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A New Polyconvolution and Its Application to Solving a Class of Toeplitz Plus Hankel Integral Equations and Systems of Integral Equations

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Abstract. In this paper, we introduce a new polyconvolution for integral transforms in order to solve a class of integral equations and also systems of two or three equations with generalized Toeplitz plus Hankel kernel, whose solutions can be obtained in a closed form. Estimation of the new polyconvolution in weighted L_p spaces is obtained.

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1. Introduction

The integral equation with the Toeplitz plus Hankel kernel is studied in [4, 20]

$$f(x) + \int_0^\infty [k_1(x+y) + k_2(x-y)]f(y)dy = \varphi(x), \quad x > 0, \tag{1}$$

where k_1, k_2, φ are given functions, and f is an unknown function. Many partial cases of this equation have been used in physics, medicine and biology (see [6, 7]). Unfortunately, the solution of this equation in general case of Hankel kernel k_1 and Toeplitz kernel k_2 in a closed form is still not known.

Recently, in [16], several classes of integral equations with Toeplitz plus Hankel kernel were studied, where solutions can be obtained in a closed form with the help of generalized convolutions.

In this paper, we will introduce a new polyconvolution with a weight function for the Fourier sine and Kontorovich-Lebedev integral transforms in order to solve several classes of integral equations and also systems of integral equations with the Toeplitz plus Hankel kernel. This class of Toeplitz plus Hankel integral equations are quite different from [16]. To a deeper view of generalized convolutions and polyconvolutions, we refer the reader to references [8, 9, 10, 15, 17, 18, 19].

The paper is organized as follows. In Section 2, we recall some known convolutions and generalized convolutions and their properties. In Section 3, we introduce a new polyconvolution (6) with a weight function of three functions f, g, h for the Fourier sine and Kontorovich-Lebedev integral transforms and prove the existence of this polyconvolution on certain function spaces as well as the factorization identity (7). Boundedness properties of the polyconvolution operator on $L_p(\mathbb{R}_+)$ are also considered. These properties are quite different from [21] because of the different technique and also, the polyconvolution (6) is quite different from those of generalized convolutions introduced in [21]. In Sections 4 and 5, with the help of new polyconvolution (6), we consider some new classes of equations and systems of integral equations with Toeplitz plus Hankel kernel that can be solved in a closed form. These equations and systems of integral equations with Toeplitz plus Hankel kernel are first introduced in this paper and seem to be difficult to be solved by other techniques.

2. Preliminaries

In this section, we recall some known results which are useful in this paper. The Fourier convolution is defined by (see [13])

$$(f \underset{F}{*} g)(x) = \int_{-\infty}^{\infty} f(x-u)g(u)du,$$

where F denotes the Fourier transform [13]

$$(Ff)(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ixy} f(x)dx.$$

The convolution with a weight function $\gamma(x) = \sin x$ of two functions f and g for the Fourier sine transform was introduced in [8] as follows

$$(f\overset{\gamma}{*}g)(x) = \frac{1}{2\sqrt{2\pi}} \int\limits_0^{+\infty} f(y)[\mathrm{sign}(x+y-1)g(|x+y-1|)+\mathrm{sign}(x-y+1)g(|x-y+1|)$$
$$- g(x+y+1) - \mathrm{sign}(x-y-1)g(|x-y-1|)]dy, \quad x > 0. \qquad (2)$$

For this convolution the following factorization identity holds

$$F_s(f\overset{\gamma}{*}g)(y) = \sin y (F_s f)(y)(F_s g)(y), \quad \forall y > 0, \ \ f, g \in L_1(\mathbb{R}_+),$$

where F_s is the Fourier sine transform, that is

$$(F_s f)(y) = \sqrt{\frac{2}{\pi}} \int\limits_0^{\infty} \sin yx.f(x)dx, y > 0.$$

The generalized convolution of two functions f, g for the Fourier sine and Fourier cosine transforms

$$(f \underset{1}{*} g)(x) = \frac{1}{\sqrt{2\pi}} \int\limits_0^{\infty} f(u)[g(|x-u|) - g(x+u)]du, \quad x > 0, \qquad (3)$$

was introduced in [13]. There the following factorization identity was proved

$$F_s(f \underset{1}{*} g)(y) = (F_s f)(y).(F_c g)(y), \quad \forall y > 0, \ \ f, g \in L_1(\mathbb{R}_+),$$

where F_c is the Fourier cosine transform defined as follows:

$$(F_c f)(y) = \sqrt{\frac{2}{\pi}} \int\limits_0^{\infty} \cos yx f(x)dx, \quad y > 0.$$

For the Fourier convolution [13], the Young's theorem states that

$$\left| \int\limits_{-\infty}^{\infty} (f \underset{F}{*} g)(x).h(x)dx \right| \leqslant \|f\|_{L_p(\mathbb{R})}.\|g\|_{L_q(\mathbb{R})}.\|h\|_{L_r(\mathbb{R})}, \qquad (4)$$

where $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 2$, $f \in L_p(\mathbb{R})$, $g \in L_q(\mathbb{R})$, $h \in L_r(\mathbb{R})$ [1]. An important corollary of this theorem is the so-called Young's inequality for the Fourier convolution

$$\|f \underset{F}{*} g\|_{L_r(\mathbb{R})} \leqslant \|f\|_{L_p(\mathbb{R})}\|g\|_{L_q(\mathbb{R})}, \qquad (5)$$

where $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$, $f \in L_p(\mathbb{R})$, $g \in L_q(\mathbb{R})$.

Note, however, that for the typical case $f, g \in L_2(\mathbb{R})$, the inequality (5) does not hold. In [11, 12], S. Saitoh et al. introduced a weighted $L_p(\mathbb{R})$ inequality for the Fourier convolution.

Extending the notion of convolutions and generalized convolutions, Kakichev V.A. [9] introduced the notion of polyconvolution:

Definition 2.1. ([9])Consider the integral transforms $K_i : U_i(X_i) \to V(Y), i = 1, ..., 4$, where $U_i(X_i)$ are linear spaces, and $V(Y)$ is an algebra. Then a polyconvolution of functions $f \in U_1(X_1)$, $g \in U_2(X_2)$, $h \in U_3(X_3)$ with a weight function γ for the integral transforms K_4, K_3, K_2, K_1 is a multi-linear operator $\xi : \prod_{i=1}^{3} U_i(X_i) \to V(Y)$ such that the following factorization equality holds

$$(K_4\xi)(x) = \gamma(x)(K_1 f)(x)(K_2 g)(x)(K_3 h)(x).$$

3. A Polyconvolution

Definition 3.1. The polyconvolution with the weight function $\gamma = \sin x$ of functions f, g, and h for the Fourier sine and the Kontorovich-Lebedev integral transforms is defined as follows

$$\overset{\gamma}{*}(f,g,h)(x) = \frac{1}{4\sqrt{2\pi}} \int\limits_{\mathbb{R}_+^3} \left(\sum_{i=1}^{4} \theta_i(x,u,v,w) \right) f(u)g(v)h(w)dudvdw, \quad x > 0, \tag{6}$$

here

$$\begin{aligned}
\theta_1(x,u,v,w) =& e^{-w\cosh(x+u+v+1)} - e^{-w\cosh(x+u+v-1)};\\
\theta_2(x,u,v,w) =& e^{-w\cosh(x-u+v-1)} - e^{-w\cosh(x-u+v+1)};\\
\theta_3(x,u,v,w) =& e^{-w\cosh(x+u-v-1)} - e^{-w\cosh(x+u-v+1)};\\
\theta_4(x,u,v,w) =& e^{-w\cosh(x-u-v+1)} - e^{-w\cosh(x-u-v-1)}.
\end{aligned}$$

First of all, we will show that this definition really gives the notion of polyconvolution, that is the operator (6) satisfies the equality in Definition 2.1 for suitable integral transforms.

We consider functions in the function space $L_p^{\alpha,\beta} \equiv L_p(\mathbb{R}_+, t^\alpha K_0(\beta t)), \alpha \in \mathbb{R}, 0 < \beta \leqslant 1$, whose norm is defined as follows [21]

$$\|f\|_{L_p^{\alpha,\beta}} = \Big(\int\limits_0^\infty |f(t)|^p K_0(\beta t) t^\alpha dt \Big)^{\frac{1}{p}} < \infty.$$

The Kontorovich-Lebedev integral transform is of the form (see [14])

$$K_{ix}[f] = \int\limits_0^\infty K_{ix}(t) f(t) dt,$$

here $K_{ix}(t)$ denotes the modified Bessel function (see [2]).

Theorem 3.2. *Let $f, g \in L_1(\mathbb{R}_+)$ and let $h \in L_1^{0,\beta}, 0 < \beta \leqslant 1$, then the polyconvolution (6) belongs to $L_1(\mathbb{R}_+)$ and the following factorization equality holds*

$$F_s(\overset{\gamma}{*}(f,g,h))(y) = \sin y (F_s f)(y)(F_s g)(y)(K_{iy}h), \ \forall y > 0. \tag{7}$$

Besides, the following norm estimation holds

$$\| \overset{\gamma}{*} (f,g,h)\|_{L_1(\mathbb{R}_+)} \leqslant \sqrt{\frac{2}{\pi}} \|f\|_{L_1(\mathbb{R}_+)} \|g\|_{L_1(\mathbb{R}_+)} \|h\|_{L_1^{0,\beta}(\mathbb{R}_+)}.$$

Moreover, in case $\beta \in (0;1)$, the polyconvolution $\overset{\gamma}{*}(f,g,h)(x)$ also belongs to $C_0(\mathbb{R}_+)$ and the following Parseval type identity holds

$$\overset{\gamma}{*}(f,g,h)(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty (F_s f)(y)(F_s g)(y)(K_{iy}h) \sin y \sin xy dy, \ \forall x > 0. \tag{8}$$

Lemma 3.3. *The following estimation holds*

$$\int_0^\infty |\theta(x,u,v,w)| dx \leqslant \sqrt{\frac{2}{\pi}} K_0(w).$$

Proof. We have

$$\begin{aligned} &\int_0^\infty |e^{-w\cosh(x+u+v+1)} - e^{-w\cosh(x-u-v-1)}| dx \\ &< \int_0^\infty e^{-w\cosh(x+u+v+1)} dx + \int_0^\infty e^{-w\cosh(x-u-v-1)} dx \\ &= \int_{-\infty}^\infty e^{-w\cosh y} dy = 2K_0(w). \end{aligned} \tag{9}$$

■

Proof of Theorem 3.2. Using Fubini theorem and the above lemma, we have

$$\begin{aligned} \int_0^\infty | \overset{\gamma}{*} (f,g,h)(x)| dx &\leqslant \sqrt{\frac{2}{\pi}} \int_{\mathbb{R}_+^3} |f(u)||g(v)||h(w)| K_0(w) dudvdw \\ &\leqslant \sqrt{\frac{2}{\pi}} \int_{\mathbb{R}_+^3} |f(u)||g(v)||h(w)| K_0(\beta w) dudvdw. \end{aligned}$$

Therefore

$$\|\overset{\gamma}{*}(f,g,h)\|_{L_1(\mathbb{R}_+)} \leqslant \sqrt{\frac{2}{\pi}}\|f\|_{L_1(\mathbb{R}_+)}\|g\|_{L_1(\mathbb{R}_+)}\|h\|_{L_1^{0,\beta}(\mathbb{R}_+)} < \infty,$$

the norm estimation is obtained.

We now prove the factorization equality. Indeed, we have

$$\begin{aligned}&\sin y\,(F_s f)(y)(F_s g)(y)(K_{iy}h) =\\ &= \frac{2}{\pi}\int_0^\infty\int_0^\infty\int_0^\infty \sin y\, \sin(yu)\sin(yv)K_{iy}(w)f(u)g(v)h(w)dudvdw.\end{aligned}$$

Using formula 1.13.75, p. 62 in [3] we get

$$\begin{aligned}&\sin y\,(F_s f)(y)(F_s g)(y)(K_{iy}h) =\\ &= \frac{2}{\pi}\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty \sin y\sin(yu)\sin(yv)\cos(y\alpha)e^{-w\cosh\alpha}f(u)g(v)h(w)dudvdwd\alpha\\ &= \frac{1}{4\pi}\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty e^{-w\cosh\alpha}[\sin y(1+\alpha+u-v)+\sin y(1+\alpha-u+v)\\ &\quad +\sin y(1-\alpha+u-v)+\sin y(1-\alpha-u+v)-\sin y(1+\alpha+u+v)\\ &\quad -\sin y(1+\alpha-u-v)-\sin y(1-\alpha+u+v)\\ &\quad -\sin y(1-\alpha-u-v)]f(u)g(v)h(w)dudvdwd\alpha. \qquad (10)\end{aligned}$$

Interchanging variables we have

$$\begin{aligned}&\int_0^\infty e^{-w\cosh\alpha}[\sin y(u-1+v+\alpha)-\sin y(u+1+v+\alpha)]d\alpha\\ &\qquad = \int_0^\infty \theta_4(x,u,v,w)\sin yxdx. \qquad (11)\end{aligned}$$

Similarly,

$$\begin{aligned}&\int_0^\infty e^{-w\cosh\alpha}[\sin y(u+1-v+\alpha)-\sin y(u-1-v+\alpha)]d\alpha\\ &\qquad = \int_0^\infty \theta_2(x,u,v,w)\sin yxdx; \qquad (12)\end{aligned}$$

$$\int_0^\infty e^{-w\cosh\alpha}[\sin y(1+\alpha-u+v)-\sin y(\alpha-u-1+v)]d\alpha$$

$$= \int_0^\infty \theta_3(x,u,v,w) \sin yx dx; \tag{13}$$

$$\int_0^\infty e^{-w\cosh\alpha}[\sin y(\alpha-u-1-v) - \sin y(\alpha-u+1-v)]d\alpha$$

$$= \int_0^\infty \theta_1(x,u,v,w) \sin yx dx. \tag{14}$$

From formulas (10) - (14) we have

$$\sin y\,(F_s f)(y)(F_s g)(y)(K_{iy}h) = F_s(\overset{\gamma}{*}(f,g,h))(y).$$

By virtue of formula 1.13.75, p. 62 in [3] and the estimation [21]

$$|K_{ix}(t) \leqslant e^{-|x|\arccos\beta} K_0(\beta t),\ 0 < \beta \leqslant 1,$$

the condition $0 < \beta \leqslant 1$ is sufficient for absolute convergence in x of the integral (6). From the help of the Fubini theorem we easily see that

$$\overset{\gamma}{*}(f,g,h)(x) = \sqrt{\left(\frac{2}{\pi}\right)^3} \int_{R_+^4} f(u)g(v)h(w)K_{it}(w) \sin t \sin xt \sin tu \sin tv du dv dw dt$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \sin xt (F_s f)(t)(F_s g)(t)(K_{it}h) \sin t dt.$$

The Parseval type identity is proved. Finally, since the integral (6) is absolutely converges in x, from Riemann-Lebesgue lemma we have $\overset{\gamma}{*}(f,g,h)(x) \in C_0(\mathbb{R}_+)$. The proof is complete. ■

For the Fourier convolution, the Young theorem is fundamental [1]. Next, we prove the Young's type theorem for the Fourier sine convolution with a weight function (2) in order to study the boundedness of the polyconvolution (6) in the spaces $L_p(\mathbb{R}_+),\ p \geqslant 1$, and establish some norm estimations.

Lemma 3.4 (An Young's type theorem). *Let $p,\ q,\ r > 1$, be such that $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 2$, and let $f \in L_p(\mathbb{R}_+)$, $g \in L_q(\mathbb{R}_+)$, $h \in L_r(\mathbb{R}_+)$, then*

$$\left| \int_0^\infty (f \overset{\gamma}{*} g)(x).h(x)dx \right| \leqslant \|f\|_{L_p(\mathbb{R}_+)}.\|g\|_{L_q(\mathbb{R}_+)}.\|h\|_{L_r(\mathbb{R}_+)}.$$

Proof. Let p_1, q_1, r_1 be the conjugate exponentials of p, q, r, respectively, i.e.,

$$\frac{1}{p}+\frac{1}{p_1}=1,\ \frac{1}{q}+\frac{1}{q_1}=1,\ \frac{1}{r}+\frac{1}{r_1}=1.$$

Then it is obvious that $\frac{1}{p_1}+\frac{1}{q_1}+\frac{1}{r_1}=1$. Put

$$\begin{aligned}U(x,u)&=|\operatorname{sign}(x+u-1)g(|x+u-1|)\\&\quad+\operatorname{sign}(x-u+1)g(|x-u+1|)|^{q/p_1}.|h(x)|^{r/p_1};\\V(x,u)&=|f(u)|^{p/q_1}.|h(x)|^{r/q_1};\\W(x,u)&=|f(y)|^{p/r_1}.|\operatorname{sign}(x+u-1)g(|x+u-1|)\\&\quad+\operatorname{sign}(x-u+1)g(|x-u+1|)|^{q/r_1}.\end{aligned}$$

We have

$$(U.V.W)(x,u)=f(u)h(x)[\operatorname{sign}(x+u-1)g(|x+u-1|)+\operatorname{sign}(x-u+1)g(|x-u+1|)]. \tag{15}$$

On the other hand, in the space $L_{p_1}(\mathbb{R}_+^2)$ we have

$$\begin{aligned}&\|U\|^{p_1}_{L_{p_1}(\mathbb{R}_+^2)}=\\&=\int_0^\infty\int_0^\infty|\operatorname{sign}(x+u-1)g(|x+u-1|)+\operatorname{sign}(x-u+1)g(|x-u+1|)|^q|h(x)|^r dxdu\\&=\int_0^\infty\left(\int_0^\infty|\operatorname{sign}(x+u-1)g(|x+u-1|)+\operatorname{sign}(x-u+1)g(|x-u+1|)|^q du\right)\times\\&\qquad\times|h(x)|^r dx.\end{aligned}$$

Note that t^q is a convex function, therefore, by changing variables we have

$$\begin{aligned}&\int_0^\infty|\operatorname{sign}(x+u-1)g(|x+u-1|)+\operatorname{sign}(x-u+1)g(|x-u+1|)|^q du\\&\leqslant 2^{q-1}\Big(\int_0^\infty|g(|x+u-1|)|^q du+\int_0^\infty|g(|x-u+1|)|^q du\Big)\\&=2^q\int_0^\infty|g(u)|^q du.\end{aligned}$$

It yields

$$\|U\|^{p_1}_{L_{p_1}(\mathbb{R}_+^2)}\leqslant 2^q\int_0^\infty\left(\int_0^\infty|g(u)|^q du\right)|h(x)|^r dx=2^q\|g\|^p_{L_q(\mathbb{R}_+)}\|h\|^r_{L_r(\mathbb{R}_+)}. \tag{16}$$

Similarly,

$$\|W\|^{r_1}_{L_{r_1}(\mathbb{R}^2_+)} \leqslant 2^q \|f\|^p_{L_p(\mathbb{R}_+)} \|g\|^q_{L_q(\mathbb{R}_+)}. \tag{17}$$

It is obvious that $\|V\|_{L_{q_1}(\mathbb{R}^2_+)} = \|f\|^{p/q_1}_{L_p(\mathbb{R}_+)} \|h\|^{r/q_1}_{L_r(\mathbb{R}_+)}$. Hence, from (16) and (17) we have

$$\|U\|_{L_{p_1}(\mathbb{R}^2_+)} \|V\|_{L_{q_1}(\mathbb{R}^2_+)} \|W\|_{L_{r_1}(\mathbb{R}^2_+)} \leqslant 2 \|f\|_{L_p(\mathbb{R}_+)} \|g\|_{L_q(\mathbb{R}_+)} \|h\|_{L_r(\mathbb{R}_+)}. \tag{18}$$

By a similar argument, we have

$$\begin{aligned} U_1(x,u) =& |g(x+u+1) + \operatorname{sign}(x-u-1) g(|x-u-1|)|^{\frac{q}{p_1}} |h(w)|^{\frac{r}{p_1}}; \\ W_1(x,u) =& |f(u)^{\frac{p}{r_1}} |g(x+u+1) + \operatorname{sign}(x-u-1) g(|x-u-1|)|^{\frac{q}{r_1}}. \end{aligned}$$

We obtain

$$\|U_1\|_{L_{p_1}(\mathbb{R}^2_+)} \|V\|_{L_{q_1}(\mathbb{R}^2_+)} \|W_1\|_{L_{r_1}(\mathbb{R}^2_+)} \leqslant 2 \|f\|_{L_p(\mathbb{R}_+)} \|g\|_{L_q(\mathbb{R}_+)} \|h\|_{L_r(\mathbb{R}_+)}. \tag{19}$$

From (15), (18) and (19), by the three-function form of Hölder's inequality (see [1]) we have

$$\begin{aligned} &\left| \int_0^\infty (f \underset{F_s}{\overset{\gamma}{*}} g)(x).h(x)dx \right| \\ &\leqslant \frac{1}{2\sqrt{2\pi}} \int_0^\infty \int_0^\infty (U(x,u).V(x,u).W(x,u) + U_1(x,u).V(x,u).W_1(x,u))dudx \\ &\leqslant \frac{1}{2\sqrt{2\pi}} \left(\|U\|_{L_{p_1}(\mathbb{R}^2_+)} \|V\|_{L_{q_1}(\mathbb{R}^2_+)} \|W\|_{L_{r_1}(\mathbb{R}^2_+)} + \|U_1\|_{L_{p_1}(\mathbb{R}^2_+)} \|V\|_{L_{q_1}(\mathbb{R}^2_+)} \|W_1\|_{L_{r_1}(\mathbb{R}^2_+)} \right) \\ &\leqslant \sqrt{\frac{2}{\pi}} \|f\|_{L_p(\mathbb{R}_+)}.\|g\|_{L_q(\mathbb{R}_+)}.\|h\|_{L_r(\mathbb{R}_+)}. \end{aligned}$$

The proof is complete. ∎

Corollary 3.5 (An Young's type inequality). *Let p, q, r be positive numbers such that $p > 1, q > 1, r > 1$ and $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$, and let $f \in L_p(\mathbb{R}_+), g \in L_q(\mathbb{R}_+)$. Then the Fourier sine convolution with a weight function* (2) $f \overset{\gamma}{*} g$ *belongs to $L_r(\mathbb{R}_+)$, moreover,*

$$\|f \overset{\gamma}{*} g\|_{L_r(\mathbb{R}_+)} \leqslant \sqrt{\frac{2}{\pi}} \|f\|_{L_p(\mathbb{R}_+)} \|g\|_{L_q(\mathbb{R}_+)}.$$

Using Fourier sine convolution (2) and formula 2.16.48.19 in [10] one can prove the following expression

Lemma 3.6. *For $f, g \in L_1(\mathbb{R}_+)$ and $h \in L_1^{0,\beta}(\mathbb{R}_+), 0 < \beta \leqslant 1$, we have*

$$\overset{\gamma}{*}(f,g,h)(x) = \frac{1}{2}\int\limits_{\mathbb{R}_+^2} (f \overset{\gamma}{*} g)(u)h(u)[e^{-w\cosh(u-x)} - e^{-w\cosh(u+x)}]dudw.$$

The boundedness of the polyconvolution operator (6) in the function spaces $L_r^{\alpha,\gamma}(\mathbb{R}_+), 1 \leqslant r,\ \alpha > -1, 0 < \gamma \leqslant 1$, is proved in the following theorem

Theorem 3.7. *Let $p > 1, q > 1, r > 1$ be such that $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$, and let s be the conjugate exponential of r, and let $f \in L_p(\mathbb{R}_+), g \in L_q(\mathbb{R}_+), h \in L_s^{0,\beta}(\mathbb{R}_+), 0 < \beta \leqslant 1$. Then the polyconvolution (6) exists, is continuous and bounded for all $x > 0$. Moreover, $\overset{\gamma}{*}(f,g,h) \in L_r^{\alpha,\gamma}(\mathbb{R}_+), 1 \leqslant r < \infty, \alpha > -1, 0 < \gamma \leqslant 1$, the coefficients r, β, γ are independent and the following estimation holds*

$$\| \overset{\gamma}{*}(f,g,h)\|_{L_r^{\alpha,\gamma}(\mathbb{R}_+)} \leqslant C\|f\|_{L_p(\mathbb{R}_+)}\|g\|_{L_q(\mathbb{R}_+)}\|h\|_{L_s^{0,\beta}(\mathbb{R}_+)},$$

where $C = \sqrt{\frac{2}{\pi}}(2\gamma)^{-\frac{1}{r}}\left(\frac{2}{\gamma}\right)^{\frac{\alpha}{r}}\Gamma^{\frac{2}{r}}\left(\frac{\alpha+1}{2}\right).$

If, in addition, $f \in L_1(\mathbb{R}_+) \cap L_p(\mathbb{R}_+), g \in L_1(\mathbb{R}_+) \cap L_q(\mathbb{R}_+)$, then the polyconvolution (6) satisfies the factorization equality (7). Moreover, in case $\beta \in (0,1)$, the polyconvolution $\overset{\gamma}{*}(f,g,h) \in C_0(\mathbb{R}_+)$ and the Parseval identity (8) holds.

Proof. From Lemma 3.6 and the Hölder inequality we obtain

$$\begin{aligned}|\overset{\gamma}{*}(f,g,h)(x)| &\leqslant \Big(\int\limits_{\mathbb{R}_+^2} |f \underset{F_s}{\overset{\gamma}{*}} g)(u)|^p e^{-w} dudw\Big)^{\frac{1}{r}} \Big(\int\limits_0^\infty |h(w)|^q K_0(w)dw\Big)^{\frac{1}{s}} \\ &\leqslant \Big(\int\limits_{\mathbb{R}_+^2} |f \underset{F_s}{\overset{\gamma}{*}} g)(u)|^r du\Big)^{\frac{1}{r}} \Big(\int\limits_0^\infty |h(w)|^s K_0(\beta w)dw\Big)^{\frac{1}{s}}.\end{aligned}$$

Then Lemma 3.4 gives us

$$|(\overset{\gamma}{*}(f,g,h)(x)| \leqslant \sqrt{\frac{2}{\pi}}\|f\|_{L_p(\mathbb{R}_+)}\|g\|_{L_q(\mathbb{R}_+)}\|h\|_{L_s^{0,\beta}(\mathbb{R}_+)}. \tag{20}$$

It shows the boundedness, absolute convergence, and continuity of the polyconvolution operator $\overset{\gamma}{*}(f,g,h)(x)$ on $\mathbb{R}_+$. Hence, by virtue of formula 2.16.2.2 in [10] we get

$$\| \overset{\gamma}{*}(f,g,h)\|_{L_r^{\alpha,\gamma}(\mathbb{R}_+)} \leqslant \Big(\int\limits_0^\infty t^\alpha K_0(\gamma t)dt\Big)^{\frac{1}{r}} \sqrt{\frac{2}{\pi}}\|f\|_{L_p(\mathbb{R}_+)}\|g\|_{L_q(\mathbb{R}_+)}\|h\|_{L_s^{0,\beta}(\mathbb{R}_+)}$$

$$= C\|f\|_{L_p(\mathbb{R}_+)}\|g\|_{L_q(\mathbb{R}_+)}\|h\|_{L_s^{0,\beta}(\mathbb{R}_+)},$$

where $C = \sqrt{\dfrac{2}{\pi}}(2\gamma)^{-\frac{1}{r}}\left(\dfrac{2}{\gamma}\right)^{\frac{\alpha}{r}}\Gamma^{\frac{2}{r}}\left(\dfrac{\alpha+1}{2}\right)$.
Besides, since $f, g \in L_1(\mathbb{R}_+)$, and the fact that $L_p^{0,\beta}(\mathbb{R}_+) \subset L_1^{0,\beta}(\mathbb{R}_+)$, Theorem 3.2 implies that $\overset{\gamma}{*}(f, g, h) \in C_0(\mathbb{R}_+) \cap L_1(\mathbb{R}_+)$, then the factorization identity (7) and the Parseval equality hold for $\beta \in (0\ ;\ 1)$. The proof is complete. ■

Corollary 3.8. *With the same hypothesis as in Theorem* 3.7*, the polyconvolution* (6) *exists for all* $x > 0$*, is continuous and belongs to* $L_l(\mathbb{R}_+)$*. Moreover, the following estimation holds*

$$\|\overset{\gamma}{*}(f,g,h)\|_{L_l(\mathbb{R}_+)} \leqslant \left(\frac{\pi}{2\beta}\right)^{\dfrac{1}{p}} \|f\|_{|L_p(\mathbb{R}_+)}\|g\|_{|L_q(\mathbb{R}_+)}\|h\|_{|L_s^{0,\beta}(\mathbb{R}_+)}. \tag{21}$$

Especially, in case $r = 2$*, we get the Parseval type equality*

$$\int_0^\infty |\overset{\gamma}{*}(f,g,h)(x)|^2 dx = \int_0^\infty |\sin x (F_s f)(x)(F_s g)(x)(K_{ix}h)|^2 dx.$$

Proof. Using formulas (9), (20), Lemmas 3.4, 3.6, and the fact that

$$\int_0^\infty K_0(\beta\theta)d\theta = \frac{\pi}{2\beta},$$

we have

$$\int_0^\infty |\overset{\gamma}{*}(f,g,h)(x)|^r dx \leqslant \Big(\int_{R_+^2} |(f \underset{F_s}{\overset{\gamma}{*}} g)(u)|^r K_0(\beta\theta)dud\theta\Big)\Big(\int_0^\infty |h(w)|^s K_0(\beta w)dw\Big)^{\dfrac{r}{s}}$$

$$\leqslant \frac{\pi}{2\beta}\big(\|f\|_{L_p(\mathbb{R}_+)}\|g\|_{L_q(\mathbb{R}_+)}\|h\|_{L_s^{0,\beta}(\mathbb{R}_+)}\big)^r.$$

Therefore the inequality (21) holds. Then using the factorization equality in Theorem 3.2 we obtain the Parseval identity of Fourier type. The corollary is proved. ■

Corollary 3.9. *For* $f, g \in L_1(\mathbb{R}_+), h \in L_2(\mathbb{R}_+)$*, the polyconvolution exists, is continuous, bounded and belongs to* $L_r^{\alpha,\gamma}(\mathbb{R}_+), 1 \leqslant r < \infty, \alpha > -1, 0 < \gamma \leqslant 1$*, moreover, the following estimation holds*

$$\|\overset{\gamma}{*}(f,g,h)\|_{L_r^{\alpha,\gamma}(\mathbb{R}_+)} \leqslant C\|f\|_{L_1(\mathbb{R}_+)}\|g\|_{L_1(\mathbb{R}_+)}\|h\|_{L_2(\mathbb{R}_+)}.$$

Proof. Using Schwarz inequality and the polyconvolution (6) we have

$$\begin{aligned}
|\overset{\gamma}{*}(f,g,h)| \leqslant \frac{1}{4\sqrt{2\pi}} \Big(\int\limits_{\mathbb{R}^3_+} 8|f(u)||g(v|e^{-w} dudvdw \Big)^{1/2} \times \\
\times \Big(\int\limits_{\mathbb{R}^3_+} |f(u)||g(v)||h(w)|^2 8e^{-w} dudvdw \Big)^{1/2} \\
\leqslant \sqrt{\frac{2}{\pi}} \Big(\int\limits_{R^2_+} |f(u)||g(v)|dudv \Big)^{1/2} \Big(\int\limits_{R^2_+} |f(u)||g(v)|h(w)|^2 dudvdw \Big)^{1/2} \\
= \sqrt{\frac{2}{\pi}} \|f\|_{L_1(\mathbb{R}_+)} \|g\|_{L_1(\mathbb{R}_+)} \|h\|_{L_2(\mathbb{R}_+)}.
\end{aligned}$$

Therefore, by virtue of formulas (9), (20) we get

$$\| \overset{\gamma}{*}(f,g,h)\|_{L_r^{\alpha,\gamma}(\mathbb{R}_+)} \leqslant C \|f\|_{L_1(\mathbb{R}_+)} \|g\|_{L_1(\mathbb{R}_+)} \|h\|_{L_2(\mathbb{R}_+)}.$$

■

The following corollary is proved similarly.

Corollary 3.10. *For $f, g, h \in L_1(\mathbb{R}_+)$, the polyconvolution* (6) *exists. Moreover, the polyconvolution operator is continuous, bounded, and belongs to $L_r^{\alpha,\gamma}(\mathbb{R}_+), 1 \leqslant r < \infty,\ \alpha > -1, 0 < \gamma \leqslant 1$ and the following estimation holds*

$$\| \overset{\gamma}{*}(f,g,h)\|_{L_r^{\alpha,\gamma}(\mathbb{R}_+)} \leqslant C \|f\|_{L_1(\mathbb{R}_+)} \|g\|_{L_1(\mathbb{R}_+)} \|h\|_{L_1(\mathbb{R}_+)}.$$

4. The Integral Equation with the Toeplitz Plus Hankel Kernel

In this section, we introduce a new class of Toeplitz plus Hankel integral equations (1) related to polyconvolution (6) which can be solved in a closed form. Namely, the integral equation (1) with the Hankel kernel k_1 and the Toeplitz kernel k_2 defined as follows

$$\begin{aligned} k_1(t) = k_{11}(t) + k_{12}(t) - k_{13}(t) - k_{14}(t), \\ k_2(t) = k_{21}(t) + k_{22}(t) - k_{23}(t) - k_{24}(t). \end{aligned} \tag{22}$$

In the case $k_{21} \equiv -k_{11},\ k_{22} \equiv -k_{12},\ k_{23} \equiv -k_{13};\ k_{24} \equiv -k_{14}$, and

$$\begin{aligned}
k_{11}(t) = \frac{1}{4\sqrt{2\pi}} \int\limits_{\mathbb{R}^2_+} e^{-w\cosh(t+v+1)} g(v)h(w)dvdw; \\
k_{12}(t) = \frac{1}{4\sqrt{2\pi}} \int\limits_{\mathbb{R}^2_+} e^{-w\cosh(t-v-1)} g(v)h(w)dvdw;
\end{aligned}$$

$$k_{13}(t) = \frac{1}{4\sqrt{2\pi}} \int\limits_{\mathbb{R}_+^2} e^{-w\cosh(t-v+1)} g(v)h(w)dvdw; \tag{23}$$

$$k_{14}(t) = \frac{1}{4\sqrt{2\pi}} \int\limits_{\mathbb{R}_+^2} e^{-w\cosh(t+v-1)} g(v)h(w)dvdw,$$

the Toeplitz plus Hankel integral equation with the kernels k_1, k_2 defined by (22) is of the form

$$f(x) + \int\limits_0^\infty f(y)[k_{11}(x+y) + k_{12}(x+y) - k_{13}(x+y) - k_{14}(x+y) - k_{11}(x-y) - k_{12}(x-y) + k_{13}(x+y) + k_{14}(x-y)]dy = \varphi(x), \quad x > 0. \tag{24}$$

Theorem 4.1. *Let $g, \varphi \in L_1(\mathbb{R}_+), h \in L_1^{0,\beta}(\mathbb{R}_+)$ be such that $1 + \sin y (F_s g)(y)(K_{iy}h) \neq 0$. Then the equation (24) has a unique solution in $L_1(\mathbb{R}_+)$, which can be written in a closed form as follows*

$$f(x) = \varphi(x) - (\varphi \underset{1}{*} \xi)(x),$$

here the function $\xi \in L_1(\mathbb{R}_+)$ is defined by

$$(F_c\xi)(y) = \frac{\sin y (F_s g)(y)(K_{iy}h)}{1 + \sin y (F_s g)(y)(K_{iy}h)}.$$

Lemma 4.2. *Let $g \in L_1(\mathbb{R}_+)$, $h \in L_1^{0,\beta}(\mathbb{R}_+)$, then the generalized convolution $(g \overset{\gamma}{\underset{2}{*}} h)(x)$ belongs to $L_1(\mathbb{R}_+)$ and the respective factorization equality is*

$$F_c(g \overset{\gamma}{\underset{2}{*}} h)(y) = \sin y\, (F_s g)(y)(K_{iy}h), \forall y > 0,$$

where

$$(g \overset{\gamma}{\underset{2}{*}} h)(x) = \frac{1}{4}\int\limits_0^\infty\int\limits_0^\infty [e^{-v\cosh(x+u-1)} + e^{-v\cosh(x-u+1)} - e^{-v\cosh(x+u+1)} - e^{-v\cosh(x-u-1)}]g(u)h(v)dudv, \; x > 0.$$

Proof. We now prove Theorem 4.1 with the help of polyconvolution (6) and convolutions (2), (3). The equation (24) can be rewritten in the following form

$$f(x) + \overset{\gamma}{*}(f, g, h)(x) = \varphi(x).$$

Using factorization identity (7), this equation becomes

$$(F_s f)(y) + \sin y (F_s f)(y).(F_s g)(y).(K_{iy}h) = (F_s\varphi)(y).$$

Therefore, by the given condition

$$(F_s f)(y) = (F_s\varphi)(y)\left(1 - \frac{\sin y (F_s g)(y)(K_{iy}h)}{1+\sin y(F_s g)(y)(K_{iy}h)}\right).$$

Using Lemma 4.2 we get

$$(F_s f)(y) = (F_s\varphi)(y)\left(1 - \frac{F_c(g \overset{\gamma}{\underset{2}{*}} h)(y)}{1+F_c(g \overset{\gamma}{\underset{2}{*}} h)(y)}\right). \tag{25}$$

We recall that the Wiener-Levy theorem ([5, p.63]) states that if f is the Fourier transform of an $L_1(\mathbb{R})$ function, and φ is analytic in a neighborhood of the origin that contains the domain $\{f(y), \forall y \in \mathbb{R}\}$, and $\varphi(0) = 0$, then $\varphi(f)$ is also the Fourier transform of an $L_1(\mathbb{R})$ function. For the Fourier cosine transform it means that if f is the Fourier cosine transform of an $L_1(\mathbb{R}_+)$ function, and φ is analytic in a neighborhood of the origin that contains the domain $\{f(y), \forall y \in \mathbb{R}_+\}$, and $\varphi(0) = 0$, then $\varphi(f)$ is also the Fourier cosine transform of an $L_1(\mathbb{R}_+)$ function.

With the given condition, the function $\theta(z) = \dfrac{z}{1+z}$ satisfies the conditions of the Wiener-Levy theorem [5], then there exists a function $\xi \in L_1(\mathbb{R}_+)$ such that

$$(F_c\xi)(y) = \frac{\sin y(F_s g)(y)(K_{iy}h)}{1+\sin y(F_s g)(y)(K_{iy}h)}. \tag{26}$$

From (25), (26) we have

$$(F_s f)(y) = (F_s\varphi)(y)[1-(F_c\xi)(y)].$$

Then the solution in $L_1(\mathbb{R}_+)$ of the equation (24) has the form

$$f(x) = \varphi(x) - (\varphi \underset{1}{*} \xi)(x),$$

here $(\cdot \underset{1}{*} \cdot)$ is defined by (3). Since φ, ξ are functions in $L_1(\mathbb{R}_+)$, one can easily see that $f \in L_1(\mathbb{R}_+)$. The proof is complete. ∎

The following corollaries of Theorem 3.2 give us a necessary condition for a solution in $L_1(\mathbb{R}_+)$ of equation (24) and a norm estimation on $L_1(\mathbb{R}_+)$ of the solution.

Corollary 4.3. *For $g, \varphi \in L_1(\mathbb{R}_+)$, $h \in L_1^{0,\beta}(\mathbb{R}_+)$, a necessary condition for the solution f in $L_1(\mathbb{R}_+)$ of equation* (24) *is*

$$\|f\|_{L_1(\mathbb{R}_+)} \geqslant \frac{\|\varphi\|_{L_1(\mathbb{R}_+)}}{1+\|g\|_{L_1(\mathbb{R}_+)}\|h\|_{L_1^{0,\beta}(\mathbb{R}_+)}}.$$

Corollary 4.4. *With the same hypothesis as in Theorem* 4.1*, the solution f of the equation* (24) *satisfies the following estimation*

$$\|f\|_{L_1(\mathbb{R}_+)} \leqslant \|\varphi\|_{L_1(\mathbb{R}_+)}(1 + \|\xi\|_{L_1(\mathbb{R}_+)}).$$

5. Systems of Integral Equations with the Toeplitz Plus Hankel Kernels

Finally, in this section, we consider two classes of systems of integral equations with the Toeplitz plus Hankel kernel related to polyconvolution (6) which can be solved in a closed form. These systems are introduced for the first time in this paper.

5.1. We consider the system of two integral equations with the Toeplitz plus Hankel kernel

$$\begin{aligned} f(x) + \int_0^\infty \xi(u)(k_1(x+u) + k_2(x-u))du &= p(x) \\ \xi(x) + \int_0^\infty f(u)(k_3(x+u) + k_4(x-u))du &= q(x),\ x > 0. \end{aligned} \tag{27}$$

The solutions in a closed form for this system in general case is still open. In this part we consider the system (27) in the case

$$\begin{aligned} k_1(t) =& k_{11}(t) + k_{12}(t) - k_{13}(t) - k_{14}(t), \\ k_2(t) =& k_{21}(t) + k_{22}(t) - k_{23}(t) - k_{24}(t), \\ k_3(t) =& k_{31}(t) + k_{32}(t) - k_{33}(t) - k_{34}(t), \\ k_4(t) =& k_{41}(t) + k_{42}(t) - k_{43}(t) - k_{44}(t), \end{aligned} \tag{28}$$

here $k_{1i}(t),\ \ i = 1, ..., 4$ is defined in (23), and $k_{3i}(t)$ as in (23) in the case $g \equiv \varphi,\ h \equiv \psi$; $k_{2i}(t) \equiv -k_{1i}, k_{4i} \equiv -k_{3i},\ i = 1, ..., 4$; f and ξ are unknown; $g, h, \varphi, \psi, p, q$ are given.

Theorem 5.1. *Let $g, \varphi, p, q \in L_1(\mathbb{R}_+)$, and $h, \psi \in L_1^{0,\beta}(\mathbb{R}_+)$ be such that*

$$1 - \sin^2 y (F_s g)(y)(F_s\varphi)(y)(K_{iy}h)(K_{iy}\psi) \neq 0.$$

Then system (27) with kernels defined by (28) has a unique solution in $L_1(\mathbb{R}_+) \times L_1(\mathbb{R}_+)$ and the solution can be expressed in a closed form as follows

$$\begin{aligned} f(x) &= p(x) + (p \underset{1}{*} l)(x) - \overset{\gamma}{*}(q, g, h)(x) - (\overset{\gamma}{*}(q, g, h) \underset{1}{*} l)(x), \quad x > 0, \\ \xi(x) &= q(x) + (q \underset{1}{*} l)(x) - \overset{\gamma}{*}(p, \varphi, \psi)(x) - (\overset{\gamma}{*}(p, \varphi, \psi) \underset{1}{*} l)(x), \quad x > 0, \end{aligned}$$

here, $(\cdot \underset{1}{*} \cdot)$ is defined by (3), $\overset{\gamma}{*}(\cdot, \cdot, \cdot)$ is defined by (6), and $l \in L_1(\mathbb{R}_+)$ is defined as

$$(F_c l)(y) = \frac{\sin^2 y(F_s g)(y)(F_s\varphi)(y)(K_{iy}h)(K_{iy}\psi)}{1-\sin^2 y(F_s g)(y)(F_s\varphi)(y)(K_{iy}h)(K_{iy}\psi)}. \tag{29}$$

Proof. The system (27) with kernels defined by (28) can be rewritten in the following form

$$f(x) + \big(\overset{\gamma}{*}(g,h,\xi)\big)(x) = p(x),\ x>0,$$
$$\big(\overset{\gamma}{*}(\varphi,\psi,f)\big)(x) + \xi(x) = q(x).$$

Using Theorem 3.2 we have

$$(F_s f)(y) + \sin y(F_s g)(y)(K_{iy}h)(F_s\xi)(y) = (F_s p)(y),$$
$$\sin y(F_s\varphi)(y)(K_{iy}\psi)(F_s f)(y) + (F_s\xi)(y) = (F_s q)(y),\ y>0.$$

Using Theorem 3.2 and Lemma 4.2 we have

$$\varDelta = \begin{vmatrix} 1 & \sin y(F_s g)(y)(K_{iy}h) \\ \sin y(F_s\varphi)(y)(K_{iy}\psi & 1 \end{vmatrix} = 1 - F_c(\overset{\gamma}{*}(\varphi,g,h)\underset{2}{\overset{\gamma}{*}}\psi)(y).$$

By virtue of Wiener-Levy theorem [5], there exists a unique function $l \in L_1(\mathbb{R}_+)$ satisfying (29), it shows that

$$\frac{1}{\varDelta} = 1 + (F_c l)(y). \tag{30}$$

On the other hand, from Theorem 3.2 we get

$$\varDelta_1 = \begin{vmatrix} (F_s p)(y) & \sin y(F_s g)(y)(K_{iy}h) \\ (F_s q)(y) & 1 \end{vmatrix} = (F_s p)(y) - F_s(\overset{\gamma}{*}(q,g,h))(y). \tag{31}$$

Therefore, from (30), (31) and (3) we obtain

$$\begin{aligned}(F_s f)(y) &= (1+(F_c l)(y))((F_s p)(y) - F_s(\overset{\gamma}{*}(q,g,h))(y))\\ &= (F_s p)(y) + F_s(p\underset{1}{*}l)(y) - F_s(\overset{\gamma}{*}(q,g,h))(y) - F_s(\overset{\gamma}{*}(q,g,h)\underset{1}{*}l)(y).\end{aligned}$$

This implies that

$$f(x) = p(x) + (p\underset{1}{*}l)(x) - \overset{\gamma}{*}(q,g,h)(x) - (\overset{\gamma}{*}(q,g,h)\underset{1}{*}l)(x),\quad x>0. \tag{32}$$

Similarly,

$$\varDelta_2 = (F_s q)(y) - F_s(\overset{\gamma}{*}(p,\varphi,\psi))(y). \tag{33}$$

From (30), (33) and (3) we have

$$\begin{aligned}(F_s\xi)(y) &= (1+(F_c l)(y))((F_s q)(y) - F_s(\overset{\gamma}{*}(p,\varphi,\psi))(y))\\ &= (F_s q)(y) + F_s(q\underset{1}{*}l)(y) - F_s(\overset{\gamma}{*}(p,\varphi,\psi))(y) - F_s(\overset{\gamma}{*}(p,\varphi,\psi)\underset{1}{*}l)(y).\end{aligned}$$

Therefore

$$\xi(x) = q(x) + (q \underset{1}{*} l)(x) - \overset{\gamma}{*}(p,\varphi,\psi)(x) - (\overset{\gamma}{*}(p,\varphi,\psi) \underset{1}{*} l)(x), \ \ x > 0. \tag{34}$$

From the hypothesis, and since $l \in L_1(\mathbb{R}_+)$, we can easily see that f and ξ belong to $L_1(\mathbb{R}_+)$. The pair (f, g) defined by (32) and (34) is the solution in a closed form in $L_1(\mathbb{R}_+) \times L_1(\mathbb{R}_+)$ of system (27) with kernels defined by (28). ■

5.2. We now consider the system of three integral equations with Toeplitz plus Hankel kernels

$$\begin{aligned}
f(x) + \int_0^\infty g(u)\Big(\sum_{i=1}^4 (k_{1i}(x+u) - k_{2i}(x-u))(-1)^{\frac{i-1}{2}i}\Big)du &= p(x), \ \ x > 0,\\
g(x) + \int_0^\infty h(u)\Big(\sum_{i=1}^4 (k_{3i}(x+u) - k_{4i}(x-u))(-1)^{\frac{i-1}{2}i}\Big)du &= q(x), \ \ x > 0,\\
h(x) + \int_0^\infty f(u)\Big(\sum_{i=1}^4 (k_{5i}(x+u) - k_{6i}(x-u))(-1)^{\frac{i-1}{2}i}\Big)du &= r(x), \ \ x > 0.
\end{aligned} \tag{35}$$

Here f, g, h are unknown functions, $p, q, r, \varphi_i, \psi_i$ $(i = 1, 2, 3)$ are given functions. Systems of three integral equations in the general case are still open.

With the same techniques as in the proof of Theorem 5.1, we obtain the solution of system (35) in the case

$$k_{2i} \equiv -k_{1i}, k_{4i} \equiv -k_{3i}, k_{6i} \equiv -k_{5i}, \ i = 1, ..., 4, \tag{36}$$

where k_{1i} $(i = 1, ..., 4)$ are respectively defined as in (23) in the case $h \equiv \varphi_1, g \equiv \psi_1$;
k_{3i} $(i = 1, ..., 4)$ are respectively defined as in (23) in the case $h \equiv \varphi_2, g \equiv \psi_2$;
k_{5i} $(i = 1, ..., 4)$ are respectively defined as in (23) in the case $h \equiv \varphi_3, g \equiv \psi_3$.

Theorem 5.2. *Let $\varphi_i, i = 1, 2, 3,$ p, q, r be functions in $L_1(\mathbb{R}_+)$, and ψ_i, $i = 1, 2, 3$ be functions in $L_1^{0,\beta}(\mathbb{R}_+)$ such that*

$$1 - \sin^3 y (F_s\varphi_1)(y)(F_s\varphi_2)(y)(F_s\varphi_3)(y)(K_{iy}\psi_1)(K_{iy}\psi_2)(K_{iy}\psi_3) \neq 0,$$

then the system (35) *with kernels defined by* (36) *has a unique solution* (f, g, h) *in* $L_1(\mathbb{R}_+) \times L_1(\mathbb{R}_+) \times L_1(\mathbb{R}_+)$ *and the solution can be expressed in a closed form as follows*

$$\begin{aligned}
f(x) = p(x) &+ (p \underset{1}{*} \eta)(x) - \overset{\gamma}{*}(r,\varphi_1,\psi_1)(x) - (\overset{\gamma}{*}(r,\varphi_1,\psi_1) \underset{1}{*} \eta)(x)\\
&+ (\overset{\gamma}{*}(\overset{\gamma}{*}(r,\varphi_1,\psi_1),\varphi_2,\psi_2)(x) + (\overset{\gamma}{*}(\overset{\gamma}{*}(r,\varphi_1,\psi_1),\varphi_2,\psi_2) \underset{1}{*} \eta)(x), \ x > 0,\\
g(x) = q(x) &+ (q \underset{1}{*} \eta)(x) - \overset{\gamma}{*}(r,\varphi_2,\psi_2)(x) - (\overset{\gamma}{*}(r,\varphi_2,\psi_2) \underset{1}{*} \eta)(x)
\end{aligned}$$

$$+ (\overset{\gamma}{*}(\overset{\gamma}{*}(r, \varphi_2, \psi_2), \varphi_3, \psi_3)(x) + (\overset{\gamma}{*}(\overset{\gamma}{*}(r, \varphi_2, \psi_2), \varphi_3, \psi_3) \underset{1}{*} \eta)(x), \ x > 0,$$

$$h(x) = r(x) + (r \underset{1}{*} \eta)(x) - \overset{\gamma}{*}(p, \varphi_3, \psi_3)(x) - (\overset{\gamma}{*}(p, \varphi_3, \psi_3) \underset{1}{*} \eta)(x)$$
$$+ (\overset{\gamma}{*}(\overset{\gamma}{*}(q, \varphi_1, \psi_1), \varphi_3, \psi_3)(x) + (\overset{\gamma}{*}(\overset{\gamma}{*}(q, \varphi_1, \psi_1), \varphi_3, \psi_3) \underset{1}{*} \eta)(x), \ x > 0.$$

Here, $(\cdot \underset{1}{*} \cdot)$ *is defined by* (3), $\overset{\gamma}{*}(\cdot, \cdot, \cdot)$ *is defined by* (6), *and* $\eta \in L_1(\mathbb{R}_+)$ *is defined by*

$$(F_c\eta)(y) = \frac{\sin^3 y \prod_{i=1}^{3} (F_s\varphi_i)(y)(K_{iy}\psi)}{\sin^3 y \prod_{i=1}^{3} (F_s\varphi_i)(y)(K_{iy}\psi)}.$$

Remark 5.3. Similar results for the polyconvolution (6) with weight functions $\gamma(x) = \sin \xi x, \ \xi \in \mathbb{R}/\{0\}$ can be obtained easily.

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References

1. Adams R. A. and Fournier J. J. F., *Sobolev Spaces,* 2nd ed., Academic Press, New York, Amsterdam, Elsevier Science, 2003.
2. A. Erdelyi, W. Magnus, F. Oberhettinger, and F. G. Tricomi, *Higher Transcendental Functions*, vol. 1, 2, 3, McGrawHill, New York, NY, USA, 1953.
3. A. Erdelyi, H. Bateman, *Table of Integral Transforms.* McGraw - Hill, New York, 1954.
4. M. G. Krein, On a new method for solving linear integral equations of the first and second kinds. *Dokl. Akad. Nauk SSSR* (*N.S.*), **100** (1955), 413-416 (in Russian).
5. Paley R.C. and Wiener N., *Fourier Transforms in the Complex Domain,* Amer. Math. Soc, 1949.
6. F. Garcia-Vicente, J. M. Delgado and C. Peraza, Experimental determination of the convolution kernel for the study of spatial response of a detector. *Med. Phys.*, **25** (1998), 202-207.
7. F. Garcia-Vicente, J. M. Delgado and C. Rodriguez, Exact analytical solution of the convolution integral equation for a general profile fitting function and Gaussian detector kernel. *Phys. Med. and Rick.*, **45**(2000), 645-650.
8. V. A. Kakichev, On the convolution for integral transforms. *Izv. Vyssh. Uchebn. Zaved. Mat.*, No. 2 (1967), 53-62. (in Russian).
9. V. A. Kakichev, *Polyconvolution*, Taganrog, TPTU, 54p, 1997. (in Russian).
10. A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev, *Integrals and Series. Vol. 2, Special Functions,* Gordon and Breach Science Publishers, New York and London, 1986.
11. S. Saitoh, Inequalities in most simple Sobolev space and convolutions of L_2 function with weight. *Proc. Amer. Math. Soc.*, **118** (1993), 515-520.
12. S. Saitoh, V. K. Tuan and M. Yamamoto, Convolution inequalities and applications. *Journal of Inequalities in Pure and Applied Mathematics*, **4**(3) (2003), Article 50. [ONLINE: http://jipam.vu.edu.au/u/v4n3/ 138-02 .html].
13. I. N. Sneddon, *Fourier Transform*, Mc. Gray Hill, New York, 1951.

14. I.N. Sneddon, *The Use of Integral Transforms*, McGray-Hill, New York, 1972.
15. H. M. Srivastava and V.K. Tuan, A new convolution theorem for the Stieltjes transform and its application to a class of singular integral equation. *Arch. Math.*, **64** (1995), 144-149.
16. N.X. Thao, V.K. Tuan, N.T. Hong, On the Toeplitz plus Hankel integral equations, Int. Tran. & Spec. Func. (2011) (To appear).
17. V.K. Tuan, Integral transforms of Fourier cosine convolution type, *J. Math. Anal. Appl.*, **229** (1999), 519-529.
18. T. Tuan, On the generalized convolution with a weight function for the Fourier cosine and the inverse Kontorovich-Lebedev integral transformations, *Nonlinear Func. Anal. Appl.*, **12** (2007), 325-341.
19. T. Tuan, N. X. Thao and N. V. Mau, On the generalized convolution for the Fourier sine and the Kontorovich-Lebedev transforms, *Acta Math. Vietnam.*, **35** (2010), 303-317.
20. J.N. Tsitsiklis and B.C. Levy, Integral equations and resolvents of Toeplitz plus Hankel kernels. *Laboratory for Information and Decision Systems, Massachusetts Institute of Technology*, Series/Report No.: LIDS-P 1170, 1981.
21. S. B. Yakubovich and L. E. Britvina, Convolution related to the Fourier and Kontorovich-Lebedev transforms revisited. *Integral Transforms and Special Functions*, **21** (2010), 259-276.