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# On Strongly Regular Graphs of Order $6(2p+1)$ where $2p+1$ is a Prime Number

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**Abstract.** We say that a regular graph $G$ of order $n$ and degree $r \geq 1$ (which is not a complete graph) is strongly regular if there exist non-negative integers $\tau$ and $\theta$ such that $|S_i \cap S_j| = \tau$ for any two adjacent vertices $i$ and $j$, and $|S_i \cap S_j| = \theta$ for any two distinct non-adjacent vertices $i$ and $j$, where $S_k$ denotes the neighborhood of the vertex $k$. We here describe the parameters $n$, $r$, $\tau$ and $\theta$ for strongly regular graphs of order $6(2p+1)$, where $2p+1$ is a prime number.

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## 1. Introduction

Let $G$ be a simple graph of order $n$. The spectrum of $G$ consists of the eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ of its (0,1) adjacency matrix $A$ and is denoted by $\sigma(G)$. We say that a regular graph $G$ of order $n$ and degree $r \geq 1$ (which is not the complete graph $K_n$) is strongly regular if there exist non-negative integers $\tau$ and $\theta$ such that $|S_i \cap S_j| = \tau$ for any two adjacent vertices $i$ and $j$, and $|S_i \cap S_j| = \theta$ for any two distinct non-adjacent vertices $i$ and $j$, where $S_k$ denotes the neighborhood of the vertex $k$. We say that a regular connected graph $G$ is strongly regular if and only if it has exactly three distinct eigenvalues [1]. Let $\lambda_1 = r$, $\lambda_2$ and $\lambda_3$ denote the distinct eigenvalues of $G$. Let $m_1 = 1$, $m_2$ and $m_3$ denote the multiplicity of $r$, $\lambda_2$ and $\lambda_3$, respectively.

**Theorem 1.1.** [2] *Let $G$ be a connected strongly regular graph of order $n$ and degree $r$. Then $m_2m_3\delta^2 = nr\overline{r}$, where $\delta = \lambda_2 - \lambda_3$ and $\overline{r} = (n-1) - r$.*

**Remark 1.2.** Let $\overline{r} = (n-1) - r$, $\overline{\lambda_2} = -\lambda_3 - 1$ and $\overline{\lambda_3} = -\lambda_2 - 1$ denote the distinct eigenvalues of the strongly regular graph $\overline{G}$, where $\overline{G}$ denotes the complement of $G$. Then $\overline{\tau} = n - 2r - 2 + \theta$ and $\overline{\theta} = n - 2r + \tau$, where $\overline{\tau} = \tau(\overline{G})$ and $\overline{\theta} = \theta(\overline{G})$.

**Remark 1.3.** (i) A strongly regular graph $G$ of order $4n+1$ and degree $r = 2n$ with $\tau = n-1$ and $\theta = n$ is called the conference graph; (ii) a strongly regular graph is the conference graph if and only if $m_2 = m_3$ and (iii) if $m_2 \neq m_3$ then $G$ is an integral[1] graph.

**Remark 1.4.** (i) If $G$ is a disconnected strongly regular graph of degree $r$ then $G = mK_{r+1}$, where $mH$ denotes the $m$-fold union of the graph $H$; (ii) $G$ is a disconnected strongly regular graph if and only if $\theta = 0$.

Due to Theorem 1.1 we have recently obtained the following results [2]: (i) there is no strongly regular graph of order $4p+3$ if $4p+3$ is a prime number; (ii) the only strongly regular graphs of order $4p+1$ are conference graphs if $4p+1$ is a prime number. Beside [2, 3, 4], we have described the parameters $n$, $r$, $\tau$ and $\theta$ for strongly regular graphs of order $2(2p+1)$, $3(2p+1)$, $4(2p+1)$ and $5(2p+1)$, where $2p+1$ is a prime number. We now proceed to establish the parameters of strongly regular graphs of order $6(2p+1)$ where $2p+1$ is a prime number, as follows. First,

**Proposition 1.5.** [1] *Let $G$ be a connected or disconnected strongly regular graph of order $n$ and degree $r$. Then*

$$r^2 - (\tau - \theta + 1)r - (n-1)\theta = 0. \tag{1}$$

**Proposition 1.6.** [1] *Let $G$ be a connected strongly regular graph of order $n$ and degree $r$. Then*

$$2r + (\tau - \theta)(m_2 + m_3) + \delta(m_2 - m_3) = 0, \tag{2}$$

*where $\delta = \lambda_2 - \lambda_3$.*

Second, in what follows $(x, y)$ denotes the greatest common divisor of integers $x, y \in \mathbb{N}$ while $x \mid y$ means that $x$ divides $y$.

## 2. Main Results

**Remark 2.1.** a) The connected strongly regular graphs of order 18 are (i) the complete bipartite graph $K_{9,9}$ of degree $r = 9$ with $\tau = 0$ and $\theta = 9$. Its

[1] We say that a connected or disconnected graph $G$ is integral if its spectrum $\sigma(G)$ consists of integral values.

eigenvalues are $\lambda_2 = 0$ and $\lambda_3 = -9$ with $m_2 = 16$ and $m_3 = 1$; (ii) the strongly regular graph $\overline{3K_6}$ of degree $r = 12$ with $\tau = 6$ and $\theta = 12$. Its eigenvalues are $\lambda_2 = 0$ and $\lambda_3 = -6$ with $m_2 = 15$ and $m_3 = 2$; (iii) the strongly regular graph $\overline{6K_3}$ of degree $r = 15$ with $\tau = 12$ and $\theta = 15$. Its eigenvalues are $\lambda_2 = 0$ and $\lambda_3 = -3$ with $m_2 = 12$ and $m_3 = 5$ and (iv) the cocktail-party graph $\overline{9K_2}$ of degree $r = 16$ with $\tau = 14$ and $\theta = 16$. Its eigenvalues are $\lambda_2 = 0$ and $\lambda_3 = -2$ with $m_2 = 9$ and $m_3 = 8$.

b) Since the strongly regular graphs of order $n = 18$ are completely described, in the sequel it will be assumed that $p \geq 2$.

c) In Theorem 2.2 the complements of strongly regular graphs appear in pairs in $(k^0)$ and $(\overline{k}^0)$ classes, where $k$ denotes the corresponding number of a class.

**Theorem 2.2.** *Let $G$ be a connected strongly regular graph of order $6(2p+1)$ and degree $r$, where $2p+1$ is a prime number. Then $G$ is one of the following strongly regular graphs:*

($1^0$) *$G$ is the complete bipartite graph $K_{6p+3,6p+3}$ of order $n = 6(2p+1)$ and degree $r = 6p+3$ with $\tau = 0$ and $\theta = 6p+3$, where $p \in \mathbb{N}$ and $2p+1$ is a prime number. Its eigenvalues are $\lambda_2 = 0$ and $\lambda_3 = -(6p+3)$ with $m_2 = 12p+4$ and $m_3 = 1$;*

($2^0$) *$G$ is the strongly regular graph $\overline{3K_{4p+2}}$ of order $n = 6(2p+1)$ and degree $r = 8p+4$ with $\tau = 4p+2$ and $\theta = 8p+4$, where $p \in \mathbb{N}$ and $2p+1$ is a prime number. Its eigenvalues are $\lambda_2 = 0$ and $\lambda_3 = -2(2p+1)$ with $m_2 = 12p+3$ and $m_3 = 2$;*

($3^0$) *$G$ is the strongly regular graph $\overline{6K_{2p+1}}$ of order $n = 6(2p+1)$ and degree $r = 10p+5$ with $\tau = 8p+4$ and $\theta = 10p+5$, where $p \in \mathbb{N}$ and $2p+1$ is a prime number. Its eigenvalues are $\lambda_2 = 0$ and $\lambda_3 = -(2p+1)$ with $m_2 = 12p$ and $m_3 = 5$;*

($4^0$) *$G$ is the strongly regular graph $\overline{(2p+1)K_6}$ of order $n = 6(2p+1)$ and degree $r = 12p$ with $\tau = 12p-6$ and $\theta = 12p$, where $p \in \mathbb{N}$ and $2p+1$ is a prime number. Its eigenvalues are $\lambda_2 = 0$ and $\lambda_3 = -6$ with $m_2 = 5(2p+1)$ and $m_3 = 2p$;*

($5^0$) *$G$ is the strongly regular graph $\overline{(4p+2)K_3}$ of order $n = 6(2p+1)$ and degree $r = 12p+3$ with $\tau = 12p$ and $\theta = 12p+3$, where $p \in \mathbb{N}$ and $2p+1$ is a prime number. Its eigenvalues are $\lambda_2 = 0$ and $\lambda_3 = -3$ with $m_2 = 4(2p+1)$ and $m_3 = 4p+1$ ;*

($6^0$) *$G$ is the cocktail-party graph $\overline{(6p+3)K_2}$ of order $n = 6(2p+1)$ and degree $r = 12p+4$ with $\tau = 12p+2$ and $\theta = 12p+4$, where $p \in \mathbb{N}$ and $2p+1$ is a prime number. Its eigenvalues are $\lambda_2 = 0$ and $\lambda_3 = -2$ with $m_2 = 3(2p+1)$ and $m_3 = 6p+2$ ;*

($7^0$) *$G$ is the strongly regular graph of order $n = 6(6k^2+6k+1)$ and degree $r = k(6k+1)$ with $\tau = k^2-4k-1$ and $\theta = k^2$, where $k \geq 5$ and $6k^2+6k+1$ is a prime number. Its eigenvalues are $\lambda_2 = k$ and $\lambda_3 = -(5k+1)$ with $m_2 = 5(6k^2+6k+1)$ and $m_3 = 6k(k+1)$;*

($\overline{7}^0$) *G is the strongly regular graph of order* $n = 6(6k^2 + 6k + 1)$ *and degree* $r = 5(k+1)(6k+1)$ *with* $\tau = 25k^2 + 34k + 4$ *and* $\theta = 5(k+1)(5k+1)$, *where* $k \geq 5$ *and* $6k^2 + 6k + 1$ *is a prime number. Its eigenvalues are* $\lambda_2 = 5k$ *and* $\lambda_3 = -(k+1)$ *with* $m_2 = 6k(k+1)$ *and* $m_3 = 5(6k^2 + 6k + 1)$;

($8^0$) *G is the strongly regular graph of order* $n = 6(6k^2 + 6k + 1)$ *and degree* $r = (k+1)(6k+5)$ *with* $\tau = k^2 + 6k + 4$ *and* $\theta = (k+1)^2$, *where* $k \in \mathbb{N}$ *and* $6k^2 + 6k + 1$ *is a prime number. Its eigenvalues are* $\lambda_2 = 5k + 4$ *and* $\lambda_3 = -(k+1)$ *with* $m_2 = 6k(k+1)$ *and* $m_3 = 5(6k^2 + 6k + 1)$;

($\overline{8}^0$) *G is the strongly regular graph of order* $n = 6(6k^2 + 6k + 1)$ *and degree* $r = 5k(6k+5)$ *with* $\tau = 25k^2 + 16k - 5$ *and* $\theta = 5k(5k+4)$, *where* $k \in \mathbb{N}$ *and* $6k^2 + 6k + 1$ *is a prime number. Its eigenvalues are* $\lambda_2 = k$ *and* $\lambda_3 = -(5k+5)$ *with* $m_2 = 5(6k^2 + 6k + 1)$ *and* $m_3 = 6k(k+1)$;

($9^0$) *G is the strongly regular graph of order* $n = 6(30k^2 - 10k + 1)$ *and degree* $r = 5(3k-1)(6k-1)$ *with* $\tau = (3k-2)(15k-2)$ *and* $\theta = 3(3k-1)(5k-1)$, *where* $k \in \mathbb{N}$ *and* $30k^2 - 10k + 1$ *is a prime number. Its eigenvalues are* $\lambda_2 = 3k - 1$ *and* $\lambda_3 = -(15k-2)$ *with* $m_2 = 5(30k^2 - 10k + 1)$ *and* $m_3 = 10k(3k-1)$;

($\overline{9}^0$) *G is the strongly regular graph of order* $n = 6(30k^2 - 10k + 1)$ *and degree* $r = 15k(6k-1)$ *with* $\tau = 3(3k+1)(5k-1)$ *and* $\theta = 3k(15k-2)$, *where* $k \in \mathbb{N}$ *and* $30k^2 - 10k + 1$ *is a prime number. Its eigenvalues are* $\lambda_2 = 15k - 3$ *and* $\lambda_3 = -3k$ *with* $m_2 = 10k(3k-1)$ *and* $m_3 = 5(30k^2 - 10k + 1)$;

($10^0$) *G is the strongly regular graph of order* $n = 6(30k^2 - 10k + 1)$ *and degree* $r = 3(5k-1)(6k-1)$ *with* $\tau = 3k(15k-4)$ *and* $\theta = 3(3k-1)(5k-1)$, *where* $k \in \mathbb{N}$ *and* $30k^2 - 10k + 1$ *is a prime number. Its eigenvalues are* $\lambda_2 = 15k - 3$ *and* $\lambda_3 = -3k$ *with* $m_2 = 30k^2 - 10k + 1$ *and* $m_3 = 2(5k-1)(15k-2)$;

($\overline{10}^0$) *G is the strongly regular graph of order* $n = 6(30k^2 - 10k + 1)$ *and degree* $r = (6k-1)(15k-2)$ *with* $\tau = (3k-1)(15k-1)$ *and* $\theta = 3k(15k-2)$, *where* $k \in \mathbb{N}$ *and* $30k^2 - 10k + 1$ *is a prime number. Its eigenvalues are* $\lambda_2 = 3k - 1$ *and* $\lambda_3 = -(15k-2)$ *with* $m_2 = 2(5k-1)(15k-2)$ *and* $m_3 = 30k^2 - 10k + 1$;

($11^0$) *G is the strongly regular graph of order* $n = 6(30k^2 + 10k + 1)$ *and degree* $r = 15k(6k+1)$ *with* $\tau = 3(3k-1)(5k+1)$ *and* $\theta = 3k(15k+2)$, *where* $k \in \mathbb{N}$ *and* $30k^2 + 10k + 1$ *is a prime number. Its eigenvalues are* $\lambda_2 = 3k$ *and* $\lambda_3 = -(15k+3)$ *with* $m_2 = 5(30k^2 + 10k + 1)$ *and* $m_3 = 10k(3k+1)$;

($\overline{11}^0$) *G is the strongly regular graph of order* $n = 6(30k^2 + 10k + 1)$ *and degree* $r = 5(3k+1)(6k+1)$ *with* $\tau = (3k+2)(15k+2)$ *and* $\theta = 3(3k+1)(5k+1)$, *where* $k \in \mathbb{N}$ *and* $30k^2 + 10k + 1$ *is a prime number. Its eigenvalues are* $\lambda_2 = 15k + 2$ *and* $\lambda_3 = -(3k+1)$ *with* $m_2 = 10k(3k+1)$ *and* $m_3 = 5(30k^2 + 10k + 1)$;

($12^0$) *G is the strongly regular graph of order* $n = 6(30k^2 + 10k + 1)$ *and degree* $r = (6k+1)(15k+2)$ *with* $\tau = (3k+1)(15k+1)$ *and* $\theta = 3k(15k+2)$, *where* $k \in \mathbb{N}$ *and* $30k^2 + 10k + 1$ *is a prime number. Its eigenvalues are* $\lambda_2 = 15k + 2$ *and* $\lambda_3 = -(3k+1)$ *with* $m_2 = 30k^2 + 10k + 1$ *and* $m_3 = 2(5k+1)(15k+2)$;

($\overline{12}^0$) *G is the strongly regular graph of order* $n = 6(30k^2 + 10k + 1)$ *and degree* $r = 3(5k+1)(6k+1)$ *with* $\tau = 3k(15k+4)$ *and* $\theta = 3(3k+1)(5k+1)$, *where*

*$k \in \mathbb{N}$ and $30k^2+10k+1$ is a prime number. Its eigenvalues are $\lambda_2 = 3k$ and $\lambda_3 = -(15k+3)$ with $m_2 = 2(5k+1)(15k+2)$ and $m_3 = 30k^2+10k+1$;*

($13^0$) *$G$ is the strongly regular graph of order $n = 6(96k^2-18k+1)$ and degree $r = (12k-1)(16k-1)$ with $\tau = 4k(16k-3)$ and $\theta = 4k(16k-1)$, where $k \in \mathbb{N}$ and $96k^2-18k+1$ is a prime number. Its eigenvalues are $\lambda_2 = 8k-1$ and $\lambda_3 = -(16k-1)$ with $m_2 = 3(8k-1)(16k-1)$ and $m_3 = 2(96k^2-18k+1)$;*

($\overline{13}^0$) *$G$ is the strongly regular graph of order $n = 6(96k^2-18k+1)$ and degree $r = 4(8k-1)(12k-1)$ with $\tau = 4(8k-1)^2+2(4k-1)$ and $\theta = 4(8k-1)^2$, where $k \in \mathbb{N}$ and $96k^2-18k+1$ is a prime number. Its eigenvalues are $\lambda_2 = 16k-2$ and $\lambda_3 = -8k$ with $m_2 = 2(96k^2-18k+1)$ and $m_3 = 3(8k-1)(16k-1)$;*

($14^0$) *$G$ is the strongly regular graph of order $n = 6(96k^2+18k+1)$ and degree $r = (12k+1)(16k+1)$ with $\tau = 4k(16k+3)$ and $\theta = 4k(16k+1)$, where $k \in \mathbb{N}$ and $96k^2+18k+1$ is a prime number. Its eigenvalues are $\lambda_2 = 16k+1$ and $\lambda_3 = -(8k+1)$ with $m_2 = 2(96k^2+18k+1)$ and $m_3 = 3(8k+1)(16k+1)$;*

($\overline{14}^0$) *$G$ is the strongly regular graph of order $n = 6(96k^2+18k+1)$ and degree $r = 4(8k+1)(12k+1)$ with $\tau = 4(8k+1)^2-2(4k+1)$ and $\theta = 4(8k+1)^2$, where $k \in \mathbb{N}$ and $96k^2+18k+1$ is a prime number. Its eigenvalues are $\lambda_2 = 8k$ and $\lambda_3 = -(16k+2)$ with $m_2 = 3(8k+1)(16k+1)$ and $m_3 = 2(96k^2+18k+1)$;*

($15^0$) *$G$ is the strongly regular graph of order $n = 6(150k^2-54k+5)$ and degree $r = (6k-1)(25k-4)$ with $\tau = 25k^2-24k+3$ and $\theta = k(25k-4)$, where $k \in \mathbb{N}$ and $150k^2-54k+5$ is a prime number. Its eigenvalues are $\lambda_2 = 5k-1$ and $\lambda_3 = -(25k-4)$ with $m_2 = 6(5k-1)(25k-4)$ and $m_3 = 150k^2-54k+5$;*

($\overline{15}^0$) *$G$ is the strongly regular graph of order $n = 6(150k^2-54k+5)$ and degree $r = 25(5k-1)(6k-1)$ with $\tau = 25(5k-1)^2+5(4k-1)$ and $\theta = 25(5k-1)^2$, where $k \in \mathbb{N}$ and $150k^2-54k+5$ is a prime number. Its eigenvalues are $\lambda_2 = 25k-5$ and $\lambda_3 = -5k$ with $m_2 = 150k^2-54k+5$ and $m_3 = 6(5k-1)(25k-4)$;*

($16^0$) *$G$ is the strongly regular graph of order $n = 6(150k^2+54k+5)$ and degree $r = (6k+1)(25k+4)$ with $\tau = 25k^2+24k+3$ and $\theta = k(25k+4)$, where $k \geq 0$ and $150k^2+54k+5$ is a prime number. Its eigenvalues are $\lambda_2 = 25k+4$ and $\lambda_3 = -(5k+1)$ with $m_2 = 150k^2+54k+5$ and $m_3 = 6(5k+1)(25k+4)$;*

($\overline{16}^0$) *$G$ is the strongly regular graph of order $n = 6(150k^2+54k+5)$ and degree $r = 25(5k+1)(6k+1)$ with $\tau = 25(5k+1)^2-5(4k+1)$ and $\theta = 25(5k+1)^2$, where $k \geq 0$ and $150k^2+54k+5$ is a prime number. Its eigenvalues are $\lambda_2 = 5k$ and $\lambda_3 = -(25k+5)$ with $m_2 = 6(5k+1)(25k+4)$ and $m_3 = 150k^2+54k+5$;*

($17^0$) *$G$ is the strongly regular graph of order $n = 6(240k^2-198k+41)$ and degree $r = (12k-5)(40k-17)$ with $\tau = 4(40k^2-29k+5)$ and $\theta = 2(2k-1)(40k-17)$, where $k \in \mathbb{N}$ and $240k^2-198k+41$ is a prime number. Its eigenvalues are $\lambda_2 = 40k-17$ and $\lambda_3 = -(8k-3)$ with $m_2 = 240k^2-198k+41$ and $m_3 = 6(5k-2)(40k-17)$;*

($\overline{17}^0$) *$G$ is the strongly regular graph of order $n = 6(240k^2-198k+41)$ and degree $r = 16(5k-2)(12k-5)$ with $\tau = 4(8k-3)(20k-9)$ and $\theta = 16(5k-2)(8k-3)$, where $k \in \mathbb{N}$ and $240k^2-198k+41$ is a prime number. Its eigenvalues are*

*$\lambda_2 = 8k-4$ and $\lambda_3 = -(40k-16)$ with $m_2 = 6(5k-2)(40k-17)$ and $m_3 = 240k^2-198k+41$;*

($18^0$) *$G$ is the strongly regular graph of order $n = 6(240k^2-102k+11)$ and degree $r = 4(5k-1)(24k-5)$ with $\tau = 2(80k^2-42k+5)$ and $\theta = 4(5k-1)(8k-1)$, where $k \in \mathbb{N}$ and $240k^2-102k+11$ is a prime number. Its eigenvalues are $\lambda_2 = 8k-2$ and $\lambda_3 = -(40k-8)$ with $m_2 = 6(5k-1)(40k-9)$ and $m_3 = 240k^2-102k+11$;*

($\overline{18}^0$) *$G$ is the strongly regular graph of order $n = 6(240k^2-102k+11)$ and degree $r = (24k-5)(40k-9)$ with $\tau = 4(4k-1)(40k-7)$ and $\theta = 4(4k-1)(40k-9)$, where $k \in \mathbb{N}$ and $240k^2-102k+11$ is a prime number. Its eigenvalues are $\lambda_2 = 40k-9$ and $\lambda_3 = -(8k-1)$ with $m_2 = 240k^2-102k+11$ and $m_3 = 6(5k-1)(40k-9)$;*

($19^0$) *$G$ is the strongly regular graph of order $n = 6(240k^2-30k+1)$ and degree $r = 5(8k-1)(12k-1)$ with $\tau = 4(40k^2-17k+1)$ and $\theta = 2(8k-1)(10k-1)$, where $k \in \mathbb{N}$ and $240k^2-30k+1$ is a prime number. Its eigenvalues are $\lambda_2 = 8k-1$ and $\lambda_3 = -(40k-3)$ with $m_2 = 5(240k^2-30k+1)$ and $m_3 = 30k(8k-1)$;*

($\overline{19}^0$) *$G$ is the strongly regular graph of order $n = 6(240k^2-30k+1)$ and degree $r = 80k(12k-1)$ with $\tau = 4(160k^2-4k-1)$ and $\theta = 16k(40k-3)$, where $k \in \mathbb{N}$ and $240k^2-30k+1$ is a prime number. Its eigenvalues are $\lambda_2 = 40k-4$ and $\lambda_3 = -8k$ with $m_2 = 30k(8k-1)$ and $m_3 = 5(240k^2-30k+1)$;*

($20^0$) *$G$ is the strongly regular graph of order $n = 6(240k^2-30k+1)$ and degree $r = 20k(24k-1)$ with $\tau = 2(80k^2+14k-1)$ and $\theta = 4k(40k-1)$, where $k \in \mathbb{N}$ and $240k^2-30k+1$ is a prime number. Its eigenvalues are $\lambda_2 = 40k-2$ and $\lambda_3 = -8k$ with $m_2 = 30k(8k-1)$ and $m_3 = 5(240k^2-30k+1)$;*

($\overline{20}^0$) *$G$ is the strongly regular graph of order $n = 6(240k^2-30k+1)$ and degree $r = 5(8k-1)(24k-1)$ with $\tau = 4(160k^2-36k+1)$ and $\theta = 4(8k-1)(20k-1)$, where $k \in \mathbb{N}$ and $240k^2-30k+1$ is a prime number. Its eigenvalues are $\lambda_2 = 8k-1$ and $\lambda_3 = -(40k-1)$ with $m_2 = 5(240k^2-30k+1)$ and $m_3 = 30k(8k-1)$;*

($21^0$) *$G$ is the strongly regular graph of order $n = 6(240k^2+30k+1)$ and degree $r = 20k(24k+1)$ with $\tau = 2(80k^2-14k-1)$ and $\theta = 4k(40k+1)$, where $k \in \mathbb{N}$ and $240k^2+30k+1$ is a prime number. Its eigenvalues are $\lambda_2 = 8k$ and $\lambda_3 = -(40k+2)$ with $m_2 = 5(240k^2+30k+1)$ and $m_3 = 30k(8k+1)$;*

($\overline{21}^0$) *$G$ is the strongly regular graph of order $n = 6(240k^2+30k+1)$ and degree $r = 5(8k+1)(24k+1)$ with $\tau = 4(160k^2+36k+1)$ and $\theta = 4(8k+1)(20k+1)$, where $k \in \mathbb{N}$ and $240k^2+30k+1$ is a prime number. Its eigenvalues are $\lambda_2 = 40k+1$ and $\lambda_3 = -(8k+1)$ with $m_2 = 30k(8k+1)$ and $m_3 = 5(240k^2+30k+1)$;*

($22^0$) *$G$ is the strongly regular graph of order $n = 6(240k^2+30k+1)$ and degree $r = 5(8k+1)(12k+1)$ with $\tau = 4(40k^2+17k+1)$ and $\theta = 2(8k+1)(10k+1)$, where $k \in \mathbb{N}$ and $240k^2+30k+1$ is a prime number. Its eigenvalues are $\lambda_2 = 40k+3$ and $\lambda_3 = -(8k+1)$ with $m_2 = 30k(8k+1)$ and $m_3 = 5(240k^2+30k+1)$;*

($\overline{22}^0$) *$G$ is the strongly regular graph of order $n = 6(240k^2 + 30k + 1)$ and degree $r = 80k(12k + 1)$ with $\tau = 4(160k^2 + 4k - 1)$ and $\theta = 16k(40k + 3)$, where $k \in \mathbb{N}$ and $240k^2 + 30k + 1$ is a prime number. Its eigenvalues are $\lambda_2 = 8k$ and $\lambda_3 = -(40k + 4)$ with $m_2 = 5(240k^2 + 30k + 1)$ and $m_3 = 30k(8k + 1)$;*

($23^0$) *$G$ is the strongly regular graph of order $n = 6(240k^2+102k+11)$ and degree $r = 4(5k+1)(24k+5)$ with $\tau = 2(80k^2+42k+5)$ and $\theta = 4(5k+1)(8k+1)$, where $k \geq 0$ and $240k^2 + 102k + 11$ is a prime number. Its eigenvalues are $\lambda_2 = 40k + 8$ and $\lambda_3 = -(8k + 2)$ with $m_2 = 240k^2 + 102k + 11$ and $m_3 = 6(5k + 1)(40k + 9)$;*

($\overline{23}^0$) *$G$ is the strongly regular graph of order $n = 6(240k^2+102k+11)$ and degree $r = (24k+5)(40k+9)$ with $\tau = 4(4k+1)(40k+7)$ and $\theta = 4(4k+1)(40k+9)$, where $k \geq 0$ and $240k^2 + 102k + 11$ is a prime number. Its eigenvalues are $\lambda_2 = 8k + 1$ and $\lambda_3 = -(40k + 9)$ with $m_2 = 6(5k + 1)(40k + 9)$ and $m_3 = 240k^2 + 102k + 11$;*

($24^0$) *$G$ is the strongly regular graph of order $n = 6(240k^2+198k+41)$ and degree $r = (12k+5)(40k+17)$ with $\tau = 4(40k^2+29k+5)$ and $\theta = 2(2k+1)(40k+17)$, where $k \geq 0$ and $240k^2 + 198k + 41$ is a prime number. Its eigenvalues are $\lambda_2 = 8k + 3$ and $\lambda_3 = -(40k + 17)$ with $m_2 = 6(5k + 2)(40k + 17)$ and $m_3 = 240k^2 + 198k + 41$;*

($\overline{24}^0$) *$G$ is the strongly regular graph of order $n = 6(240k^2+198k+41)$ and degree $r = 16(5k+2)(12k+5)$ with $\tau = 4(8k+3)(20k+9)$ and $\theta = 16(5k+2)(8k+3)$, where $k \geq 0$ and $240k^2 + 198k + 41$ is a prime number. Its eigenvalues are $\lambda_2 = 40k + 16$ and $\lambda_3 = -(8k + 4)$ with $m_2 = 240k^2 + 198k + 41$ and $m_3 = 6(5k + 2)(40k + 17)$.*

In order to prove Theorem 2.2, we need some propositions below:

**Proposition 2.3.** *Let $G$ be a connected strongly regular graph of order $6(2p+1)$ and degree $r$, where $2p + 1$ is a prime number. If $\delta = 2p + 1$ then $G$ belongs to the class ($3^0$) represented in Theorem 2.2.*

*Proof.* Using Theorem 1.1 we have $(2p + 1)m_2m_3 = 6r\,\overline{r}$, which means that $(2p+1) \mid r$ or $(2p+1) \mid \overline{r}$. Without loss of generality we may consider only the case when $(2p+1) \mid r$.

*Case 1.* ($r = 2p + 1$). Then $m_2m_3 = 12(5p + 2)$ and $m_2 + m_3 = 12p + 5$, which provides that $m_2$ and $m_3$ are the roots of the quadratic equation $m^2 - (12p + 5)m + 12(5p + 2) = 0$. So we find that $m_2, m_3 = \frac{12p+5\pm\Delta}{2}$ where $\Delta^2 = (12p - 5)^2 - 96$, a contradiction because $\Delta^2$ is not a perfect square.

*Case 2.* ($r = 2(2p + 1)$). Then $m_2m_3 = 12(8p + 3)$ and $m_2 + m_3 = 12p + 5$. So we obtain $m_2, m_3 = \frac{12p+5\pm\Delta}{2}$ where $\Delta^2 = (12p - 11)^2 - 240$. We can easily see that $\Delta^2$ is a perfect square only for $p = 6$. In this case we find that $m_2 = 68$ and $m_3 = 9$. Using (2) we obtain $77(\tau - \theta) + 819 = 0$, a contradiction because $77 \nmid 819$.

*Case 3.* ($r = 3(2p+1)$). Then $m_2 m_3 = 36(3p+1)$ and $m_2 + m_3 = 12p+5$. So we obtain $m_2, m_3 = \frac{12p+5\pm\Delta}{2}$ where $\Delta^2 = (12p-13)^2 - 288$, a contradiction because $\Delta^2$ is not a perfect square.

*Case 4.* ($r = 4(2p+1)$). Then $m_2 m_3 = 24(4p+1)$ and $m_2 + m_3 = 12p+5$. So we obtain $m_2, m_3 = \frac{12p+5\pm\Delta}{2}$ where $\Delta^2 = (12p-11)^2 - 192$. We can easily see that $\Delta^2$ is a perfect square only for $p = 5$. In this case we find that $m_2 = 56$ and $m_3 = 9$. Using (2) we obtain $65(\tau - \theta) + 605 = 0$, a contradiction because $65 \nmid 605$.

*Case 5.* ($r = 5(2p+1)$). Then $m_2 m_3 = 60p$ and $m_2 + m_3 = 12p+5$, which yields that $m_2 = 12p$ and $m_3 = 5$ or $m_2 = 5$ and $m_3 = 12p$. Consider first the case when $m_2 = 12p$ and $m_3 = 5$. Using (2) we obtain $\tau - \theta = -(2p+1)$. Since $\lambda_{2,3} = \frac{\tau-\theta\pm\delta}{2}$ we get easily $\lambda_2 = 0$ and $\lambda_3 = -(2p+1)$, which proves that $G$ is the strongly regular graph $\overline{6K_{2p+1}}$ of degree $r = 10p+5$ with $\tau = 8p+4$ and $\theta = 10p+5$. Consider the case when $m_2 = 5$ and $m_3 = 12p$. Using (2) we obtain $\tau - \theta = \frac{(2p+1)(12p-15)}{12p+5}$, a contradiction because $(12p+5) \nmid 12p-15$. ■

**Proposition 2.4.** *Let $G$ be a connected strongly regular graph of order $6(2p+1)$ and degree $r$, where $2p+1$ is a prime number. If $\delta = 2(2p+1)$ then $G$ belongs to the class $(2^0)$ represented in Theorem 2.2.*

*Proof.* Using Theorem 1.1 we have $2(2p+1)m_2 m_3 = 3r\overline{r}$, which means that $(2p+1) \mid r$ or $(2p+1) \mid \overline{r}$. We shall here consider only the case when $(2p+1) \mid r$.

*Case 1.* ($r = 2p+1$). Then $m_2 m_3 = 3(5p+2)$ and $m_2 + m_3 = 12p+5$ which yields that $m_2, m_3 = \frac{12p+5\pm\Delta}{2}$, where $\Delta^2 = (12p+2)^2 + 12p - 3$ and $\Delta^2 = (12p+3)^2 - (12p+8)$. So we obtain $(12p+2) < \Delta < (12p+3)$, a contradiction.

*Case 2.* ($r = 2(2p+1)$). Then $m_2 m_3 = 3(8p+3)$ and $m_2 + m_3 = 12p+5$. So we obtain $m_2, m_3 = \frac{12p+5\pm\Delta}{2}$ where $\Delta^2 = (12p+1)^2 - 12$, a contradiction because $\Delta^2$ is not a perfect square.

*Case 3.* ($r = 3(2p+1)$). Then $m_2 m_3 = 9(3p+1)$ and $m_2 + m_3 = 12p+5$ which yields that $m_2, m_3 = \frac{12p+5\pm\Delta}{2}$, where $\Delta^2 = 144p^2 + 12p - 11$ and $\Delta^2 = (12p+1)^2 - 12(p+1)$. So we obtain $12p < \Delta < 12p+1$, a contradiction.

*Case 4.* ($r = 4(2p+1)$). Then $m_2 m_3 = 24p+6$ and $m_2 + m_3 = 12p+5$, which means that $m_2 = 12p+3$ and $m_3 = 2$ or $m_2 = 2$ and $m_3 = 12p+3$. Consider first the case when $m_2 = 12p+3$ and $m_3 = 2$. Using (2) we obtain $\tau - \theta = -2(2p+1)$, which provides that $\lambda_2 = 0$ and $\lambda_3 = -2(2p+1)$. So we obtain that $G$ is the strongly regular graph $\overline{3K_{4p+2}}$ of degree $r = 8p+4$ with $\tau = 4p+2$ and $\theta = 8p+4$. Consider the case when $m_2 = 2$ and $m_3 = 12p+3$. Using Using (2) we obtain $\tau - \theta = \frac{2(2p+1)(12p-3)}{12p+5}$, a contradiction because $(12p+5) \nmid 12p-3$.

*Case 5.* ($r = 5(2p+1)$). Then $m_2 m_3 = 15p$ and $m_2 + m_3 = 12p+5$ which yields that $m_2, m_3 = \frac{12p+5\pm\Delta}{2}$, where $\Delta^2 = (12p+2)^2 + 3(4p+7)$ and $\Delta^2 = (12p+3)^2 - 4(3p-4)$. So we obtain $(12p+2) < \Delta < (12p+3)$ for $p \geq 2$, a contradiction. ■

**Proposition 2.5.** *Let $G$ be a connected strongly regular graph of order $6(2p+1)$ and degree $r$, where $2p+1$ is a prime number. If $\delta = 3(2p+1)$ then $G$ belongs to the class $(1^0)$ represented in Theorem 2.2.*

*Proof.* Using Theorem 1.1 we have $3(2p+1)m_2m_3 = 2r\,\overline{r}$, which means that $(2p+1) \mid r$ or $(2p+1) \mid \overline{r}$.

*Case 1.* ($r = 2p+1$). In this case we find that $3m_2m_3 = 20p+8$ and $3(m_2+m_3) = 36p+15$, a contradiction.

*Case 2.* ($r = 2(2p+1)$). In this case we find that $3m_2m_3 = 32p+12$ and $3(m_2+m_3) = 36p+15$, a contradiction.

*Case 3.* ($r = 3(2p+1)$). Then $m_2m_3 = 12p+4$ and $m_2+m_3 = 12p+5$, which means that $m_2 = 12p+4$ and $m_3 = 1$ or $m_2 = 1$ and $m_3 = 12p+4$. Consider first the case when $m_2 = 12p+4$ and $m_3 = 1$. Using (2) we obtain $\tau - \theta = -3(2p+1)$, which provides that $\lambda_2 = 0$ and $\lambda_3 = -3(2p+1)$. So we obtain that $G$ is the complete bipartite graph $K_{6p+3,6p+3}$ of degree $r = 6p+3$ with $\tau = 0$ and $\theta = 6p+3$. Consider the case when $m_2 = 1$ and $m_3 = 12p+4$. Using Using (2) we obtain $\tau - \theta = \frac{3(2p+1)(12p+1)}{12p+5}$, a contradiction because $(12p+5) \nmid 12p+1$.

*Case 4.* ($r = 4(2p+1)$). In this case we find that $3m_2m_3 = 32p+8$ and $3(m_2+m_3) = 36p+15$, a contradiction.

*Case 5.* ($r = 5(2p+1)$). In this case we find that $3m_2m_3 = 20p$ and $3(m_2+m_3) = 36p+15$, a contradiction. ■

**Proposition 2.6.** *There is no connected strongly regular graph $G$ of order $6(2p+1)$ and degree $r$ with $\delta = 4(2p+1)$, where $2p+1$ is a prime number.*

*Proof.* Contrary to the statement, assume that $G$ is a strongly regular graph with $\delta = 4(2p+1)$. Using Theorem 1.1 we have $8(2p+1)m_2m_3 = 3r\,\overline{r}$, which means that $(2p+1) \mid r$ or $(2p+1) \mid \overline{r}$. Consider the case when $r = 2p+1$ and $\overline{r} = 10p+4$. Then $4m_2m_3 = 15p+6$ and $4(m_2+m_3) = 48p+20$, a contradiction. Consider the case when $r = 2(2p+1)$ and $\overline{r} = 8p+3$. Then $4m_2m_3 = 24p+9$ and $4(m_2+m_3) = 48p+20$, a contradiction. Consider the case when $r = 3(2p+1)$ and $\overline{r} = 6p+2$. Then $4m_2m_3 = 27p+9$ and $4(m_2+m_3) = 48p+20$, a contradiction. Consider the case when $r = 4(2p+1)$ and $\overline{r} = 4p+1$. Then $2m_2m_3 = 12p+3$ and $m_2+m_3 = 12p+5$, a contradiction. Consider the case when $r = 5(2p+1)$ and $\overline{r} = 2p$. Then $4m_2m_3 = 15p$ and $4(m_2+m_3) = 48p+20$, a contradiction. ■

**Proposition 2.7.** *There is no connected strongly regular graph $G$ of order $6(2p+1)$ and degree $r$ with $\delta = 5(2p+1)$, where $2p+1$ is a prime number.*

*Proof.* Contrary to the statement, assume that $G$ is a strongly regular graph with $\delta = 5(2p+1)$. Using Theorem 1.1 we have $25(2p+1)m_2m_3 = 6r\,\overline{r}$, which means that $(2p+1) \mid r$ or $(2p+1) \mid \overline{r}$. Consider the case when $r = 2p+1$ and $\overline{r} = 10p+4$. Then $25m_2m_3 = 12(5p+2)$, a contradiction because $5 \nmid (5p+2)$. Consider the case when $r = 2(2p+1)$ and $\overline{r} = 8p+3$. Then $25m_2m_3 = 12(8p+3)$ and $25(m_2+m_3) = 25(12p+5)$, a contradiction. Consider the case when $r = 3(2p+1)$

and $\overline{r} = 6p+2$. Then $25m_2m_3 = 36(3p+1)$ and $25(m_2+m_3) = 25(12p+5)$, a contradiction. Consider the case when $r = 4(2p+1)$ and $\overline{r} = 4p+1$. Then $25m_2m_3 = 24(4p+1)$ and $25(m_2+m_3) = 25(12p+5)$, a contradiction. Consider the case when $r = 5(2p+1)$ and $\overline{r} = 2p$. Then $5m_2m_3 = 12p$ and $m_2+m_3 = 12p+5$, a contradiction. ■

**Proposition 2.8.** *Let $G$ be a connected strongly regular graph of order $6(2p+1)$ and degree $r$, where $2p+1$ is a prime number. If $m_2 = 2p+1$ and $m_3 = 10p+4$ then $G$ belongs to the class $(10^0)$ or $(12^0)$ or $(\overline{15}^0)$ or $(16^0)$ or $(17^0)$ or $(\overline{18}^0)$ or $(23^0)$ or $(\overline{24}^0)$ represented in Theorem 2.2.*

*Proof.* Using (2) we obtain $2r-3\delta+5(\tau-\theta) = 4p(|\lambda_3|-\lambda_2)$. Since $\delta = \lambda_2-\lambda_3$ and $\tau-\theta = \lambda_2+\lambda_3$ we arrive at $2p(5|\lambda_3|-\lambda_2) = r+\lambda_2+4\lambda_3$. Since $\lambda_2 \le \lfloor\frac{12p+6}{2}\rfloor-1$ and $|\lambda_3| \le \lfloor\frac{12p+6}{2}\rfloor$ (see [2]) it follows that $-20p \le r+\lambda_2+4\lambda_3 \le 20p$. Let $5|\lambda_3|-\lambda_2 = t$ where $t = 0, \pm1, \ldots, \pm10$. Let $\lambda_3 = -k$ where $k$ is a positive integer. Then (i) $\lambda_2 = 5k-t$; (ii) $\tau-\theta = 4k-t$; (iii) $\delta = 6k-t$ and (iv) $r = (2p+1)t-k$. Since $\delta^2 = (\tau-\theta)^2+4(r-\theta)$ (see [1]) we obtain (v) $\theta = (2p+1)t-(5k^2-(t-1)k)$. Using (ii), (iv) and (v) it is not difficult to see that (1) is transformed into

$$(p+1)t^2-3(2p+1)t+15k^2-3k(2t-1) = 0. \tag{3}$$

*Case 1.* ($t=1$). Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = 5k-1$ and $\lambda_3 = -k$, $\tau-\theta = 4k-1$, $\delta = 6k-1$, $r = (2p+1)-k$ and $\theta = (2p+1)-5k^2$. Using (3) we find that $5p+2 = 3k(5k-1)$. Replacing $k$ with $5k+1$ we arrive at $p = 75k^2+27k+2$, where $k$ is a non-negative integer. So we obtain that $G$ is a strongly regular graph of order $6(150k^2+54k+5)$ and degree $r = (6k+1)(25k+4)$ with $\tau = 25k^2+24k+3$ and $\theta = k(25k+4)$.

*Case 2.* ($t=2$). Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = 5k-2$ and $\lambda_3 = -k$, $\tau-\theta = 4k-2$, $\delta = 6k-2$, $r = 2(2p+1)-k$ and $\theta = 2(2p+1)-(5k^2-k)$. Using (3) we find that $2(4p+1) = 3k(5k-3)$. Replacing $k$ with $8k+2$ we arrive at $p = 120k^2+51k+5$, where $k$ is a non-negative integer. So we obtain that $G$ is a strongly regular graph of order $6(240k^2+102k+11)$ and degree $r = 4(5k+1)(24k+5)$ with $\tau = 2(80k^2+42k+5)$ and $\theta = 4(5k+1)(8k+1)$. Replacing $k$ with $8k-3$ we arrive at $p = 120k^2-99k+20$, where $k$ is a positive integer. So we obtain that $G$ is a strongly regular graph of order $6(240k^2-198k+41)$ and degree $r = (12k-5)(40k-17)$ with $\tau = 4(40k^2-29k+5)$ and $\theta = 2(2k-1)(40k-17)$.

*Case 3.* ($t=3$). Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = 5k-3$ and $\lambda_3 = -k$, $\tau-\theta = 4k-3$, $\delta = 6k-3$, $r = 3(2p+1)-k$ and $\theta = 3(2p+1)-(5k^2-2k)$. Using (3) we find that $3p = 5k(k-1)$. Replacing $k$ with $3k$ we arrive at $p = 15k^2-5k$, where $k$ is a positive integer. So we obtain that $G$ is a strongly regular graph of order $6(30k^2-10k+1)$ and degree $r = 3(5k-1)(6k-1)$ with $\tau = 3k(15k-4)$ and $\theta = 3(3k-1)(5k-1)$. Replacing $k$ with $3k+1$ we arrive at $p = 15k^2+5k$, where $k$ is a positive integer. So we obtain that $G$ is a strongly regular graph of order $6(30k^2+10k+1)$ and degree $r = (6k+1)(15k+2)$ with $\tau = (3k+1)(15k+1)$ and $\theta = 3k(15k+2)$.

*Case 4.* ($t = 4$). Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = 5k - 4$ and $\lambda_3 = -k$, $\tau - \theta = 4k - 4$, $\delta = 6k - 4$, $r = 4(2p+1) - k$ and $\theta = 4(2p+1) - (5k^2 - 3k)$. Using (3) we find that $4(2p-1) = 3k(5k-7)$. Replacing $k$ with $8k+4$ we arrive at $p = 120k^2 + 99k + 20$, where $k$ is a non-negative integer. So we obtain that $G$ is a strongly regular graph of order $6(240k^2 + 198k + 41)$ and degree $r = 16(5k + 2)(12k + 5)$ with $\tau = 4(8k+3)(20k+9)$ and $\theta = 16(5k+2)(8k+3)$. Replacing $k$ with $8k - 1$ we arrive at $p = 120k^2 - 51k + 5$, where $k$ is a positive integer. So we obtain that $G$ is a strongly regular graph of order $6(240k^2 - 102k + 11)$ and degree $r = (24k-5)(40k-9)$ with $\tau = 4(4k-1)(40k-7)$ and $\theta = 4(4k-1)(40k-9)$.

*Case 5.* ($t = 5$). Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = 5k - 5$ and $\lambda_3 = -k$, $\tau - \theta = 4k - 5$, $\delta = 6k - 5$, $r = 5(2p+1) - k$ and $\theta = 5(2p+1) - (5k^2 - 4k)$. Using (3) we find that $5(p-2) = 3k(5k-9)$. Replacing $k$ with $5k$ we arrive at $p = 75k^2 - 27k + 2$, where $k$ is a positive integer. So we obtain that $G$ is a strongly regular graph of order $6(150k^2 - 54k + 5)$ and degree $r = 25(5k-1)(6k-1)$ with $\tau = 25(5k-1)^2 + 5(4k-1)$ and $\theta = 25(5k-1)^2$.

*Case 6.* ($t = 6$). Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = 5k - 6$ and $\lambda_3 = -k$, $\tau - \theta = 4k - 6$, $\delta = 6k - 6$, $r = 6(2p+1) - k$ and $\theta = 6(2p+1) - (5k^2 - 5k)$. Using (3) we find that $(k-1)(5k-6) = 0$, a contradiction.

*Case 7.* ($t \geq 7$). Using (3) we find that (a) $7p + 15k^2 - 39k + 28 = 0$; (b) $16p + 15k^2 - 45k + 40 = 0$; (c) $9p + 5k^2 - 17k + 18 = 0$ and (d) $40p + 15k^2 - 57k + 70 = 0$ for $t = 7$, $t = 8$, $t = 9$ and $t = 10$, respectively, a contradiction.

*Case 8.* ($t \leq 0$). In this case we find that $(p+1)t^2 + 3(2p+1)|t| + 15k^2 + 3k(2|t| + 1) = 0$, a contradiction (see (3)). ∎

**Proposition 2.9.** *Let $G$ be a connected strongly regular graph of order $6(2p+1)$ and degree $r$, where $2p+1$ is a prime number. If $m_2 = 2(2p+1)$ and $m_3 = 8p+3$ then $G$ belongs to the class $(\overline{13}^0)$ or $(14^0)$ represented in Theorem 2.2.*

*Proof.* Using (2) we obtain $8p(2|\lambda_3| - \lambda_2) = 2r + 5(\tau - \theta) - \delta$. Since $\delta = \lambda_2 - \lambda_3$ and $\tau - \theta = \lambda_2 + \lambda_3$ we obtain $4p(|2\lambda_3| - \lambda_2) = r + 2\lambda_2 + 3\lambda_3$. Let $2|\lambda_3| - \lambda_2 = t$ where $t = 0, \pm1, \pm2, \ldots, \pm6$. Let $\lambda_3 = -k$ where $k$ is a positive integer. Then (i) $\lambda_2 = 2k - t$; (ii) $\tau - \theta = k - t$; (iii) $\delta = 3k - t$, (iv) $r = 2(2p+1)t - k$ and (v) $\theta = 2(2p+1)t - (2k^2 - (t-1)k)$. Using (ii), (iv) and (v) we can easily see that (1) is reduced to

$$(4p+3)t^2 - 6(2p+1)t + 6k^2 - 3k(2t-1) = 0. \tag{4}$$

*Case 1.* ($t = 1$). Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = 2k - 1$ and $\lambda_3 = -k$, $\tau - \theta = k - 1$, $\delta = 3k - 1$, $r = 2(2p+1) - k$ and $\theta = 2(2p+1) - 2k^2$. Using (4) we find that $8p + 3 = 3k(2k-1)$. Replacing $k$ with $8k+1$ we arrive at $p = 48k^2 + 9k$, where $k$ is a positive integer. So we obtain that $G$ is a strongly regular graph of order $6(96k^2 + 18k + 1)$ and degree $r = (12k+1)(16k+1)$ with $\tau = 4k(16k+3)$ and $\theta = 4k(16k+1)$.

*Case 2.* ($t = 2$). Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = 2k - 2$ and $\lambda_3 = -k$, $\tau - \theta = k - 2$, $\delta = 3k - 2$, $r = 4(2p+1) - k$ and $\theta = 4(2p+1) - (2k^2 - k)$.

Using (4) we find that $8p = 3k(2k-3)$. Replacing $k$ with $8k$ we arrive at $p = 48k^2 - 9k$, where $k$ is a positive integer. So we obtain that $G$ is a strongly regular graph of order $6(96k^2 - 18k + 1)$ and degree $r = 4(8k-1)(12k-1)$ with $\tau = 4(8k-1)^2 + 2(4k-1)$ and $\theta = 4(8k-1)^2$.

*Case 3.* ($t = 3$). Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = 2k-3$ and $\lambda_3 = -k$, $\tau - \theta = k-3$, $\delta = 3k-3$, $r = 6(2p+1)-k$ and $\theta = 6(2p+1)-(2k^2-2k)$. Using (4) we find that $(k-1)(2k-3) = 0$, a contradiction.

*Case 4.* ($t \geq 4$). Using (4) we find that (a) $16p + 6k^2 - 21k + 24 = 0$; (b) $40p + 6k^2 - 27k + 45 = 0$ and (c) $24p + 2k^2 - 11k + 24 = 0$ for $t = 4$, $t = 5$ and $t = 6$, respectively, a contradiction.

*Case 5.* ($t \leq 0$). In this case we find that $(4p+3)t^2 + 6(2p+1)|t| + 6k^2 + 3k(2|t| + 1) = 0$, a contradiction (see (4)). ■

**Proposition 2.10.** *Let* $G$ *be a connected strongly regular graph of order* $6(2p+1)$ *and degree* $r$*, where* $2p+1$ *is a prime number. If* $m_2 = 3(2p+1)$ *and* $m_3 = 6p+2$ *then* $G$ *belongs to the class* $(6^0)$ *represented in Theorem* 2.2*.*

*Proof.* Using (2) we obtain $12p(|\lambda_3| - \lambda_2) = 2r + 5(\tau - \theta) + \delta$. Since $2r + 5(\tau - \theta) + \delta = 2r + 6\lambda_2 + 4\lambda_3$ it follows that $-24p \leq 2r + 5(\tau - \theta) + \delta \leq 60p$. Let $|\lambda_3| - \lambda_2 = t$ where $-2 \leq t \leq 5$. Let $\lambda_3 = -k$ where $k$ is a positive integer. Then (i) $\lambda_2 = k - t$; (ii) $\tau - \theta = -t$; (iii) $\delta = 2k - t$; (iv) $r = 3(2p+1)t - k$ and (v) $\theta = 3(2p+1)t - (k^2 - (t-1)k)$. Using (ii), (iv) and (v) we can easily see that (1) is reduced to

$$(3p+2)t^2 - 3(2p+1)t + k^2 - k(2t-1) = 0. \tag{5}$$

*Case 1.* ($t = 1$). Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = k-1$ and $\lambda_3 = -k$, $\tau - \theta = -1$, $\delta = 2k-1$, $r = 3(2p+1) - k$ and $\theta = 3(2p+1) - k^2$. Using (5) we find that $3p+1 = k(k-1)$, a contradiction because $3 \nmid k^2 - k - 1$.

*Case 2.* ($t = 2$). Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = k-2$ and $\lambda_3 = -k$, $\tau - \theta = -2$, $\delta = 2k-2$, $r = 6(2p+1) - k$ and $\theta = 6(2p+1) - (k^2 - k)$. Using (5) we find that $(k-1)(k-2) = 0$. So we obtain that $G$ is the cocktail-party graph $\overline{(6p+3)K_2}$ of degree $r = 12p+4$ with $\tau = 12p+2$ and $\theta = 12p+4$.

*Case 3.* ($t \geq 3$). Using (5) we find that (a) $9p + k^2 - 5k + 9 = 0$; (b) $24p + k^2 - 7k + 20 = 0$ and (c) $45p + k^2 - 9k + 35 = 0$ for $t = 3$, $t = 4$ and $t = 5$, respectively, a contradiction.

**Case 4.** ($t \leq 0$). In this case we find that $(3p+2)t^2 + 3(2p+1)|t| + k^2 + k(2|t|+1) = 0$, a contradiction (see (5)). ■

**Proposition 2.11.** *Let* $G$ *be a connected strongly regular graph of order* $6(2p+1)$ *and degree* $r$*, where* $2p+1$ *is a prime number. If* $m_2 = 4(2p+1)$ *and* $m_3 = 4p+1$ *then* $G$ *belongs to the class* $(5^0)$ *represented in Theorem* 2.2*.*

*Proof.* Using (2) we obtain $8p(|\lambda_3| - 2\lambda_2) = 2r + 5(\tau - \theta) + 3\delta$. Let $|\lambda_3| - 2\lambda_2 = t$ where $t \in \mathbb{N}$. Let $\lambda_2 = k$ where $k$ is a non-negative integer. Then (i) $\lambda_3 =$

$-(2k+t)$; (ii) $\tau-\theta=-(k+t)$; (iii) $\delta=3k+t$, (iv) $r=2(2p+1)t-(2k+t)$ and (v) $\theta=2(2p+1)t-(k+1)(2k+t)$. Using (ii), (iv) and (v) we can easily see that (1) is reduced to

$$(4p+1)(t-3)t+6k(k+1)=0. \tag{6}$$

*Case 1.* ($t=1$). Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2=k$ and $\lambda_3=-(2k+1)$, $\tau-\theta=-(k+1)$, $\delta=3k+1$, $r=2(2p+1)-(2k+1)$ and $\theta=2(2p+1)-(k+1)(2k+1)$. Using (6) we find that $4p+1=3k(k+1)$, a contradiction because $2\nmid 4p+1$.

*Case 2.* ($t=2$). Using (6) we find that $4p+1=3k(k+1)$, a contradiction because $2\nmid 4p+1$.

*Case 3.* ($t=3$). Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2=k$ and $\lambda_3=-(2k+3)$, $\tau-\theta=-(k+3)$, $\delta=3k+3$, $r=6(2p+1)-(2k+3)$ and $\theta=6(2p+1)-(k+1)(2k+3)$. Using (6) we find that $k(k+1)=0$. So we obtain that $G$ is the strongly regular graph $\overline{(4p+2)K_3}$ of degree $r=12p+3$ with $\tau=12p$ and $\theta=12p+3$.

*Case 4.* ($t\geq 4$). In this case we find that $(4p+1)(t-3)t+6k(k+1)>0$, a contradiction (see (6)). ■

**Proposition 2.12.** *Let $G$ be a connected strongly regular graph of order $6(2p+1)$ and degree $r$, where $2p+1$ is a prime number. If $m_2=5(2p+1)$ and $m_3=2p$ then $G$ belongs to the class $(4^0)$ or $(7^0)$ or $(\overline{8}^0)$ or $(9^0)$ or $(11^0)$ or $(19^0)$ or $(\overline{20}^0)$ or $(21^0)$ or $(\overline{22}^0)$ represented in Theorem 2.2.*

*Proof.* Using (2) we obtain $4p(|\lambda_3|-5\lambda_2)=2r+5(\tau-\theta)+5\delta$. Let $|\lambda_3|-5\lambda_2=t$ where $t\in\mathbb{N}$. Let $\lambda_2=k$ where $k$ is a non-negative integer. Then (i) $\lambda_3=-(5k+t)$; (ii) $\tau-\theta=-(4k+t)$; (iii) $\delta=6k+t$, (iv) $r=(2p+1)t-(5k+t)$ and (v) $\theta=(2p+1)t-(k+1)(5k+t)$. Using (ii), (iv) and (v) we can easily see that (1) is reduced to

$$p(t-6)t+15k(k+1)=0. \tag{7}$$

*Case 1.* ($t=1$). Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2=k$ and $\lambda_3=-(5k+1)$, $\tau-\theta=-(4k+1)$, $\delta=6k+1$, $r=(2p+1)-(5k+1)$ and $\theta=(2p+1)-(k+1)(5k+1)$. Using (7) we find that $p=3k(k+1)$. So we obtain that $G$ is a strongly regular graph of order $6(6k^2+6k+1)$ and degree $r=k(6k+1)$ with $\tau=k^2-4k-1$ and $\theta=k^2$.

*Case 2.* ($t=2$). Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2=k$ and $\lambda_3=-(5k+2)$, $\tau-\theta=-(4k+2)$, $\delta=6k+2$, $r=2(2p+1)-(5k+2)$ and $\theta=2(2p+1)-(k+1)(5k+2)$. Using (7) we find that $8p=15k(k+1)$. Replacing $k$ with $8k$ we arrive at $p=120k^2+15k$, where $k$ is a positive integer. So we obtain that $G$ is a strongly regular graph of order $6(240k^2+30k+1)$ and degree $r=20k(24k+1)$ with $\tau=2(80k^2-14k-1)$ and $\theta=4k(40k+1)$. Replacing $k$ with $8k-1$ we arrive at $p=120k^2-15k$, where $k$ is a positive integer. So we obtain that $G$ is a strongly regular graph of order $6(240k^2-30k+1)$ and degree $r=5(8k-1)(12k-1)$ with $\tau=4(40k^2-17k+1)$ and $\theta=2(8k-1)(10k-1)$.

*Case 3.* ($t = 3$). Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = k$ and $\lambda_3 = -(5k+3)$, $\tau - \theta = -(4k+3)$, $\delta = 6k+3$, $r = 3(2p+1)-(5k+3)$ and $\theta = 3(2p+1)-(k+1)(5k+3)$. Using (7) we find that $3p = 5k(k+1)$. Replacing $k$ with $3k$ we arrive at $p = 15k^2+5k$, where $k$ is a positive integer. So we obtain that $G$ is a strongly regular graph of order $6(30k^2+10k+1)$ and degree $r = 15k(6k+1)$ with $\tau = 3(3k-1)(5k+1)$ and $\theta = 3k(15k+2)$. Replacing $k$ with $3k-1$ we arrive at $p = 15k^2-5k$, where $k$ is a positive integer. So we obtain that $G$ is a strongly regular graph of order $6(30k^2-10k+1)$ and degree $r = 5(3k-1)(6k-1)$ with $\tau = (3k-2)(15k-2)$ and $\theta = 3(3k-1)(5k-1)$.

*Case 4.* ($t = 4$). Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = k$ and $\lambda_3 = -(5k+4)$, $\tau - \theta = -(4k+4)$, $\delta = 6k+4$, $r = 4(2p+1)-(5k+4)$ and $\theta = 4(2p+1)-(k+1)(5k+4)$. Using (7) we find that $8p = 15k(k+1)$. Replacing $k$ with $8k$ we arrive at $p = 120k^2+15k$, where $k$ is a positive integer. So we obtain that $G$ is a strongly regular graph of order $6(240k^2+30k+1)$ and degree $r = 80k(12k+1)$ with $\tau = 4(160k^2+4k-1)$ and $\theta = 16k(40k+3)$. Replacing $k$ with $8k-1$ we arrive at $p = 120k^2-15k$, where $k$ is a positive integer. So we obtain that $G$ is a strongly regular graph of order $6(240k^2-30k+1)$ and degree $r = 5(8k-1)(24k-1)$ with $\tau = 4(160k^2-36k+1)$ and $\theta = 4(8k-1)(20k-1)$.

*Case 5.* ($t = 5$). Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = k$ and $\lambda_3 = -(5k+5)$, $\tau - \theta = -(4k+5)$, $\delta = 6k+5$, $r = 5(2p+1)-(5k+5)$ and $\theta = 5(2p+1)-(k+1)(5k+5)$. Using (7) we find that $p = 3k(k+1)$. So we obtain that $G$ is a strongly regular graph of order $6(6k^2+6k+1)$ and degree $r = 5k(6k+5)$ with $\tau = 25k^2+16k-5$ and $\theta = 5k(5k+4)$.

*Case 6.* ($t = 6$). Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = k$ and $\lambda_3 = -(5k+6)$, $\tau - \theta = -(4k+6)$, $\delta = 6k+6$, $r = 6(2p+1)-(5k+6)$ and $\theta = 6(2p+1)-(k+1)(5k+6)$. Using (7) we find that $k(k+1) = 0$. So we obtain that $G$ is the strongly regular graph $\overline{(2p+1)K_6}$ of degree $r = 12p$ with $\tau = 12p-6$ and $\theta = 12p$.

*Case 7.* ($t \geq 7$). In this case we find that $p(t-6)t+15k(k+1) > 0$, a contradiction (see (7)). ∎

**Proposition 2.13.** *Let $G$ be a connected strongly regular graph of order $6(2p+1)$ and degree $r$, where $2p+1$ is a prime number. If $m_3 = 2p+1$ and $m_2 = 10p+4$ then $G$ belongs to the class $(\overline{10}^0)$ or $(\overline{12}^0)$ or $(15^0)$ or $(\overline{16}^0)$ or $(\overline{17}^0)$ or $(18^0)$ or $(\overline{23}^0)$ or $(24^0)$ represented in Theorem 2.2.*

*Proof.* Using (2) we obtain $2p(|\lambda_3|-5\lambda_2) = r+4\lambda_2+\lambda_3$. Let $|\lambda_3|-5\lambda_2 = t$ where $t \in \mathbb{N}$. Let $\lambda_2 = k$ where $k$ is a non-negative integer. Then (i) $\lambda_3 = -(5k+t)$; (ii) $\tau - \theta = -(4k+t)$; (iii) $\delta = 6k+t$ and (iv) $r = (2p+1)t+k$ and (v) $\theta = (2p+1)t-(5k^2+(t-1)k)$. Using (ii), (iv) and (v) we can easily see that (1) is reduced to

$$(p+1)t^2 - 3(2p+1)t + 15k^2 + 3k(2t-1) = 0. \tag{8}$$

*Case 1.* ($t = 1$). Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = k$ and $\lambda_3 = -(5k+1)$, $\tau - \theta = -(4k+1)$, $\delta = 6k+1$, $r = (2p+1)+k$ and $\theta = (2p+1)-5k^2$.

Using (8) we find that $5p+2 = 3k(5k+1)$. Replacing $k$ with $5k-1$ we arrive at $p = 75k^2-27k+2$, where $k$ is a positive integer. So we obtain that $G$ is a strongly regular graph of order $6(150k^2-54k+5)$ and degree $r = (6k-1)(25k-4)$ with $\tau = 25k^2-24k+3$ and $\theta = k(25k-4)$.

*Case 2.* ($t = 2$). Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = k$ and $\lambda_3 = -(5k+2)$, $\tau - \theta = -(4k+2)$, $\delta = 6k+2$, $r = 2(2p+1)+k$ and $\theta = 2(2p+1)-(5k^2+k)$. Using (8) we find that $2(4p+1) = 3k(5k+3)$. Replacing $k$ with $8k+3$ we arrive at $p = 120k^2+99k+20$, where $k$ is a non-negative integer. So we obtain that $G$ is a strongly regular graph of order $6(240k^2+198k+41)$ and degree $r = (12k+5)(40k+17)$ with $\tau = 4(40k^2+29k+5)$ and $\theta = 2(2k+1)(40k+17)$. Replacing $k$ with $8k-2$ we arrive at $p = 120k^2-51k+5$, where $k$ is a positive integer. So we obtain that $G$ is a strongly regular graph of order $6(240k^2-102k+11)$ and degree $r = 4(5k-1)(24k-5)$ with $\tau = 2(80k^2-42k+5)$ and $\theta = 4(5k-1)(8k-1)$.

*Case 3.* ($t = 3$). Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = k$ and $\lambda_3 = -(5k+3)$, $\tau - \theta = -(4k+3)$, $\delta = 6k+3$, $r = 3(2p+1)+k$ and $\theta = 3(2p+1)-(5k^2+2k)$. Using (8) we find that $3p = 5k(k+1)$. Replacing $k$ with $3k$ we arrive at $p = 15k^2+15k$, where $k$ is a positive integer. So we obtain that $G$ is a strongly regular graph of order $6(30k^2+10k+1)$ and degree $r = 3(5k+1)(6k+1)$ with $\tau = 3k(15k+4)$ and $\theta = 3(3k+1)(5k+1)$. Replacing $k$ with $3k-1$ we arrive at $p = 15k^2-5k$, where $k$ is a positive integer. So we obtain that $G$ is a strongly regular graph of order $6(30k^2-10k+1)$ and degree $r = (6k-1)(15k-2)$ with $\tau = (3k-1)(15k-1)$ and $\theta = 3k(15k-2)$.

*Case 4.* ($t = 4$). Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = k$ and $\lambda_3 = -(5k+4)$, $\tau - \theta = -(4k+4)$, $\delta = 6k+4$, $r = 4(2p+1)+k$ and $\theta = 4(2p+1)-(5k^2+3k)$. Using (8) we find that $4(2p-1) = 3k(5k+7)$. Replacing $k$ with $8k+1$ we arrive at $p = 120k^2+51k+5$, where $k$ is a non-negative integer. So we obtain that $G$ is a strongly regular graph of order $6(240k^2+102k+11)$ and degree $r = (24k+5)(40k+9)$ with $\tau = 4(4k+1)(40k+7)$ and $\theta = 4(4k+1)(40k+9)$. Replacing $k$ with $8k-4$ we arrive at $p = 120k^2-99k+20$, where $k$ is a positive integer. So we obtain that $G$ is a strongly regular graph of order $6(240k^2-198k+41)$ and degree $r = 16(5k-2)(12k-5)$ with $\tau = 4(8k-3)(20k-9)$ and $\theta = 16(5k-2)(8k-3)$.

*Case 5.* ($t = 5$). Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = k$ and $\lambda_3 = -(5k+5)$, $\tau - \theta = -(4k+5)$, $\delta = 6k+5$, $r = 5(2p+1)+k$ and $\theta = 5(2p+1)-(5k^2+4k)$. Using (8) we find that $5(p-1) = 3k(5k+9)$. Replacing $k$ with $5k$ we arrive at $p = 75k^2+27k+2$, where $k$ is a non-negative integer. So we obtain that $G$ is a strongly regular graph of order $6(150k^2+54k+5)$ and degree $r = 25(5k+1)(6k+1)$ with $\tau = 25(5k+1)^2-5(4k+1)$ and $\theta = 25(5k+1)^2$.

*Case 6.* ($t = 6$). Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = k$ and $\lambda_3 = -(5k+6)$, $\tau - \theta = -(4k+6)$, $\delta = 6k+6$, $r = 5(2p+1)+6$ and $\theta = 5(2p+1)-(5k^2+5k)$. Using (8) we find that $(k+1)(5k+6) = 0$, a contradiction.

*Case 7.* ($t \geq 7$). In this case we find that $(p+1)t^2 - 3(2p+1)t + 15k^2 + 3k(2t-1) > 0$, a contradiction (see (8)). ■

**Proposition 2.14.** *Let $G$ be a connected strongly regular graph of order $6(2p+1)$ and degree $r$, where $2p+1$ is a prime number. If $m_3 = 2(2p+1)$ and $m_2 = 8p+3$ then $G$ belongs to the class $(13^0)$ or $(\overline{14}^0)$ represented in Theorem 2.2.*

*Proof.* Using (2) we obtain $8p(|\lambda_3| - 2\lambda_2) = 2r + 5(\tau - \theta) + \delta$. Since $\delta = \lambda_2 - \lambda_3$ and $\tau - \theta = \lambda_2 + \lambda_3$ we obtain $4p(|\lambda_3| - 2\lambda_2) = r + 3\lambda_2 + 2\lambda_3$. Let $2|\lambda_3| - \lambda_2 = t$ where $-2 \leq t \leq 8$. Let $\lambda_2 = k$ where $k$ is a non-negative integer. Then (i) $\lambda_3 = -(2k+t)$; (ii) $\tau - \theta = -(k+t)$; (iii) $\delta = 3k + t$, (iv) $r = 2(2p+1)t + k$ and (v) $\theta = 2(2p+1)t - (2k^2 + (t-1)k)$. Using (ii), (iv) and (v) we can easily see that (1) is reduced to

$$(4p+3)t^2 - 6(2p+1)t + 6k^2 + 3k(2t-1) = 0. \tag{9}$$

*Case 1.* ($t = 1$). Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = k$ and $\lambda_3 = -(2k+1)$, $\tau - \theta = -(k+1)$, $\delta = 3k+1$, $r = 2(2p+1) + k$ and $\theta = 2(2p+1) - 2k^2$. Using (9) we find that $8p + 3 = 3k(2k+1)$. Replacing $k$ with $8k - 1$ we arrive at $p = 48k^2 - 9k$, where $k$ is a positive integer. So we obtain that $G$ is a strongly regular graph of order $6(96k^2 - 18k + 1)$ and degree $r = (12k-1)(16k-1)$ with $\tau = 4k(16k-3)$ and $\theta = 4k(16k-1)$.

*Case 2.* ($t = 2$). Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = k$ and $\lambda_3 = -(2k+2)$, $\tau - \theta = -(k+2)$, $\delta = 3k+2$, $r = 4(2p+1) + k$ and $\theta = 4(2p+1) - (2k^2 + k)$. Using (9) we find that $8p = 3k(2k+3)$. Replacing $k$ with $8k$ we arrive at $p = 48k^2 + 9k$, where $k$ is a positive integer. So we obtain that $G$ is a strongly regular graph of order $6(96k^2 + 18k + 1)$ and degree $r = 4(8k+1)(12k+1)$ with $\tau = 4(8k+1)^2 - 2(4k+1)$ and $\theta = 4(8k+1)^2$.

*Case 3.* ($t = 3$). Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = k$ and $\lambda_3 = -(2k+3)$, $\tau - \theta = -(k+3)$, $\delta = 3k+3$, $r = 6(2p+1) + k$ and $\theta = 6(2p+1) - (2k^2 + 2k)$. Using (9) we find that $(k+1)(2k+3) = 0$, a contradiction.

*Case 4.* ($t \geq 4$). In this case we find that $(4p+3)t^2 - 6(2p+1)t + 6k^2 + 3k(2t-1) > 0$, a contradiction (see (9)).

*Case 5.* ($t \leq 0$). Using (9) we find that (a) $k(2k-1) = 0$; (b) $16p + 6k^2 - 9k + 9 = 0$ and (c) $40p + 6k^2 - 15k + 24 = 0$ for $t = 0$, $t = -1$ and $t = -2$, respectively, a contradiction. ■

**Proposition 2.15.** *There is no connected strongly regular graph $G$ of order $6(2p+1)$ and degree $r$ with $m_3 = 3(2p+1)$ and $m_2 = 6p+2$, where $2p+1$ is a prime number.*

*Proof.* Contrary to the statement, assume that $G$ is a strongly regular graph with $m_3 = 3(2p+1)$ and $m_2 = 6p+2$. Using (2) we obtain $12p(|\lambda_3| - \lambda_2) = 2r + 5(\tau - \theta) - \delta$. Let $|\lambda_3| - \lambda_2 = t$ where $t \in \mathbb{Z}$. Let $\lambda_3 = -k$ where $k$ is a positive integer. Then (i) $\lambda_2 = k - t$; (ii) $\tau - \theta = -t$; (iii) $\delta = 2k - t$; (iv)

$r = 3(2p+1)t + k - t$ and (v) $\theta = 3(2p+1)t - (k-1)(k-t)$. Using (ii), (iv) and (v) we can easily see that (1) is reduced to

$$(3p+1)(t-2)t + k(k-1) = 0. \tag{10}$$

*Case 1.* ($t = 1$). Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = k - 1$ and $\lambda_3 = -k$, $\tau - \theta = -1$, $\delta = 2k-1$, $r = 3(2p+1)+k-1$ and $\theta = 3(2p+1)-(k-1)^2$. Using (10) we find that $3p+1 = k(k-1)$, a contradiction because $3 \nmid k^2 - k - 1$.

*Case 2.* ($t = 2$). Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = k-2$ and $\lambda_3 = -k$, $\tau - \theta = -2$, $\delta = 2k-2$, $r = 6(2p+1)+k-2$ and $\theta = 6(2p+1)-(k-1)(k-2)$. Using (10) we find that $k(k-1) = 0$, a contradiction.

*Case 3.* ($t \geq 3$). In this case we find that $(3p+1)(t-2)t + k(k-1) > 0$, a contradiction (see (10)).

*Case 4.* ($t \leq 0$). In this case we find that $(3p+1)(|t|+2)|t| + k(k-1) = 0$, a contradiction (see (10)). ■

**Proposition 2.16.** *There is no connected strongly regular graph $G$ of order $6(2p+1)$ and degree $r$ with $m_3 = 4(2p+1)$ and $m_2 = 4p+1$, where $2p+1$ is a prime number.*

*Proof.* Contrary to the statement, assume that $G$ is a strongly regular graph with $m_3 = 4(2p+1)$ and $m_2 = 4p+1$. Using (2) we obtain $8p(2|\lambda_3| - \lambda_2) = 2r + 5(\tau - \theta) - 3\delta$. Let $2|\lambda_3| - \lambda_2 = t$ where $t \in \mathbb{Z}$. Let $\lambda_3 = -k$ where $k$ is a positive integer. Then (i) $\lambda_2 = 2k - t$; (ii) $\tau - \theta = k - t$; (iii) $\delta = 3k - t$; (iv) $r = 2(2p+1)t + 2k - t$ and (v) $\theta = 2(2p+1)t - (k-1)(2k-t)$. Using (ii), (iv) and (v) we can easily see that (1) is reduced to

$$(4p+1)(t-3)t + 6k(k-1) = 0. \tag{11}$$

*Case 1.* ($t = 1$). Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = 2k-1$ and $\lambda_3 = -k$, $\tau - \theta = k-1$, $\delta = 3k-1$, $r = 2(2p+1)+2k-1$ and $\theta = 2(2p+1)-(k-1)(2k-1)$. Using (11) we find that $4p+1 = 3k(k-1)$, a contradiction because $2 \nmid 4p+1$.

*Case 2.* ($t = 2$). Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = 2k-2$ and $\lambda_3 = -k$, $\tau - \theta = k-2$, $\delta = 3k-2$, $r = 4(2p+1)+2k-2$ and $\theta = 4(2p+1)-(k-1)(2k-2)$. Using (11) we find that $4p+1 = 3k(k-1)$, a contradiction because $2 \nmid 4p+1$.

*Case 3.* ($t = 3$). Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = 2k-3$ and $\lambda_3 = -k$, $\tau - \theta = k-3$, $\delta = 3k-3$, $r = 6(2p+1)+2k-3$ and $\theta = 6(2p+1)-(k-1)(2k-3)$. Using (11) we find that $k(k-1) = 0$, a contradiction.

*Case 4.* ($t \geq 4$). In this case we find that $(4p+1)(t-3)t + 6k(k-1) > 0$, a contradiction (see (11)).

*Case 5.* ($t \leq 0$). In this case we find that $(4p+1)(|t|+3)|t| + 6k(k-1) = 0$, a contradiction (see (11)). ■

**Proposition 2.17.** *Let $G$ be a connected strongly regular graph of order $6(2p+1)$ and degree $r$, where $2p+1$ is a prime number. If $m_3 = 5(2p+1)$ and $m_2 = 2p$ then $G$ belongs to the class $(\overline{7}^0)$ or $(8^0)$ or $(\overline{9}^0)$ or $(\overline{11}^0)$ or $(\overline{19}^0)$ or $(20^0)$ or $(\overline{21}^0)$ or $(22^0)$ represented in Theorem 2.2.*

*Proof.* Using (2) we obtain $4p(5|\lambda_3| - \lambda_2) = 2r + 5(\tau - \theta) - 5\delta$. Let $5|\lambda_3| - \lambda_2 = t$ where $t \in \mathbb{Z}$. Let $\lambda_2 = -k$ where $k$ is a positive integer. Then (i) $\lambda_2 = 5k - t$; (ii) $\tau - \theta = 4k - t$; (iii) $\delta = 6k - t$, (iv) $r = (2p+1)t + (5k - t)$ and (v) $\theta = (2p+1)t - (k-1)(5k-t)$. Using (ii), (iv) and (v) we can easily see that (1) is reduced to

$$p(t-6)t + 15k(k-1) = 0. \tag{12}$$

*Case 1.* ($t = 1$). Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = 5k - 1$ and $\lambda_3 = -k$, $\tau - \theta = 4k - 1$, $\delta = 6k - 1$, $r = (2p+1) + (5k-1)$ and $\theta = (2p+1) - (k-1)(5k-1)$. Using (12) we find that $p = 3k(k-1)$. Replacing $k$ with $k+1$ we arrive at $p = 3k^2 + 3k$, where $k$ is a positive integer. So we obtain that $G$ is a strongly regular graph of order $6(6k^2 + 6k + 1)$ and degree $r = (k+1)(6k+5)$ with $\tau = k^2 + 6k + 4$ and $\theta = (k+1)^2$.

*Case 2.* ($t = 2$). Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = 5k - 2$ and $\lambda_3 = -k$, $\tau - \theta = 4k - 2$, $\delta = 6k - 2$, $r = 2(2p+1) + (5k-2)$ and $\theta = 2(2p+1) - (k-1)(5k-2)$. Using (12) we find that $8p = 15k(k-1)$. Replacing $k$ with $8k$ we arrive at $p = 120k^2 - 15k$, where $k$ is a positive integer. So we obtain that $G$ is a strongly regular graph of order $6(240k^2 - 30k + 1)$ and degree $r = 20k(24k-1)$ with $\tau = 2(80k^2 + 14k - 1)$ and $\theta = 4k(40k-1)$. Replacing $k$ with $8k+1$ we arrive at $p = 120k^2 + 15k$, where $k$ is a positive integer. So we obtain that $G$ is a strongly regular graph of order $6(240k^2 + 30k + 1)$ and degree $r = 5(8k+1)(12k+1)$ with $\tau = 4(40k^2 + 17k + 1)$ and $\theta = 2(8k+1)(10k+1)$.

*Case 3.* ($t = 3$). Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = 5k - 3$ and $\lambda_3 = -k$, $\tau - \theta = 4k - 3$, $\delta = 6k - 3$, $r = 3(2p+1) + (5k-3)$ and $\theta = 3(2p+1) - (k-1)(5k-3)$. Using (12) we find that $3p = 5k(k-1)$. Replacing $k$ with $3k$ we arrive at $p = 15k^2 - 5k$, where $k$ is a positive integer. So we obtain that $G$ is a strongly regular graph of order $6(30k^2 - 10k + 1)$ and degree $r = 15k(6k-1)$ with $\tau = 3(3k+1)(5k-1)$ and $\theta = 3k(15k-2)$. Replacing $k$ with $3k+1$ we arrive at $p = 15k^2 + 5k$, where $k$ is a positive integer. So we obtain that $G$ is a strongly regular graph of order $6(30k^2 + 10k + 1)$ and degree $r = 5(3k+1)(6k+1)$ with $\tau = (3k+2)(15k+2)$ and $\theta = 3(3k+1)(5k+1)$.

*Case 4.* ($t = 4$). Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = 5k - 4$ and $\lambda_3 = -k$, $\tau - \theta = 4k - 4$, $\delta = 6k - 4$, $r = 4(2p+1) + (5k-4)$ and $\theta = 4(2p+1) - (k-1)(5k-4)$. Using (12) we find that $8p = 15k(k-1)$. Replacing $k$ with $8k$ we arrive at $p = 120k^2 - 15k$, where $k$ is a positive integer. So we obtain that $G$ is a strongly regular graph of order $6(240k^2 - 30k + 1)$ and degree $r = 80k(12k-1)$ with $\tau = 4(160k^2 - 4k - 1)$ and $\theta = 16k(40k-3)$. Replacing $k$ with $8k+1$ we arrive at $p = 120k^2 + 15k$, where $k$ is a positive integer. So we obtain that $G$ is a strongly regular graph of order $6(240k^2 + 30k + 1)$ and degree $r = 5(8k+1)(24k+1)$ with $\tau = 4(160k^2 + 36k + 1)$ and $\theta = 4(8k+1)(20k+1)$.

*Case 5.* ($t = 5$). Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = 5k - 5$ and $\lambda_3 = -k$, $\tau - \theta = 4k - 5$, $\delta = 6k - 5$, $r = 5(2p+1) + (5k-5)$ and $\theta = 5(2p+1) - (k-1)(5k-5)$. Using (12) we find that $p = 3k(k-1)$. Replacing $k$ with $k+1$ we arrive at $p = 3k^2 + 3k$, where $k$ is a positive integer. So we obtain that $G$ is a strongly regular graph of order $6(6k^2+6k+1)$ and degree $r = 5(k+1)(6k+1)$ with $\tau = 25k^2 + 34k + 4$ and $\theta = 5(k+1)(5k+1)$.

*Case 6.* ($t = 6$). Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = 5k - 5$ and $\lambda_3 = -k$, $\tau - \theta = 4k - 6$, $\delta = 6k - 6$, $r = 6(2p+1) + (5k-6)$ and $\theta = 6(2p+1) - (k-1)(5k-6)$. Using (12) we find that $k(k-1) = 0$, a contradiction.

*Case 7.* ($t \geq 7$). In this case we find that $p(t-6)t + 15k(k-1) > 0$, a contradiction (see (12)).

*Case 8.* ($t \leq 0$). In this case we find that $p(|t|+6)|t| + 15k(k-1) = 0$, a contradiction (see (12)). ■

*Proof of Theorem 2.2.* Using Theorem 1.1 we have $m_2 m_3 \delta^2 = 6(2p+1) r \overline{r}$. We shall now consider the following three cases.

*Case 1.* ($(2p+1) \mid \delta^2$). In this case $(2p+1) \mid \delta$ because $G$ is an integral graph. Since $\delta = \lambda_2 + |\lambda_3| < 12p + 6$ (see [2]) it follows that $\delta = 2p+1$ or $\delta = 2(2p+1)$ or $\delta = 3(2p+1)$ or $\delta = 4(2p+1)$ or $\delta = 5(2p+1)$. Using Propositions 2.3, 2.4, 2.5, 2.6 and 2.7 it turns out that $G$ belongs to the class $(1^0)$ or $(2^0)$ or $(3^0)$.

*Case 2.* ($(2p+1) \mid m_2$). Since $m_2 + m_3 = 12p + 5$ it follows that $m_2 = 2p+1$ and $m_3 = 10p + 4$ or $m_2 = 2(2p+1)$ and $m_3 = 8p + 3$ or $m_2 = 3(2p+1)$ and $m_3 = 6p+2$ or $m_2 = 4(2p+1)$ and $m_3 = 4p+1$ or $m_2 = 5(2p+1)$ and $m_3 = 2p$. Using Propositions 2.8, 2.9, 2.10, 2.11 and 2.12 it turns out that $G$ belongs to the class $(4^0)$ or $(5^0)$ or $(6^0)$ or $(7^0)$ or $(\overline{8}^0)$ or $(9^0)$ or $(10^0)$ or $(11^0)$ or $(12^0)$ or $(\overline{13}^0)$ or $(14^0)$ or $(\overline{15}^0)$ or $(16^0)$ or $(17^0)$ or $(\overline{18}^0)$ or $(19^0)$ or $(\overline{20}^0)$ or $(21^0)$ or $(\overline{22}^0)$ or $(23^0)$ or $(\overline{24}^0)$.

*Case 3.* ($(2p+1) \mid m_3$). Since $m_3 + m_2 = 12p + 5$ it follows that $m_3 = 2p+1$ and $m_2 = 10p + 4$ or $m_3 = 2(2p+1)$ and $m_2 = 8p + 3$ or $m_3 = 3(2p+1)$ and $m_2 = 6p+2$ or $m_3 = 4(2p+1)$ and $m_2 = 4p+1$ or $m_3 = 5(2p+1)$ and $m_2 = 2p$. Using Propositions 2.13, 2.14, 2.15, 2.16 and 2.17 it turns out that $G$ belongs to the class $(\overline{7}^0)$ or $(8^0)$ or $(\overline{9}^0)$ or $(\overline{10}^0)$ or $(\overline{11}^0)$ or $(\overline{12}^0)$ or $(13^0)$ or $(\overline{14}^0)$ or $(15^0)$ or $(\overline{16}^0)$ or $(\overline{17}^0)$ or $(18^0)$ or $(\overline{19}^0)$ or $(20^0)$ or $(\overline{21}^0)$ or $(22^0)$ or $(\overline{23}^0)$ or $(24^0)$. ■

## References


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