acknowledgements. We express our sincere gratitude to the referees for their very careful reading and suggestions that brought substantial improvements to our work.

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