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Extinction of Species in Discrete Models of Lotka-Volterra Type with Infinite Delay

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Abstract. A discrete model of nonautonomous competitive Lotka-Volterra type with infinite delay for d species is considered. It is shown that if coefficients satisfy some certain inequalities, then any solution with positive components at some point will have all of its last $d-1$ components tending to zero, while the first one will approach to a positive solution of a logistic equation.

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1. Introduction

Consider the following discrete competitive Lotka - Volterra model

$$u_i(n+1) = u_i(n)\exp[a_i(n) - \sum_{j=1}^{d} b_{ij}(n) \sum_{s=-\infty}^{n} H_{ij}(n-s)u_j(s)], \quad i = 1, ..., d, \quad (1)$$

where $u_i(n)$ is the density of population of species i at n^{th} time step (year, month, day), $a_i(n)$ represents the intrinsic growth rate of species i at n^{th} time step, and $b_{ij}(n)$ reflects the interspecific or intraspecific competitive intensity of species j to species i at n^{th} time step. It is assumed that $a_i(n)$ and $b_{ij}(n)$ $(i, j = 1,\ ...,\ d)$

are defined and bounded on $\mathbb{Z}$, $H_{ij}(n) \geqslant 0$, $\sum\limits_{n=0}^{+\infty} H_{ij}(n) = 1$. Many authors (see for e.g. [1, 7]) have argued that the discrete time models governed by difference equations are more appropriate than the continuous ones when the populations have nonoverlapping generations. Discrete time models can also provide efficient computational models of continuous models for numerical simulations (see [4] - [16]). System (1) is a counterpart of continuous Lotka - Volterra model. In [2] the authors considered the following continuous Lotka - Volterra model

$$v_i'(t) = v_i(t)[r_i(t) + p_i(t) - \sum_{j=1}^{d} q_{ij}(t)v_j(t)],\ i = 1,\ ...,\ d, \tag{2}$$

where $q_{ij}(t)$ is continuous and bounded above and below by positive constants, $r_i(t)$ is continuous, T-periodic and $\bar{r}_i = \frac{1}{T}\int\limits_0^T r_i(t)dt > 0$, $p_i(t)$ is continuous and $|p_i(t)| \leqslant \alpha_i e^{-\beta_i t}$, where α_i and β_i are positive constants. In [2], Ahmad has shown that if for each $i = 2,\ ...,\ d$, there exist numbers $\lambda_{i1},\ ...,\ \lambda_{ii-1} \geqslant 0$, $\lambda_{i1} + ... + \lambda_{ii-1} > 0$ such that

$$\frac{\lambda_{i1}\bar{r}_{i1} + ... + \lambda_{ii-1}\bar{r}_{i-1}}{\lambda_{i1}q_{1j}(t) + ... + \lambda_{ii-1}q_{i-1j}(t)} > \frac{\bar{r}_i}{q_{ij}(t)},\ j = 1,\ ...,\ i \tag{3}$$

for $t \geqslant t_0$, then $v_i(t) \to 0$ exponentially for $i = 2,\ ...,\ d$ and $v_1(t) - V^*(t) \to 0$ as $t \to +\infty$, where $(v_1(t),\ ...,\ v_d(t))$ is any solution of (2) with $v_i(t_0) > 0$, $(i = 1,\ ...,\ d)$ and $V^*(t)$ is the unique positive solution of the logistic equation $V'(t) = V(t)[r_1(t) - q_{11}(t)V(t)]$. In [7] Muroya extended this result to discrete models of pure-delay nonautonomous Lotka - Volterra type. The purpose of this paper is to extend the Ahmad's results in [2] to discrete system with infinite delay (1). The paper is organized as follows: In section 2 we study the discrete logistic equation and in section 3, we state and prove our main result on an extinction of species in discrete model which is expessed by system (1).

2. Discrete logistic equation

Consider the dynamics of the logistic equation on $[0, +\infty)$:

$$x(n+1) = x(n)\exp[a(n) - b(n)x(n)], \tag{4}$$

where $a,\ b : \mathbb{Z} \to \mathbb{R}$ are bounded and $b(n) \geqslant 0$. Let $\mathbb{Z}_+$ be the set of nonnegative intergers and $a_M = \sup\limits_{n\in\mathbb{Z}} a(n)$, $b_M = \sup\limits_{n\in\mathbb{Z}} b(n)$. Equation (4) is complete forward, i.e., any solution $x(n)$ of (4) corresponding to initial value $x(n_0) \geqslant 0$ is defined for all $n > n_0$. Moreover, the interval $(0, +\infty)$ is positive invariant with respect to (4), i.e, any solution $x(n)$ of (4) corresponding to positive initial value $x(n_0)$ remains positive for all $n > n_0$.

Lemma 2.1. *Assume that there exist positive intergers λ and ω such that*

$$\liminf_{p\to+\infty}\sum_{n=p}^{p+\lambda-1} a(n) > 0 \tag{5}$$

and

$$\liminf_{p\to+\infty}\sum_{n=p}^{p+\omega-1} b(n) > 0. \tag{6}$$

Then there exist positive constants m, M such that $m \leqslant \liminf_{n\to+\infty} x(n) \leqslant \limsup_{n\to+\infty} x(n) \leqslant M$ for any solution $x(n)$ of (4) with initial value $x(n_0) > 0$.

Proof. Let $x(n)$ be a solution of (4) with $x(n_0) > 0$. By (5) and (6), there exist $n_1 \geqslant n_0$, $m_1 > 0$, $M_1 > 0$ and $\delta > 0$ such that for all $p \geqslant n_1$ we have

$$\sum_{n=p}^{p+\lambda-1} [a(n) - b(n)m_1] > \delta \tag{7}$$

and

$$\sum_{n=p}^{p+\omega-1} [a(n) - b(n)M_1] < -\delta. \tag{8}$$

Claim 1. There exists $n_2 \geqslant n_1$ such that $x(n_2) < M_1$. Indeed, suppose on the contrary that $x(n) \geqslant M_1$ for all $n \geqslant n_1$. By (4) and (8), for any positive interger j we have $x(n_1 + \omega j) \leqslant x(n_1)\exp(-\delta j)$. Therefore $\lim_{j\to+\infty} x(n_1 + \omega j) = 0$. This contradiction proves our claim.

Let us put $M_2 = M_1 \exp(a_M)$, $M = M_2 \exp(a_M \omega)$, we have $M_1 < M_2 \leqslant M$.

Claim 2. If there exist $p \geqslant n_2$ and $\mu \in \mathbb{Z}_+$ such that $M_1 < x(p) \leq M_2$ and $x(p + i) > M_2$ for $i = 1, ..., \mu$, then $\mu < \omega$. To this end, in the way of contradiction, we assume that $\mu \geqslant \omega$. Then by (4) and (8) it follows that $M_2 < x(p + \omega) \leqslant x(p)\exp\left\{\sum_{n=p}^{p+\omega-1}[a(n) - b(n)M_1]\right\} \leqslant M_2 \exp(-\delta) < M_2$. This contradiction proves the claim.

Claim 3. $x(n) \leqslant M$ for all $n \geqslant n_2$. In deed, we assume that there exists $n_3 > n_2$ such that $x(n_3) > M$ and $x(n) \leqslant M$ for $n = n_2,\ n_2 + 1,\ ...,\ n_3 - 1$. Clearly that if $x(q) \leqslant M_1$ for some $q \geqslant n_2$, then $x(q + 1) \leqslant x(q)\exp(a_M) \leqslant M_1 \exp(a_M) = M_2$. Thus, there exitst $p \geqslant n_2$ such that $M_1 < x(p) \leqslant M_2$, $x(n) > M_2$ for $n = p+1,\ p+2,\ ...,\ n_3$. By Claim 2, $n_3 - p < \omega$. Thus we obtain the claim from the following contradiction: $M < x(n_3) \leqslant x(p)\exp[(n_3 - p)a_M] \leqslant M_2 \exp[(n_3 - p)a_M] < M_2 \exp(\omega a_M) = M$.

Claim 4. There exists $n_4 \geqslant n_2$ such that $x(n_4) > m_1$. To this end, in the way of contradiction, we assume that $x(n) \leqslant m_1$ for all $n \geqslant n_2$. By (4) and (7), for any positive integer j we have $x(n_2 + j\lambda) \geqslant x(n_2)\exp(j\delta)$. Thus $\lim_{j\to+\infty} x(n_2 + j\lambda) = +\infty$. This contradiction proves the claim.

Let us put $m_2 = m_1 \exp(-b_M M)$, $m = m_2 \exp(-b_M M\lambda)$, we have $m_1 > m_2 \geqslant m$.

Claim 5. If there exist $p \geqslant n_4$ and $\nu \in \mathbb{Z}_+$ such that $m_2 \leqslant x(p) < m_1$ and $x(p+i) < m_2$ for all $i = 1, \ ..., \nu$, then $\nu < \lambda$. To prove the claim, by the way of contradiction we assume that $\nu \geqslant \lambda$. Then by (4) and (7) we have $m_2 > x(p+\lambda) \geqslant \exp\{\sum_{n=p}^{p+\lambda-1}[a(n) - b(n)m_1]\} \geqslant m_2 \exp(\delta) > m_2$. This contradiction proves the claim.

Claim 6. $x(n) \geqslant m$ for all $n \geqslant n_4$. To prove the claim, we assume that there exists $n_5 > n_4$ such that $x(n_5) < m$ and $x(n) \geqslant m$ for $n = n_4, \ n_4+1, \ ..., \ n_5-1$. Clearly that if $x(q) \geqslant m_1$ for some $q > n_4$, then $x(q+1) \geqslant x(q)\exp[-b_M x(q)] \geqslant m_1 \exp[-b_M M] = m_2$. Thus there exists $p \geqslant n_4$ such that $m_1 > x(p) \geqslant m_2$ and $x(n) < m_2$ for $n = p+1, \ p+2, \ ..., \ n_5$. By Claim 5, $n_5 - p < \lambda$. Thus $m > x(n_5) \geqslant x(p)\exp[-b_M M(n_5-p)] > m_2 \exp(-\lambda b_M M) = m$. This contradiction proves the claim.

The Lemma follows from Claim 3 and Claim 6. ∎

Remark. A result similar to Lemma 2.1 is given in [16] when the coefficients of the equation are bounded above and below by positive constants.

Let $\mathcal{B}_+$ be the set $\{g : \mathbb{Z} \to \mathbb{R} | 0 < \inf_{n\in\mathbb{Z}} g(n) \leqslant \sup_{n\in\mathbb{Z}} g(n) < +\infty\}$.

Lemma 2.2. *Let $a(n)$ and $b(n)$ satisfy conditions* (5) *and* (6)*. If there exist positive intergers $\bar{\lambda}$ and $\bar{\omega}$ such that*

$$\liminf_{p\to-\infty} \sum_{n=p}^{p+\bar{\lambda}-1} a(n) > 0 \tag{9}$$

$$\textit{and} \quad \liminf_{p\to-\infty} \sum_{n=p}^{p+\bar{\omega}-1} b(n) > 0, \tag{10}$$

then equation (4) *has at least one solution $x^*(.) \in \mathcal{B}_+$.*

Proof. By (9) and (10) exist $n_1 \in \mathbb{Z}$, $\bar{m}_1 > 0$, $\bar{M}_1 > 0$ and $\bar{\bar{\delta}} > 0$ such that for all $p \leqslant n_1$ we have

$$\sum_{n=p}^{p+\bar{\lambda}-1} [a(n) - b(n)m_1] > \bar{\bar{\delta}} \ \text{ and } \sum_{n=p}^{p+\bar{\omega}-1} [a(n) - b(n)M_1] < -\bar{\bar{\delta}}. \tag{11}$$

Let us put

$$\bar{M}_2 = \bar{M}_1 \exp(a_M), \ \bar{M} = \bar{M}_2 \exp(a_M \bar{\omega}),$$

$$\bar{m}_2 = \bar{m}_1 \exp(-b_M \bar{M}), \ \bar{m} = \bar{m}_2 \exp(-b_M \bar{M}\bar{\lambda}),$$

then $\bar{m} \leqslant \bar{m}_2 < \bar{m}_1 < \bar{M}_1 < \bar{M}_2 \leqslant \bar{M}$. By the same argument as given in the proofs of Claims 2, 3, 4, 6 of Lemma 2.1, we can show that if $x(n_0) \in (\bar{m}_1, \bar{M}_1)$ for some $n_0 < n_1$, then $x(n) \in [\bar{m}, \bar{M}]$ for $n_0 \leqslant n \leqslant n_1$.

Let $\gamma \in (\bar{m}_1, \bar{M}_1)$. For each positive interger k such that $-k \leqslant n_1$, let $x^k(n)$ $(n \geqslant n_1)$ be the solution of (4) with the initial condition $x^k(-k) = \gamma$. By the same argument as given in the proofs of Claim 1 and Claim 2 of Lemma 2.1, we can show that $x^k(n) \in [\bar{m}, \bar{M}]$ for $-k \leqslant n \leqslant n_1$. Define a function $\bar{x}^k(t)$ on $(-\infty, n_1]$ by putting

$$\bar{x}^k(t) = \begin{cases} \bar{M}, \text{ if } t \leqslant -k, \\ [x^k(n+1) - x^k(n)](t-n) + x^k(n), \text{ if } t \in (n, n+1], \quad (-k < n \leqslant n_1). \end{cases}$$

It is easy to see that $\bar{x}^k(t) \in [\bar{m}, \bar{M}]$ for all $t \in (-\infty, n_1]$ and $\{\bar{x}^k(.)\}$ is equicontinuous on $(-\infty, n_1]$. By Ascoli's theorem (see [6]), there exists a subsequence $\{\bar{x}^{k_s}(.)\}$ of $\{\bar{x}^k(.)\}$ which converges to some function $\bar{x}^*(t)$, uniformly on any compact subset of $(-\infty, n_1]$. Put $\hat{x}^*(n) = \bar{x}^*(n)$ for $n \in \mathbb{Z} \cap (-\infty, n_1]$. Then $\hat{x}^*(n)$ is a solution of (4). Moreover, $\hat{x}^*(n) \in [\bar{m}, \bar{M}]$ for all $n \leqslant n_1$. Let $\tilde{x}^*(n)$ (for $n \geqslant n_1$) be the solution of (4) with $\tilde{x}^*(n_1) = \hat{x}^*(n_1)$. By Lemma 2.1, $0 < \inf\limits_{n \in [n_1, +\infty)} \tilde{x}^*(t) \leqslant \sup\limits_{n \in [n_1, +\infty)} \tilde{x}^*(t) < +\infty$. Let

$$x^*(n) = \begin{cases} \hat{x}^*(n), & \text{if } n \in \mathbb{Z} \cap (-\infty, n_1], \\ \tilde{x}^*(n), & \text{if } n \in \mathbb{Z} \cap (n_1, +\infty), \end{cases}$$

then $x^*(.) \in \mathcal{B}_+$ and $x^*(.)$ is a solution of (4). The lemma is proved. ∎

Lemma 2.3. *Let* (5)*,* (6) *and* (9) *hold. If*

$$\liminf_{n \to -\infty} b(n) > 0 \tag{12}$$

$$\text{and} \qquad \limsup_{n \to -\infty} \Big[a(n) + \ln \frac{b(n+1)}{b(n)} \Big] < 1 + \ln 2, \tag{13}$$

then equation (4) *has a unique solution* $x^*(.) \in \mathcal{B}_+$.

Proof. Clearly, condition (12) implies condition (10) for any positive interger $\bar{\omega}$. Thus the existence follows from Lemma 2.2. In order to show the uniqueness, we assume that $x_1^*(n)$ and $x_2^*(n)$ are two distinct solutions of (4) which are defined on $\mathbb{Z}$ and $x_1^*(n), x_2^*(n) \in [m', M']$ for all $n \in \mathbb{Z}$ $(0 < m' < M' < +\infty)$. By (12) there exists n_1 such that $b(n) > 0$ for all $n \leqslant n_1$. For $n \leqslant n_1$ and $i = 1,\ 2$, in view of (4), it follows that

$$b(n)x_i^*(n) = b(n)x_i^*(n-1)\exp[a(n-1) - b(n-1)x_i^*(n)] \leqslant \frac{b(n)}{b(n-1)} \exp[a(n-1) - 1],$$

where we used $\max\limits_{x \in \mathbb{R}} \{x \exp(r - hx)\} = \frac{\exp(r-1)}{h}$ for $h > 0$. Thus, by (12) and (13), for $0 < \mu < \liminf\limits_{n \to -\infty} b(n)$, there exist $\alpha \in (0, 2)$ and $n_2 \leqslant n_1$ such that

$$\inf_{n \leqslant n_2} b(n) \geq \mu \text{ and } 0 < \mu m' \leqslant b(n)x_i^*(n) \leqslant \alpha < 2 \text{ for } \ n \leqslant n_1,\ i = 1,\ 2. \tag{14}$$

Let $x_1^*(n_3) \neq x_2^*(n_3)$ for some $n_3 \in \mathbb{Z}$, then $x_1^*(n) \neq x_2^*(n)$ for all $n \leqslant n_3$. Put $n_4 = \min\{n_3, n_2\}$. By the mean value theorem of differential calculus, for each n there exists $\theta(n)$ lying between $x_1^*(n)$ and $x_2^*(n)$ such that

$$\begin{aligned} \ln x_1^*(n+1) - \ln x_2^*(n+1) &= [\ln x_1^*(n) - \ln x_2^*(n)] - b(n)[x_1^*(n) - x_2^*(n)] \\ &= (1 - b(n)\theta(n))[\ln x_1^*(n) - \ln x_2^*(n)]. \end{aligned} \tag{15}$$

Thus for each $l \in \mathbb{Z}_+$

$$\ln x_1^*(n_4) - \ln x_2^*(n_4) = \Big\{ \prod_{s=1}^{l} (1 - b(n_4 - s)\theta(n_4 - s)) \Big\} \frac{\ln x_1^*(n_4 - l)}{\ln x_2^*(n_4 - l)}. \tag{16}$$

It follows from (14) that there exists $\gamma \in (0, 1)$ such that

$$|1 - b(n_4 - s)\theta(n_4 - s)| < \gamma, \text{ for } s \geqslant 1. \tag{17}$$

It reduces from (16) and (17) that

$$|\ln x^*(n_4) - \ln x^{**}(n_4)| \leqslant \gamma^l |\ln M' - \ln m'| \quad \text{for } l \geqslant 1.$$

Thus $\ln x_1^*(n_4) = \ln x_2^*(n_4)$. This contracdiction implies the uniqueness. The lemma is proved. ■

Lemma 2.4. *Let $c : \mathbb{Z} \to \mathbb{R}$ be a function with $\lim\limits_{n\to+\infty} c(n) = 0$. Let (5) hold. If*

$$\liminf_{n\to+\infty} b(n) > 0 \tag{18}$$

$$\text{and} \qquad \limsup_{n\to+\infty} \Big[a(n) + \ln \frac{b(n+1)}{b(n)}\Big] < 1 + \ln 2, \tag{19}$$

then $\lim\limits_{n\to+\infty} |x(n) - y(n)| = 0$ for any two solutions $x(n)$ and $y(n)$ respectively of equation (4) and

$$y(n+1) = x(n) \exp[a(n) + c(n) - b(n)y(n)], \tag{20}$$

with initial values $x(n_0) > 0$ and $y(n_0) > 0$.

Proof. Clearly, condition (18) implies condition (6) for any positive interger ω. By Lemma 2.1, there exist $n_1 \geqslant n_0$, $\quad m, M \in (0, +\infty)$ $(m < M)$ such that

$$m \leqslant x(n) \leqslant M, \quad m \leqslant y(n) \leqslant M, \quad \text{for } n \geqslant n_1. \tag{21}$$

By (18) and (19), for $0 < \mu < \liminf\limits_{n\to+\infty} b(n)$, there exists $n_2 \geqslant n_1$ such that

$$\inf_{n \geqslant n_2} b(n) > \mu, \ \sup_{n \geqslant n_2} [a(n) + \ln \frac{b(n+1)}{b(n)}] < 1 + \ln 2,$$

$$\text{and} \qquad \sup_{n\geqslant n_2}[a(n)+c(n)+\ln\frac{b(n+1)}{b(n)}] < 1+\ln 2. \tag{22}$$

Since $x(n)$ and $y(n)$ respectively satisfy equations (4) and (20), for $n > n_2$ we have

$$x(n) = x(n-1)\exp[a(n-1)-b(n-1)x(n)] \leqslant \frac{\exp[a(n-1)-1]}{b(n-1)},$$

$$y(n) = y(n-1)\exp[a(n-1)+c(n-1)-b(n-1)y(n)] \leqslant \frac{\exp[a(n-1)+c(n-1)-1]}{b(n-1)},$$

where we used $\max\limits_{x\in\mathbb{R}}\{x\exp(r-hx)\} = \frac{\exp(r-1)}{h}$ for $h > 0$. Thus by (21) and (22), there exists $\alpha \in (0,2)$ such that

$$0 < \mu m \leqslant b(n)x(n) \leqslant \alpha < 2,\ 0 < \mu m \leqslant b(n)y(n) \leqslant \alpha < 2, \quad \text{for } n > n_2. \tag{23}$$

By the mean value theorem, there exists $\theta(n)$ lying between $x(n)$ and $y(n)]$ such that

$$\begin{aligned}\ln x(n+1)-\ln y(n+1) &= \ln x(n)-\ln y(n)-b(n)[x(n)-y(n)]-c(n)\\ &= (1-b(n)\theta(n))[\ln x(n)-\ln y(n)]-c(n) \text{ for } n \geqslant n_2.\end{aligned} \tag{24}$$

By (23) there exists $\gamma \in (0,1)$ such that

$$|1-b(n)\theta(n)| < \gamma, \quad \text{for } n > n_2. \tag{25}$$

(24) and (25) imply that

$$\begin{aligned}&|\ln x(n+1)-\ln y(n+1)| \leqslant \gamma|\ln x(n)-\ln y(n)|+|c(n)|\\ \leqslant&\gamma^{n+1-n_2}|\ln x(n_2)-\ln y(n_2)|+\sum_{k=n_2}^{n}|c(k)|\gamma^{n-k} \text{ for } n > n_2.\end{aligned} \tag{26}$$

Let $\varepsilon > 0$. Let $\delta > 0$ such that $\frac{\delta}{1-\gamma} < \frac{\varepsilon}{2}$. Since $\lim\limits_{n\to+\infty} c(n) = 0$ there exists $n_3 \geqslant n_2$ such that $|c(n)| < \delta$ for all $n \geqslant n_3$. Since $\gamma \in (0,1)$ there exists $n_4 \geqslant n_3$ such that

$$\sum_{k=n_2}^{n}|c(k)|\gamma^{n-k} = \sum_{k=n_2}^{n_3}|c(k)|\gamma^{n-k}+\sum_{k=n_3+1}^{n}|c(k)|\gamma^{n-k} \leqslant \frac{\varepsilon}{2}+\frac{\delta}{1-\gamma} \leqslant \varepsilon \ \text{ for } n \geqslant n_4.$$

Thus $\lim\limits_{n\to+\infty}\sum\limits_{k=n_2}^{n}|c(k)|\gamma^{n-k} = 0$, and then by (26), $\lim\limits_{n\to+\infty}|\ln x(n)-\ln y(n)| = 0$. Then by (21) we have $\lim\limits_{n\to+\infty}|x(n)-y(n)| = 0$. The lemma is proved. ■

Definition 2.5. (see [3])A sequence $z : \mathbb{Z} \to \mathbb{R}^d$ is said to be almost periodic if the ε-translation set of z:

$$E\{\varepsilon, z\} := \{\tau \in \mathbb{Z} : \|z(k+\tau) - z(k)\| < \varepsilon, \quad \text{for all} \quad k \in \mathbb{Z}\}$$

is a relatively dense set in $\mathbb{Z}$ for all $\varepsilon > 0$, that is, for any given $\varepsilon > 0$ there exists a positive integer $l(\varepsilon)$ such that each discrete interval of length $l(\varepsilon)$ contains an integer $\tau = \tau(\varepsilon) \in E\{\varepsilon, z\}$ such that $\|z(k+\tau) - z(k)\| < \varepsilon$ for all $k \in \mathbb{Z}$.

Definition 2.6. (see [3]) Let $f : \mathbb{Z} \times D \to \mathbb{R}^d$, where D is an open set in $\mathbb{R}^d$. The function $f(k, z)$ is said to be almost periodic in k uniformly for $z \in D$, or uniformly almost periodic for short, if for any $\varepsilon > 0$ and any compact set S in D, there exists a positive integer $l(\varepsilon, S)$ such that any interval of length $l(\varepsilon, S)$ contains an integer τ for which $\|f(k+\tau, z) - f(k, z)\| < \varepsilon$ for all $k \in \mathbb{Z}$ and $z \in S$.

In [15], Zhang considered the following almost periodic difference system

$$v(n+1) = f(n, v(n)), \tag{27}$$

where $f : \mathbb{Z} \times S_B \to \mathbb{R}^d$, $S_B = \{x \in \mathbb{R}^d : \|x\| < B\}$ and $f(n, v)$ is almost periodic in n uniformly for $v \in S_B$ and is continuous in v. Related to system (27), the author also considered the following product system

$$v(n+1) = f(n, v(n)), \quad w(n+1) = f(n, w(n)), \tag{28}$$

and obtained the following theorem:

Theorem 2.7. (see [15]) *Suppose that there exists a Liapunov function $V(n, v, w)$ which is defined for $n \in \mathbb{Z}_+$, $\|v\| < B$, $\|w\| < B$ and satisfies the following conditions:*

(i) *$g(\|v - w\|) \leqslant V(n, v, w) \leqslant h(\|v - w\|)$ for all $n \in \mathbb{Z}_+$, $\|v\| < B$, $\|w\| < B$, where g, $h : [0, +\infty) \to [0, +\infty)$ are continuous, increasing and $g(0) = h(0) = 0$;*

(ii) *$|V(n, v, w) - V(n, \bar{v}, \bar{w})| \leqslant L(\|v - \bar{v}\| + \|w - \bar{w}\|)$ for all $n \in \mathbb{Z}_+$, $\|v\| < B$, $\|w\| < B$, where L is a positive constant;*

(iii) *$\Delta V_{(28)}(n, v, w) \leqslant -\alpha V(n, v, w)$ for all $n \in \mathbb{Z}_+$, $\|v\| < B$, $\|w\| < B$, where α is a positive constant and $\Delta V_{(28)} = V(n+1, f(n, v), f(n, w)) - V(n, v, w)$.*

If there exists a solution $\hat{v}(n)$ of (27) such that $\|\hat{v}(n)\| \leqslant B^* < B$ for all $n \in \mathbb{Z}_+$, then there exists a unique uniformly asymptotically stable almost periodic solution $v^*(n)$ of system (27) satisfying $\|v^*(n)\| \leqslant B^*$ for all $n \in \mathbb{Z}$. In particular, if $f(n, v)$ is ω-periodic in n and continuous in v, then there exists a unique uniformly asymptotically stable ω-periodic solution $v^*(n)$ of system (27) with $\|v^*(n)\| \leqslant B^*$ for all $n = 1, ..., \omega$.

Applying Theorem 2.7 we can prove the following result on the existence of an almost periodic solution of equation (4).

Lemma 2.8. *Assume that $a(n)$ and $b(n)$ are almost periodic. Let (18) and (19) hold. If*

$$\lim_{N\to+\infty}\frac{1}{N}\sum_{n=0}^{N-1}a(n)>0, \tag{29}$$

then there exists a unique uniformly asymptotically stable almost periodic solution $x^*(.)\in\mathcal{B}_+$ of equation (4)*. In particular, if $a(n)$ and $b(n)$ are ω - periodic, then there exists a unique uniformly asymptotically stable ω - periodic positive solution of equation* (4).

Proof. Since $b(n)$ is almost periodic, (18) implies $\inf_{n\in\mathbb{Z}} b(n)=\liminf\limits_{n\to+\infty} b(n)>0$. Thus, by (29), there exist positive numbers $M_1,\ m_1\ (M_1>m_1),\ \delta$ and a positive interger λ such that

$$\sum_{n=p}^{p+\lambda-1}[a(n)-b(n)m_1]>\delta,\ \sum_{n=p}^{p+\lambda-1}[a(n)-b(n)M_1]<-\delta,\ \text{for all } p\in\mathbb{Z}. \tag{30}$$

Let us put

$$M_2=M_1\exp(a_M),\ M=M_2\exp(a_M\lambda),$$

$$m_2=m_1\exp(-b_M M),\ m=m_2\exp(-b_M M\lambda),$$

then $m\leqslant m_2<m_1<M_1<M_2\leqslant M$. Let $\hat{x}(n)$ be a solution of equation (4) with $\hat{x}(0)\in(m_1,M_1)$. By the same argument as given in the proofs of Claims 2, 3, 5 and 6 of Lemma 2.1, we get $\hat{x}(n)\in[m,M]$ for all $n\in\mathbb{Z}_+$. By the change of variables $y(n)=\ln x(n)$ equation (4) is transformed into

$$y(n+1)=y(n)+a(n)-b(n)\exp y(n). \tag{31}$$

Clearly, (31) has a bounded solution $\hat{y}(n)=\ln\hat{x}(n)\in[\ln m,\ln M]$ for all $n\in\mathbb{Z}_+$. Put $B=\max\{|\ln m|,|\ln M|\}$. Consider the product system of equation (31)

$$y(n{+}1)=y(n){+}a(n){-}b(n)\exp y(n),\ z(n{+}1)=z(n){+}a(n){-}b(n)\exp z(n). \tag{32}$$

Define a Liapunov function $V(n,y,z)$ on $\mathbb{Z}_+\times[-B,B]\times[-B,B]$ by $V(n,y,z)=|y-z|$. Clearly V satisfies conditions (i) and (ii) in Theorem 2.7. By the mean value theorem there exists $\theta(n)$ lying between $y(n)$ and $z(n)$ such that

$$\begin{aligned}\varDelta V_{(32)}(n)&=|[y(n)-z(n)]-b(n)[\exp y(n)-\exp z(n)]|-|y(n)-x(n)|\\&=(|1-b(n)\exp\theta(n)|-1)V(n),\ n\in\mathbb{Z}_+.\end{aligned} \tag{33}$$

Since $a(n),\ b(n)$ are almost periodic, it follows from (18) and (19) that

$$\mu:=\sup_{n\in\mathbb{Z}}\Big[a(n)-1+\ln\frac{b(n+1)}{b(n)}\Big]=\limsup_{n\to+\infty}\Big[a(n)-1+\ln\frac{b(n+1)}{b(n)}\Big]<\ln 2. \tag{34}$$

Put $\nu=\inf\limits_{n\in\mathbb{Z}} b(n),\ \gamma=\nu\exp(-B),\ \varepsilon=\exp\frac{[\mu+\ln 2]}{2}$. Clearly, $0<\varepsilon<2$. Since $y(n)$ satisfies equation (31) for $n\geqslant 1$ we have

$$\begin{aligned}\gamma &\leqslant b(n)\exp y(n) = b(n)\exp[y(n-1)+a(n-1)-b(n-1)\exp y(n-1)]\\ &\leqslant b(n)\max_{x\in\mathbb{R}}\exp[x+a(n-1)-b(n-1)\exp x]\\ &= \frac{b(n)}{b(n-1)}\exp[a(n-1)-1]\leqslant \varepsilon < 2.\end{aligned} \tag{35}$$

Similarly,

$$\gamma \leqslant b(n)\exp z(n) \leqslant \varepsilon, \quad \text{for } n \geqslant 1. \tag{36}$$

Since $\theta(n)$ lies between $y(n)$ and $z(n)$, it follows from (35) and (36) that

$$0 < \gamma \leqslant b(n)\exp\theta(n) \leqslant \varepsilon < 2 \quad \text{for } n \geqslant 1. \tag{37}$$

This implies that there exists $\alpha \in (0,1)$ such that $|1 - b(n)\exp\theta(n)| - 1 \leqslant -\alpha$ for $n \geqslant 1$. Thus it follows from (33) that $\Delta V_{(32)}V(n) \leqslant -\alpha V(n)$ for $n \geqslant 1$. By Theorem 2.7 there exists a unique uniformly asymptotically stable almost periodic solution $y^*(n)$ of equation (31) with $-B \leqslant y^*(n) \leqslant B$. By Lemma 2.3 equation (4) has a unique uniformly asymptotically stable almost periodic solution $x^*(.) = \exp y^*(.) \in \mathcal{B}_+$. Similarly, if $a(n)$ and $b(n)$ are ω - periodic, then there exists a unique uniformly asymptotically stable ω - periodic positive solution of equation (4). The lemma is valid. ∎

3. Extinction of species in discrete models of Lotka-Volterra type with infinite delay

In this section we consider the following Lotka-Volterra model

$$u_i(n+1) = u_i(n)\exp\Big[a_i(n) + c_i(n) - \sum_{j=1}^{d} b_{ij}(n)\sum_{s=-\infty}^{n} H_{ij}(n-s)h_j(s)u_j(s)\Big], \quad i = 1, \ \dots, \ d, \tag{38}$$

where $a_i, \ c_i : \mathbb{Z} \to \mathbb{R}$, $b_{ij} : \mathbb{Z} \to (0, +\infty)$ and $H_{ij} : \mathbb{Z}_+ \to [0, +\infty)$ are bounded, $h_j : \ \mathbb{Z} \to \mathbb{R}$ is bounded above and below by positive constants.

We assume that for each $i = 1, \ \dots, \ d$ there exist $\alpha_i > 0$ and $\beta_i > 0$ such that

$$|c_i(n)| \leqslant \alpha_i \exp[-\beta_i n] \quad \text{for all large } n. \tag{39}$$

In addition, we assume that for $i, j = 1, \ \dots, \ d$

$$a^*_{iL} = \liminf_{n\to+\infty} a_i(n) > 0, \ b^*_{iiL} = \liminf_{n\to+\infty} b_{ii}(n) > 0 \quad \text{for } n \in \mathbb{Z} \tag{40}$$

$$\text{and} \qquad \sum_{n=0}^{+\infty} H_{ij}(n) = 1, \ H_{ii}(0) > 0. \tag{41}$$

For $i, j = 1, \ \dots, \ d$ let us put

$$h_{jM} = \sup_{n\in\mathbb{Z}} h_j(n),\ h_{jL} = \inf_{n\in\mathbb{Z}} h_j(n),\ a^*_{iM} = \limsup_{n\to+\infty} a_i(n),\ b_{ijM} = \sup_{n\in\mathbb{Z}} b_{ij}(n). \tag{42}$$

Let $\mathbb{R}^d_+ = \{u = (u_1, ..., u_d) : u_i \geqslant 0,\ i = 1, ..., d\}$. Denote by $\mathrm{int}\mathbb{R}^d_+$ the interior of $\mathbb{R}^d_+$.

From the point of view of biology, in the sequel, we assume that

$$u_i(s) = \phi_i(s) \geqslant 0,\ \phi_i(0) > 0,\ i = 1,\ ...,\ d,\ s = ..., -n, -n+1, ..., -1, 0. \tag{43}$$

Clearly, problem (38) and (43) has a unique solution $(u_1(n), ..., u_d(n))$. Moreover, this solution is defined for all $n \geqslant 0$ and $u_i(n) > 0$ for all $n \geqslant 0$ and $i = 1,\ ...,\ d$.

Lemma 3.1. (see [4]) *Let $x : \mathbb{Z} \to \mathbb{R}$ be nonnegative and bounded, $H : \mathbb{Z}_+ \to \mathbb{R}$ be nonnegative such that $\sum_{n=0}^{+\infty} H(n) = 1$. Then*

$$\begin{aligned}\liminf_{n\to+\infty} x(n) &\leqslant \liminf_{n\to+\infty} \sum_{s=-\infty}^{n} H(n-s)x(s)\\ &\leqslant \limsup_{n\to+\infty} \sum_{s=-\infty}^{n} H(n-s)x(s)\\ &\leqslant \limsup_{n\to+\infty} x(n).\end{aligned}$$

Theorem 3.2. *Let* (39), (40) *and* (41) *hold. Then*

(i) *There exist positive constants $M_1,\ M_2,\ ...,\ M_d$ such that for any solution $u(n) = (u_1(n), ..., u_d(n))$ of* (38) *with the initial condition* (43) *there exists $n_1 \geqslant 0$ such that $u_i(n) \leqslant M_i$ for all $n \geqslant n_1$ and $i = 1,\ \ldots,\ d$.*

(ii) *There exists $\gamma > 0$ such that for any solution $u(n) = (u_1(n), ..., u_d(n))$ of* (38) *with the initial condition* (43) *there exists $n_1 \geqslant 0$ such that $\sum_{i=1}^{d} u_i(n) \geqslant \gamma$ for all $n \geqslant n_1$.*

Proof. Let $u(n) = (u_1(n), ..., u_d(n))$ be a solution of (38) with the initial condition (43). There exists $p \geqslant 0$ such that for $i = 1,\ \ldots,\ d$,

$$\sup_{n\geqslant p}[a_i(n) + c_i(n)] \leqslant 2a^*_{iM},\ \inf_{n\geqslant p} b_{ii}(n) \geqslant \frac{b^*_{iiL}}{2},\ \inf_{n\geqslant p}[a_i(n) + c_i(n)] \geqslant \frac{a^*_{iL}}{2}. \tag{44}$$

(i) For $n \geqslant p$ we have

$$u_i(n+1) \leqslant u_i(n) \exp\left[2a^*_{iM} - \frac{b^*_{iiL} H_{ii}(0) h_{iL} u_i(n)}{2}\right] \leqslant 2\frac{\exp[2a^*_{iM} - 1]}{b^*_{iiL} H_{ii}(0) h_{iL}},\ i = 1,\ ...,\ d.$$

Here we used $\max_{x\in\mathbb{R}}\{x \exp(r - hx)\} = \frac{\exp(r-1)}{h}$ for $h > 0$. Therefore $u_i(n) \leqslant M_i$ for all $i = 1,\ ...,\ d$ and $n \geqslant p+1$, where $M_i = 2\frac{\exp[2a^*_{iM} - 1]}{b^*_{iiL} H_{ii}(0) h_{iL}}$.

(ii) There exists $\varepsilon > 0$ such that

$$\frac{a_{iL}^*}{2} - \sum_{j=1}^{d} 2b_{ijM}h_{jM}\varepsilon > 0, \ i = 1, \ ..., \ d. \tag{45}$$

Claim 1.

$$\frac{a_{iiL}^*}{2} - 2b_{iiM}h_{iM}M_i < 0 \text{ for } i = 1, \ ..., \ d. \tag{46}$$

To prove the claim, we first consider the case of $a_{iM}^* \geqslant 1/2$. Since $0 < H_{ii}(0) \leqslant 1$,

$$\frac{a_{iiL}^*}{2} - 2b_{iiM}h_{iM}M_i \leqslant \frac{a_{iiL}^*}{2} - 4b_{iiM}h_{iM}\frac{\exp[2a_{iM}^* - 1]}{b_{iiL}^*h_{iL}} \leqslant \frac{a_{iiL}^*}{2} - 8\frac{b_{iiM}h_{iM}a_{iM}^*}{b_{iiL}^*h_{iL}} < 0.$$

If $a_{iM}^* < 1/2$ then, since $0 < H_{ii}(0) \leqslant 1$,

$$\begin{aligned}\frac{a_{iiL}^*}{2} - 2b_{iiM}h_{iM}M_i \leqslant \frac{a_{iiL}^*}{2} - 4b_{iiM}h_{iM}\frac{\exp[2a_{iM}^* - 1]}{b_{iiL}^*h_{iL}} &< \frac{a_{iiL}^*}{2} - \frac{4b_{iiM}h_{iM}}{eb_{iiL}^*h_{iL}} \\ &< \frac{a_{iiL}^*}{2} - 1 < 0.\end{aligned}$$

Thus the claim is proved.

Let us put

$$\gamma_i = \varepsilon \exp\Big[\frac{a_{iL}^*}{2} - \sum_{j=1}^{d} 2b_{ijM}h_{jM}M_j\Big], \ i = 1, \ \ldots, \ d. \tag{47}$$

By (46) it follows that $\gamma_i < \varepsilon$ for $i = 1, \ \ldots, \ d$. Put $\mathcal{A}_1 = \prod_{i=1}^{d}[0, \gamma_i]$, $\mathcal{A}_2 = \prod_{i=1}^{d}[0, \varepsilon] \setminus \mathcal{A}_1$ and $\mathcal{A}_3 = \prod_{i=1}^{d}[0, M_i] \setminus \prod_{i=1}^{d}[0, \varepsilon]$. Let us consider $n \geqslant p + 1$. Since $u_i(n) \leqslant M_i$ for all $i = 1, \ \ldots, \ d$ and $n \geqslant p+1$, it follows that $u(n) \in \mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{A}_3$ for all $n \geqslant p + 1$.

Claim 2. There exists $q \geqslant p + 1$ such that $u(q) \in \mathcal{A}_2 \cup \mathcal{A}_3$. To this end, in the way of contradiction, we assume that $u(n) \in \mathcal{A}_1$ for all $n \geqslant p+1$. Thus, by (38), (41) and (45), for all $n \geqslant p + 1$ we have

$$\frac{u_i(n+1)}{u_i(n)} \geqslant \exp\Big[\frac{a_{iL}^*}{2} - \sum_{j=1}^{d} 2b_{ijM}h_{jM}\varepsilon\Big] > 1.$$

This implies that $u(n)$ is unbounded, which is impossible and thus the claim is proved.

Claim 3. If $u(n) \in \mathcal{A}_2$ for some $n \geqslant p+1$, then $u(n+1) \in \mathcal{A}_2 \cup \mathcal{A}_3$. To this end, we know that $u_i(n) \leqslant \varepsilon$ for all $i = 1, \ \ldots, \ d$ and there exists $i_0 \in \{1, \ \ldots, \ d\}$ such that $u_{i_0}(n) > \gamma_{i_0}$. Thus, by (38), (41) and (45) we have

$$u_{i_0}(n+1) \geqslant u_{i_0}(n) \exp\Big[\frac{a^*_{i_0L}}{2} - \sum_{j=1}^{d} 2b_{i_0jM}h_{jM}\varepsilon\Big] > u_{i_0}(n) > \gamma_{i_0},$$

hence $u(n+1) \in \mathcal{A}_2 \cup \mathcal{A}_3$, since $u_i(n+1) \leqslant M_i$ for all $i = 1, \ldots, d$. The claim is proved.

Claim 4. If $u(n) \in \mathcal{A}_3$ for some $n \geqslant p+1$, then $u(n+1) \in \mathcal{A}_2 \cup \mathcal{A}_3$. To this end, we can see that $u_i(n) \leqslant M_i$ for $i = 1, \ldots, d$ and there exists $i_0 \in \{1, \ldots, d\}$ such that $u_{i_0}(n) > \varepsilon$. Therefore, by (38), (41)and (47) we have

$$u_{i_0}(n+1) > \varepsilon \exp\Big[\frac{a^*_{i_0L}}{2} - \sum_{j=1}^{d} 2b_{i_0jM}h_{jM}M_j\Big] = \gamma_{i_0},$$

hence $u(n+1) \in \mathcal{A}_2 \cup \mathcal{A}_3$, since $u_i(n+1) \leqslant M_i$ for all $i = 1, \ldots, d$. The claim is proved.

By Claims 2, 3 and 4, it follows that there exists $q \geqslant n_0$ such that $u(n) \in \mathcal{A}_2 \cup \mathcal{A}_3$ for all $n \geqslant q$.

The theorem is proved. ■

Theorem 3.3. *Let* (39), (40) *and* (41) *hold. If for each* $i = 2, \ldots, d$, *there exist* $\lambda_{i1}, \ldots, \lambda_{ii-1} \geqslant 0$, $\sum_{j=1}^{i-1} \lambda_{ij} > 0$, $n_1 \in \mathbb{Z}_+$ *and* $\delta > 0$ *such that for all* $s \leqslant n$ *and* $n \geqslant n_1$

$$b_{ij}(n)H_{ij}(n-s)\sum_{k=1}^{i-1} \lambda_{ik}a^*_{kL} \geqslant a^*_{iM}\sum_{k=1}^{i-1} \lambda_{ik}b_{kj}(n)H_{kj}(n-s) + \delta\xi_j(n-s),\ j = 1, ..., i, \tag{48}$$

where $\xi_{ij}(n) = \max_{k=1,\ldots,i}\{H_{kj}(n)\}$, *then* $u_i(n)$ *tends to zero exponentially for* $i = 2, \ldots, d$ *as* $n \to +\infty$, *where* $(u_1(n), \ldots, u_d(n))$ *is any solution of* (38) *with the initial condition* (43). *If, in addition,* $H_{11}(0) = 1$ *and*

$$\limsup_{n\to+\infty}\Big[a_1(n) + \ln\frac{b_{11}(n+1)h_1(n+1)}{b_{11}(n)h_1(n)}\Big] < 1 + \ln 2, \tag{49}$$

then $\lim_{n\to+\infty}[u_1(n) - U^*(n)] = 0$, *where* $U^*(.) \in \mathcal{B}_+$ *is the unique solution of the equation*

$$U(n+1) = U(n)\exp[a_1(n) - b_{11}(n)h_1(n)U(n)]. \tag{50}$$

Proof. By (48), we can choose positive numbers $\hat{a}_{iL} < a^*_{iL}$ and $\hat{a}_{iM} > a^*_{iM}$ ($i = 1, \ldots, d$) such that for $n \geqslant n_1$, $s \leqslant n$ and $i = 2, \ldots, d$

$$b_{ij}(n)H_{ij}(n-s)\sum_{k=1}^{i-1} \lambda_{ik}\hat{a}_{kL} \geqslant \hat{a}_{iM}\sum_{k=1}^{i-1} \lambda_{ik}b_{kj}(n)H_{kj}(n-s) + \frac{\delta\xi_j(n-s)}{2}, j = 1, \ldots, i. \tag{51}$$

Since $\lim_{n\to+\infty} c_i(n) = 0$, there exists $n_2 \geqslant n_1$ such that

$$\inf_{n\geqslant n_2}[a_i(n) + c_i(n)] \geqslant \hat{a}_{iL},\ \sup_{n\geqslant n_2}[a_i(n) + c_i(n)] \leqslant \hat{a}_{iM},\ i = 1,\ \ldots,\ d.$$

(i) First, we prove that $u_d(n) \to 0$ exponentially as $n \to +\infty$. Let us put

$$a_d^* = \sum_{i=1}^{d-1} \lambda_{di}\hat{a}_{iL},\ b_{dj}^*(n, s) = \sum_{i=1}^{d-1} \lambda_{di} b_{ij}(n) H_{ij}(n - s),\ j = 1,\ \ldots,\ d. \tag{52}$$

For $i = d$, condition (51) can now be written as

$$a_d^* b_{ij}(n) H_{ij}(n - s) - \hat{a}_{dM} b_{dj}^*(n, s) > \frac{\delta}{2}\xi_j(n - s),\ j = 1,\ \ldots,\ d,\ n \geqslant n_2,\ s \leqslant n. \tag{53}$$

System (38) can be written for $i = 1,\ \ldots,\ d$, as

$$\ln u_i(n + 1) - \ln u_i(n) = a_i(n) + c_i(n) - \sum_{j=1}^{d} b_{ij}(n) \sum_{s=-\infty}^{n} H_{ij}(n - s) h_j(s) u_j(s). \tag{54_i}$$

Multiplying (54_i) by λ_{di} for $i = 1,\ \ldots, d - 1$ and summing over $1 \leqslant i \leqslant d - 1$, we obtain

$$\sum_{i=1}^{d-1}[\ln u_i^{\lambda_{di}}(n+1) - \ln u_i^{\lambda_{di}}(n)] = \sum_{i=1}^{d-1} \lambda_{di}[a_i(n)+c_i(n)] - \sum_{j=1}^{d} \sum_{s=-\infty}^{n} b_{dj}^*(n, s) h_j(s) u_j(s). \tag{55}$$

For $n \geqslant n_2$, put $A(n) = \ln \dfrac{u_d^{a_d^*}(n)}{\prod_{i=1}^{d-1} u_i^{\lambda_{di}\hat{a}_{dM}}(n)}$. Multiplying ($54_d$) by a_d^* and (55) by $\hat{a}_{dM}$, and subtracting them, we obtain

$$\begin{aligned} A(n + 1) - A(n) = {} & [a_d^* a_d(n) - \hat{a}_{dM} \sum_{i=1}^{d-1} \lambda_{di} a_i(n)] + [a_d^* c_d(n) - \hat{a}_{dM} \sum_{i=1}^{d-1} \lambda_{di} c_i(n)] \\ & - \sum_{j=1}^{d} \sum_{s=-\infty}^{n} [a_d^* b_{dj}(n) H_{dj}(n - s) - \hat{a}_{dM} b_{dj}^*(n, s)] h_j(s) u_j(s) \text{ for all } n \geqslant n_2. \end{aligned} \tag{56}$$

For $n \geqslant n_2$ we have

$$a_d^* a_d(n) - \hat{a}_{dM} \sum_{i=1}^{d-1} \lambda_{di} a_i(n) \leqslant \sum_{i=1}^{d-1} \lambda_{di}[\hat{a}_{iL} a_d(n) - a_i(n)\hat{a}_{dM}] \leqslant 0. \tag{57}$$

By (39) there exist $n_3 > n_2$, $\mu > 0$ and $\nu > 0$ such that

$$\left|a_d^* c_d(n) - \hat{a}_{dM} \sum_{i=1}^{d-1} \lambda_{di} c_i(n)\right| \leqslant \mu \exp[-\nu n] \quad \text{for all } n \geqslant n_3. \tag{58}$$

Let $L = \min\{h_{1L},\ \ldots,\ h_{dM}\}$ and $\xi(0) = \min\{\xi_1(0),\ \ldots, \xi_d(0)\}$. Clearly, $L > 0$ and $\xi(0) > 0$. From (53), (56), (57) and (58) we obtain

$$A(n+1) - A(n) \leqslant -\frac{L\delta\xi(0)}{2} \sum_{j=1}^{d} u_j(n) + \mu \exp[-\nu n] \quad \text{for all } n \geqslant n_3. \tag{59}$$

By the part (ii) of Theorem 3.2, there exist $n_4 \geqslant n_3$ and $\gamma > 0$ such that $\sum\limits_{j=1}^{d} u_j(n) \geqslant \gamma$ for all $n \geqslant n_4$. Thus (59) implies that

$$A(n+1) - A(n) \leqslant -\frac{L\delta\xi(0)}{2}\gamma + \mu \exp[-\nu n] \text{ for all } n \geqslant n_4. \tag{60}$$

Therefore, for $n \geqslant n_4$ we have

$$A(n) \leqslant -(n - n_4)\frac{L\delta\xi(0)}{2}\gamma + \mu \sum_{k=n_4}^{n-1} \exp[-\nu k] + A(n_4),$$

and thus, for $n > n_4$

$$\begin{aligned} u_d^{a_d^*}(n) \leqslant & \frac{u_d^{a_d^*}(n_4) \prod\limits_{i=1}^{d-1} u_i^{\lambda_{di}\hat{a}_{dM}}(n)}{\prod\limits_{i=1}^{d-1} u_i^{\lambda_{di}\hat{a}_{dM}}(n_4)} \left\{\exp\left[\mu \sum_{k=n_4}^{n-1} \exp(-\nu k)\right]\right\} \\ & \times \exp\left[-\frac{L\delta\xi(0)}{2}\gamma(n-n_4)\right]. \end{aligned} \tag{61}$$

By the part (i) of Theorem 3.2, there exist $n_5 \geqslant n_4$ and $M_1,\ \ldots,\ M_d > 0$ such that $u_i(n) \leqslant M_i$ $(i = 1,\ \ldots,\ d)$ for all $n \geqslant n_5$. Thus (61) implies that for $n > n_5$

$$\begin{aligned} u_d(n) \leqslant & \left\{\frac{u_d^{a_d^*}(n_4) \prod\limits_{i=1}^{d-1} M_i^{\lambda_{di}\hat{a}_{dM}}}{\prod\limits_{i=1}^{d-1} u_i^{\lambda_{di}\hat{a}_{dM}}(n_4)}\right\}^{\frac{1}{a_d^*}} \left\{\exp\left[\frac{\mu \exp[-\nu n_4]}{a_d^*(1 - \exp[-\nu])}\right]\right\} \\ & \times \exp\left[\frac{-\frac{L\delta\xi(0)}{2}\gamma}{a_d^*}(n-n_4)\right]. \end{aligned}$$

Since $\xi(0) > 0$, $u_d(n) \to 0$ exponentially as $n \to +\infty$.

(ii) Next, we will show that $u_i(n) \to 0$ exponentially as $n \to +\infty$ for $i = 2,\ \ldots,\ d-1$. To this end, we rewrite the system (38) for $u_i, 1 \leqslant i \leqslant d-1$ as

$$u_i(n{+}1) = u_i(n)\exp[a_i(n){+}c_i^*(n){-}\sum_{j=1}^{d-1} b_{ij}(n)\sum_{s=-\infty} H_{ij}(n{-}s)h_j(s)u_j(s)], \quad (62)$$

where $c_i^*(n) = c_i(n) - b_{id}(n)\sum\limits_{s=-\infty}^{n} H_{id}(n-s)h_d(s)u_d(s)$. By Lemma 3.1,

$$\begin{aligned}\limsup_{n\to+\infty} |b_{id}(n)\sum_{s=-\infty}^{n} H_{id}(n-s)h_d(s)u_d(s)| &\leqslant \limsup_{n\to+\infty} b_{idM}h_{dM}\sum_{s=-\infty}^{n} H_{id}(n-s)u_d(s)\\ &\leqslant \limsup_{n\to+\infty} b_{idM}h_{dM}u_d(n).\end{aligned}$$

Thus it follows that for each $i = 1, \ldots, d-1$, $c_i^*(n)$ satisfies the hypothesis (39), since $c_i(n)$ and $u_d(n)$ both tend to zero exponentially as $n \to +\infty$. We note that the inequalities in (48) are independent of d^{th} in the sense that by dropping the d^{th} case, the coefficients of the smaller system (62) still satisfy inequalities (48). Thus, applying the same argument as above, we obtain that $u_{d-1}(n) \to 0$ exponentially as $n \to +\infty$. By induction we get $u_i(n) \to 0$ exponentially as $n \to +\infty$ for $i = 2, \ldots, d-2$.

(iii) We now show that $\lim\limits_{n\to+\infty}[u_1(n) - U^*(n)] = 0$, where $U^*(\cdot) \in \mathcal{B}_+$. To this end, we know that

$$u_1(n+1) = u_1(n)\exp[a_1(n) + \tilde{c}_1(n) - b_{11}(n)h_1(n)u_1(n)],$$

where $\tilde{c}_1(n) = c_1(n) - \sum\limits_{j=2}^{d} b_{1j}(n)\sum\limits_{s=-\infty}^{n} H_{1j}(n-s)h_j(s)u_j(s)$. Since $c_1(n)$ and $u_j(n) \to 0$ exponentially as $n \to +\infty$ for $j = 2, \ldots, d$, Lemma 3.1 implies that $\lim\limits_{n\to+\infty} \tilde{c}_1(n) \to 0$. By Lemma 2.4, $\lim\limits_{n\to+\infty}[u_1(n) - U^*(n)] = 0$. The theorem is proved. ■

Example 3.4. Consider the system

$$\begin{aligned}u_1(n+1) &= u_1(n)\exp\Big[a_1(n) - b_{11}(n)\sum_{s=-\infty}^{n} H_{11}(n-s)u_1(s) - b_{12}(n)\\ &\qquad \times \sum_{s=-\infty}^{n} H_{12}(n-s)u_2(s)\Big],\\ u_2(n+1) &= u_2(n)\exp\Big[a_2(n) - b_{21}(n)\sum_{s=-\infty}^{n} H_{21}(n-s)u_1(s) - b_{22}(n)\\ &\qquad \times \sum_{s=-\infty}^{n} H_{22}(n-s)u_2(s)\Big],\end{aligned}$$

where $a_1(n) = \dfrac{|n|+1}{|n|+2}$, $a_2(n) = \dfrac{n^2+1}{n^2+2}$, $b_{11}(n) = \dfrac{\sqrt{n^1+1}}{\sqrt{n^2+1}+1}$, $b_{12}(n) =$

$\dfrac{2n^2+1}{2n^2+2}$, $b_{21}(n) = \dfrac{4\sqrt{2n^2+1}}{\sqrt{2n^2+1}+1}$, $b_{22}(n) = 4\dfrac{n^4+1}{n^4+2}$, $H_{11}(n) = \dfrac{2}{3^{n+1}}$, $H_{21}(n) = \dfrac{1}{2^{n+1}}$, $H_{12}(n) = \dfrac{4}{5^{n+1}}$, $H_{22}(n) = \dfrac{3}{4^{n+1}}$. We have $a^*_{iL} = a^*_{iM} = 1$ $(i = 1, 2)$. By letting $\lambda_{21} = 1$, $\delta = 1/3$, it is easy to see that condition (48) in Theorem 3.3 holds. Thus species u_2 in the system is extinct.

Remark. In [7] Muroya considered discrete models of nonautonomous Lotka - Volterra type with finite delays. Theorem 3.3 is an extention of Muroya's result in [7] to discrete Lotka - Volterra models with infinite delay.

Theorem 3.5. *Assume that for each* $i = 1, \ldots, d$, $a_i(n)$ *is almost periodic with* $\bar{a}_i = \lim_{\omega\to+\infty} \frac{1}{\omega}\sum_{n=0}^{\omega-1} a_i(n) > 0$ *and* $c_i(n)$ *satisfies* (39), *and* $\liminf_{n\to+\infty} b_{ij}(n) > 0$ *for* $i, j = 1, \ldots, d$. *If for each* $i = 2, \ldots, d$, *there exist* $\lambda_{i1}, \ldots, \lambda_{ii-1} \geqslant 0$, $\lambda_{i1} + \cdots + \lambda_{ii-1} > 0$, $n_1 \in \mathbb{Z}_+$ *and* $\delta > 0$ *such that for all* $s \leqslant n$ *and* $n \geqslant n_1$

$$b_{ij}(n)H_{ij}(n-s)\sum_{k=1}^{i-1}\lambda_{ik}\bar{a}_k \geqslant \bar{a}_i\sum_{k=1}^{i-1}\lambda_{ik}b_{kj}(n)H_{kj}(n-s)+\delta\xi_j(n-s),\ j = 1,\ldots,i, \tag{63}$$

where $\xi_{ij}(n) = \max_{k=1,\ldots,i}\{H_{kj}(n)\}$, *then* $u_i(n) \to 0$ *exponentially for* $i = 2, \ldots, d$ *as* $n \to +\infty$, *where* $(u_1(n), \ldots, u_d(n))$ *is any solution of* (38) *with the initial condition* (43). *If, in addition,* $H_{11}(0) = 1$ *and* (49) *is satisfied, then* $\lim_{n\to+\infty}[u_1(n) - U^*(n)] = 0$ *where* $U^*(.)$ *is the unique solution of equation* (50) *in* $\mathcal{B}_+$.

Proof. There exists $\varepsilon > 0$ such that for $i = 2, \ldots, d$, $j = 1, \ldots, i$, $n \geqslant n_1$, $s \leqslant n$ we have

$$b_{ij}(n)H_{ij}(n-s)\sum_{k=1}^{i-1}\lambda_{ik}(\bar{a}_k-2\varepsilon) \geqslant (\bar{a}_i+2\varepsilon)\sum_{k=1}^{i-1}\lambda_{ik}b_{kj}(n)H_{kj}(n-s)+\frac{\delta\xi_j(n-s)}{2}. \tag{64}$$

Since $a_i(n)$ is almost periodic, for $i = 1, \ldots, d$ there exists a trigonometric polynomial $\mathit{\Delta}_i(n)$ such that $\sup_{n\in\mathbb{Z}}|a_i(n) - \mathit{\Delta}_i(n)| \leqslant \varepsilon$. Then $\sum_{k=0}^{n-1}[\mathit{\Delta}_i(k) - \bar{\mathit{\Delta}}_i]$ is bounded and $|\bar{a}_i - \bar{\mathit{\Delta}}_i| \leqslant \varepsilon$, where $\bar{\mathit{\Delta}}_i = \lim_{\omega\to+\infty}\frac{1}{\omega}\sum_{n=0}^{\omega-1}\mathit{\Delta}_i(n)$. By the change of variables

$$u_i(n) = v_i(n)\exp\Big\{\sum_{k=0}^{n-1}[\mathit{\Delta}_i(k) - \bar{\mathit{\Delta}}_i]\Big\},\quad i = 1, \ldots, d,$$

(38) leads to the following system $(i = 1, \ldots\ d)$:

$$v_i(n+1) = v_i(n)\exp\Big[\tilde{a}_i(n)+c_i(n)-\sum_{j=1}^{d}b_{ij}(n)\sum_{s=-\infty}^{n}H_{ij}(n-s)\tilde{h}_j(s)v_j(s)\Big], \tag{65}$$

where $\tilde{a}_i(n) = a_i(n) - \Delta_i(n) + \bar{\Delta}_i$, $\tilde{h}_j(s) = h_j(s)\exp\Big\{\sum_{k=0}^{s-1}[\Delta_i(k) - \bar{\Delta}_i]\Big\}$. Since $\sum_{k=0}^{s-1}[\Delta_j(k) - \bar{\Delta}_j]$ is bounded, it follows that $\tilde{h}_j(.) \in \mathcal{B}_+$. It is easy to see that $\tilde{a}^*_{iL} = \liminf_{n\to+\infty} a_i(n) \geqslant \bar{a}_i - 2\varepsilon$, $\tilde{a}^*_{iM} = \limsup_{n\to+\infty} \tilde{a}_i(n) \leqslant \bar{a}_i + 2\varepsilon$ for $i = 1, \ldots, d$. Thus (64) implies that system (65) satisfies condition (48) in Theorem 3.3. By Theorem 3.3 $v_i(n) \to 0$ exponentially for $i = 2, \ldots, d$ as $n \to +\infty$. Thus $u_i(n) \to 0$ exponentially for $i = 2, \ldots, d$ as $n \to +\infty$.

If, in addition, (49) holds then

$$\limsup_{n\to+\infty}\Big[\tilde{a}_1(n) + c_i(n) + \ln\frac{\tilde{b}_{11}(n+1)\tilde{h}_1(n+1)}{\tilde{b}_{11}(n)\tilde{h}_1(n)}\Big] < 1 + \ln 2. \tag{66}$$

By Theorem 3.3 $\lim_{n\to+\infty}[v_1(n) - V^*(n)] = 0$, where $V^*(n)$ is the unique solution of the logistic equation $V(n+1) = V(n)\exp[\tilde{a}_1(n) - \tilde{b}_{11}(n)\tilde{h}_1(n)V(n)]$, which is defined on $\mathbb{Z}$ and bouded above and below by positive constants. This implies that $\lim_{n\to+\infty}[u_1(n) - U^*(n)] = 0$. The theorem is proved. ■

Remark. If $H_{ij}(n) = 0$ for all $i \neq j$, $n \in \mathbb{Z}_+$ and $H_{ii}(0) = 1$, i.e., there is no delay in system (38), then condition (63) becomes condition (3) given by Admad in [2].

Example 3.6. Consider the system

$$\begin{aligned} u_1(n+1) &= u_1(n)\exp\Big[a_1(n) - f(n)\sum_{s=-\infty}^{n} H_{11}(n-s)u_1(s) - \frac{g(n)}{2} \\ &\quad \times \sum_{s=-\infty}^{n} H_{12}(n-s)u_2(s)\Big], \\ u_2(n+1) &= u_2(n)\exp\Big[a_2(n) - \frac{5f(n)}{2}\sum_{s=-\infty}^{n} H_{21}(n-s)u_1(s) - g(n) \\ &\quad \sum_{s=-\infty}^{n} H_{22}(n-s)u_2(s)\Big], \end{aligned}$$

where $a_1(n) = \frac{1}{2} + \sin n$, $a_2(n) = \frac{1}{2} + \sin\sqrt{2}n$, $f(n) = 1 + \frac{1}{2}\cos\sqrt{3}n$, $g(n) = 1 + \frac{1}{2}\cos\sqrt{5}n$, $H_{11}(n) = \frac{2}{3^{n+1}}$, $H_{21}(n) = \frac{1}{2^{n+1}}$, $H_{12}(n) = \frac{4}{5^{n+1}}$, $H_{22}(n) = \frac{3}{4^{n+1}}$. We have $\bar{a}_1 = \bar{a}_2 = \frac{1}{2}$. By letting $\lambda_{21} = 1$ and $\delta = \frac{1}{16}$, it is easy to see that condition (63) in Theorem 3.5 holds. Therefore, by Theorem 3.5 we obtain that species u_2 in the system are extinct.

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