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An Observation on Definable Bi-Lipschitz Homeomorphism

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Abstract. This short note gives an observation on the directional derivatives of bi-Lipschitz homeomorphisms that are definable in o-minimal structures and, as a consequence, it implies that the dimension of directional sets of definable sets is invariant under definable bi-Lipschitz homeomorphisms.

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1. Introduction

Let $A \subset \mathbb{R}^n$ be such that $0 \in \overline{A}$. Let S^{n-1} denote the unit sphere centered at 0 in $\mathbb{R}^n$. The *directional set of A at 0* is defined by

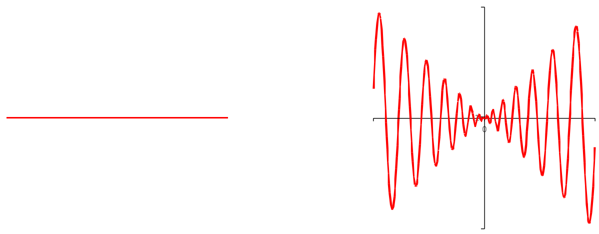
$$D(A) = \left\{ a \in S^{n-1} : \exists (x_k) \subset A \setminus \{0\}, x_k \to 0, \frac{x_k}{\|x_k\|} \to a, \text{when } k \to \infty \right\}.$$

Let $h : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ be a homeomorphism or a bi-Lipschitz homeomorphism. We consider the relation between $D(A)$ and $D(h(A))$.

First let us examine some examples (c.f. [7]).

Example 1.1. Let $h : \mathbb{R}^3 \to \mathbb{R}^3$, $h(x,y,z) = (x,y,z^3)$, and $A = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 - z^6 = 0\}$. Then h is a polynomial homeomorphism, A and $h(A) = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 - z^2 = 0\}$ are algebraic sets. It is easy to see that $\dim D(A) = 0, \dim D(h(A)) = 1$.

Example 1.2 (Oscillation). Let $h : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$, $h(x, y) = (x, y + f(x))$, where $f(x) = x \sin(\ln |x|)$, and $A = \mathbb{R} \times 0$. Then h is a bi-Lipschitz homeomorphism and $h(A)$ is the graph of f. In this case we have $\dim D(A) = 0, \dim(D(h(A)) = 1$.

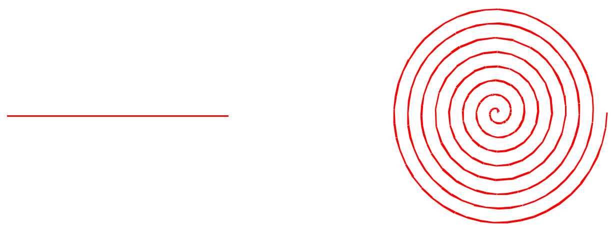


dimD(A)=0 dimD(h(A))=1

Example 1.3 (Spiral). Let $h : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ be the map defined by

$$\begin{aligned} h_1(x, y) &= \ \ x \cos(\ln \sqrt{x^2 + y^2}) + y \sin(\ln \sqrt{x^2 + y^2}), \\ h_2(x, y) &= -x \sin(\ln \sqrt{x^2 + y^2}) + y \cos(\ln \sqrt{x^2 + y^2}), \end{aligned}$$

in other words, $h(r, \theta) = (r, \theta - \ln r)$ in the polar coordinates. Then h is a bi-Lipschitz homeomorphism. Let A, B be two different segments with an end point at $0 \in \mathbb{R}^2$. Then we have $D(A) \cap D(B) = \emptyset$, and hence $\dim(D(A) \cap D(B)) = -1$. But $h(A), h(B)$ are the spirals, so $D(h(A)) = D(h(B)) = S^1$ and hence $\dim(D(h(A) \cap D(h(B)) = 1$.



dimD(A)=0 dimD(h(A))=1

The preceding examples show that the dimension of directional sets is not a homeomorphic nor a bi-Lipschitzian invariant. In this short note we give an observation on the directional derivatives of bi-Lipschitz homeomorphisms that are definable in o-minimal structures and, as a consequence, it implies that the dimension of directional sets of definable sets is invariant under definable bi-Lipschitz homeomorphisms. This result is motivated in [7] and relates to the invariants of bi-Lipschitz equivalence (see [9, 4, 5]).

2. o-minimal structures

An *o-minimal structure* on $\mathbb{R}$ is a sequence $\mathcal{D} = (\mathcal{D}_n)_{n\in\mathbb{N}}$ where each $\mathcal{D}_n$ is a Boolean algebra of subsets of $\mathbb{R}^n$ that contains all algebraic sets and such that $A \times B \in \mathcal{D}_{n+m}$ if $A \in \mathcal{D}_n, B \in \mathcal{D}_m$, and $\pi(A) \in \mathcal{D}_n$ if $\pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$ is the projection on the first n coordinates and $A \in \mathcal{D}_{n+1}$, and elements in $\mathcal{D}_1$ are precisely the finite unions of intervals and points. A set belonging to $\mathcal{D}$ is said to be *definable* (in that structure). *Definable maps* in structure $\mathcal{D}$ are maps whose graphs are definable sets in $\mathcal{D}$.

We refer the reader to [2, 3, 1, 8] for the basic properties of o-minimal structures used in this note. In particular, the class of semi-algebraic sets and the class of global sub-analytic sets are examples of such structures. We note that the dimension of definable sets is well defined. Moreover, we will use Motonicity [2, Chapter 3 (1.2)] and Curve selection [2, Chapter 6 (1.5)] in our arguments. In this note we fix an o-minimal structure on $\mathbb{R}$. "Definable" means definable in this structure.

Theorem 2.1. *Let $h : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ be a definable bi-Lipschitz homeomorphisc germ. Define $\bar{h} : S^{n-1} \to S^{n-1}$, by $\bar{h}(a) = \lim_{t\to 0^+} \frac{h(ta)}{\|h(ta)\|}$. Then*

(i) *$\bar{h}$ is well-defined and depends only on the direction of curves in the sense that if $\gamma : (0,1) \to \mathbb{R}^n$ is a definable curve with $\lim_{t\to 0^+} \gamma(t) = 0$ and $\lim_{t\to 0^+} \frac{\gamma(t)}{\|\gamma(t)\|} = a$, then $\lim_{t\to 0^+} \frac{h(\gamma(t))}{\|h(\gamma(t))\|} = \bar{h}(a)$.*

(ii) *$\bar{h}$ is a definable bi-Lipschitz homeomorphism.*

Proof. Let $r > 0$ and $l, L > 0$ be such that

$$l\|x - x'\| \leq \|h(x) - h(x')\| \leq L\|x - x'\|,$$

when $\|x\| \leq r, \|x'\| \leq r$.

(i) Let $a \in S^{n-1}$. Let $\gamma : (0,1) \to \mathbb{R}^n$ be a definable curve with $\lim_{t\to 0^+} \gamma(t) = 0$ and $\lim_{t\to 0^+} \frac{\gamma(t)}{\|\gamma(t)\|} = a$. First, note that, by Monotonicity, $\lim_{t\to 0^+} \frac{h(\gamma(t))}{\|h(\gamma(t))\|}$ exists and hence $\bar{h}$ is well-defined. For $k \in \mathbb{N}, k \gg 1$, take $t_k \in (0,1)$ such that $\|\gamma(t_k)\| = \frac{1}{k}$. Then $t_k \to 0$, when $k \to \infty$, and

$$\left\|\frac{h(\gamma(t_k))}{\|h(\gamma(t_k))\|} - \frac{h(\frac{1}{k}a)}{\|h(\frac{1}{k}a)\|}\right\| = \frac{\|h(\gamma(t_k)) - h(\frac{1}{k}a)\|}{\|h(\frac{1}{k}a)\|}$$
$$\leq \frac{k}{l}L\|\gamma(t_k) - \tfrac{1}{k}a\| \leq \frac{L}{l}\|k\gamma(t_k) - a\|.$$

Since $k\gamma(t_k) = k\|\gamma(t_k)\|\dfrac{\gamma(t_k)}{\|\gamma(t_k)\|} \to a$, when $k \to \infty$, we have

$$\lim_{t\to 0^+} \frac{h(\gamma(t))}{\|h(\gamma(t))\|} = \lim_{t\to 0^+} \frac{h(ta)}{\|h(ta)\|} = \bar{h}(a).$$

(ii) It is easy to check that $\bar{h}$ is definable and bijective with $(\bar{h})^{-1} = \overline{h^{-1}}$. It remains to prove that $\bar{h}$ is Lipschitzian. Let $a, b \in S^{n-1}$, then for $t > 0$, we have

$$\left\|\frac{h(ta))}{\|h(ta)\|} - \frac{h(tb)}{\|h(tb)\|}\right\| \leq \frac{\|h(ta) - h(tb)\|}{\min(\|h(ta)\|, \|h(tb)\|)}$$
$$\leq \frac{L\|ta - tb\|}{\min(l\|ta\|, l\|tb\|)} \leq \frac{L}{l}\|a - b\|.$$

Letting $t \to 0^+$, we get $\|\bar{h}(a) - \bar{h}(b)\| \leq \dfrac{L}{l}\|a - b\|$. ∎

Note. If h is a definable bi-Lipschitz homeomorphism at 0, then, by Monotonicity, the directional derivative of h at 0 corresponding to the direction $a \in S^{n-1}$, $D_a h(0) = \lim\limits_{t\to 0^+} \dfrac{h(ta)}{t}$ exists. Therefore, $\bar{h}(a) = \dfrac{D_a h(0)}{\|Dh_a(0)\|}$ is the direction of the directional derivative $D_a h(0)$. The bi-Lipschitz homeomorphisms h given in Examples 1.1 and 1.2 are not definable in any structure and $\bar{h}$ are not defined.

Proposition 2.2. *If A is a germ at 0 in $\mathbb{R}^n$, then $D(A) = D(\overline{A})$ is a closed subset of S^{n-1}.*

Proof. Let $a \in D(\overline{A})$. Then there exists a sequence (x_k) in $\overline{A}$, such that $\dfrac{x_k}{\|x_k\|} \to a$ when $k \to \infty$. Choose $a_k \in A$ such that $\|a_k - x_k\| \ll \|x_k\|$. We have

$$\left\|\frac{a_k}{\|a_k\|} - \frac{x_k}{\|x_k\|}\right\| \leq \frac{\|a_k - x_k\|}{\min(\|a_k\|, \|x_k\|)} \to 0$$

when $k \to \infty$. Hence

$$a = \lim_{k\to\infty} \frac{a_k}{\|a_k\|} \in D(A).$$

Similarly, to prove $D(A) = \overline{D(A)}$, let $a \in \overline{D(A)}$. Then there exists a sequence (a_k) in $D(A)$, such that $a_k \to a$ when $k \to \infty$. For each k, there exists a sequence $(b_{k,m})$ in $A \setminus \{0\}$ such that $b_{k,m} \to 0$ and $\dfrac{b_{k,m}}{\|b_{k,m}\|} \to a_k$, when $m \to \infty$. We can choose a subsequence (m_k) of (m) such that $\left\|\dfrac{b_{k,m_k}}{\|b_{k,m_k}\|} - a_k\right\| < \dfrac{1}{k}$. We have

$c_k = b_{k,m_k} \in A \setminus \{0\}$, $\dfrac{c_k}{\|c_k\|} \to a$, and hence $a \in D(A)$. So $D(A)$ is a closed set. ∎

Note. When A is a definable set, using the interpretation of the logical symbols in terms of operations on sets, one can check that $D(A)$ is a definable set. Moreover, by Curve selection, $a \in D(A)$ if and only if there exists a definable curve $\gamma : (0,1) \to \mathbb{R}^n$ with $\lim_{t\to 0^+} \gamma(t) = 0$ and $\lim_{t\to 0^+} \dfrac{\gamma(t)}{\|\gamma(t)\|} = a$. From this we get

Corollary 2.3. *Let A, B be definable set-germs at 0 in $\mathbb{R}^n$ such that $0 \in \overline{A} \cap \overline{B}$. Let $h : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ be a definable bi-Lipschitz homeomorphism. Then $\bar{h} : (S^{n-1}, D(A)) \to (S^{n-1}, D(h(A)))$ is a bi-Lipschitz homepmorphism. In particular,*

$$\dim(D(h(A)) \cap D(h(B))) = \dim(D(A) \cap D(B)).$$

Remark 2.4. Dropping the supposition of definability of h but assuming that $h(A), h(B)$ are definable, we still have $\dim(D(h(A)) \cap D(h(B))) = \dim(D(A) \cap D(B))$. This is a generalization of the main theorem in [7] but the proof requires much more effort than that of the corollary. Note that bi-Lipschitz equivalence does not always imply definable one. In fact, Shiota constructs an example of two compact polyhedra that are bi-Lipschitz homeomorphic but not definably homeomorphic in any o-minimal structure. These results are in preparation (see [6]).

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