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Insertion of a Continuous Function Between Two Comparable $\alpha-$Continuous ($C-$Continuous) Functions ⋆

Majid Mirmiran

Department of Mathematics, University of Isfahan, Isfahan 81746-73441, Iran.

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Abstract. A necessary and sufficient conditions in terms of lower cut sets are given for the insertion of a continuous function between two comparable real-valued functions.

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1. Introduction

The concept of a $C-$open set in a topological space was introduced by Hatir, Noiri and Yksel in [5]. The authors define a set S to be a $C-$*open* set if $S = U \cap A$, where U is open and A is semi-preclosed. A set S is a $C-$*closed* set if its complement (denoted by S^c) is a $C-$open set or equivalently if $S = U \cup A$, where U is closed and A is semi-preopen. The authors show that a subset of a topological space is open if and only if it is an $\alpha-$open set and a $C-$open set. This enables them to provide the following decomposition of continuity: a function is continuous if and only if it is $\alpha-$continuous and $C-$continuous.

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Recall that a subset A of a topological space (X, τ) is called $\alpha-$*open* if A is the difference of an open and a nowhere dense subset of X. A set A is called $\alpha-$*closed*, if its complement is $\alpha-$open or equivalently, if A is the union of a closed and a nowhere dense set. Sets which are dense in some regular closed subspace are called *semi-preopen* or $\beta-$*open*. A set is *semi-preclosed* or $\beta-$*closed* if its complement is semi-preopen or $\beta-$open.

In [3] it was shown that a set A is $\beta-$open if and only if $A \subseteq \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A)))$.

Recall that a real-valued function f defined on a topological space X is called $A-$continuous [10] if the preimage of every open subset of $\mathbb{R}$ belongs to A, where A is a collection of subsets of X. Most of the definitions of function used throughout this paper are consequences of the definition of $A-$continuity. However, for unknown concepts the reader may refer to [2, 4].

Hence, a real-valued function f defined on a topological space X is called $C-$continuous (resp. $\alpha-$continuous) if the preimage of every open subset of $\mathbb{R}$ is a $C-$open (resp. $\alpha-$open) subset of X.

Results of Katětov [6, 7] concerning binary relations and the concept of an indefinite lower cut set for a real-valued function, which is due to Brooks [1], are used in order to give necessary and sufficient conditions for the strong insertion of a continuous function between two comparable real-valued functions.

If g and f are real-valued functions defined on a space X, we write $g \leq f$ in case $g(x) \leq f(x)$ for all x in X.

The following definitions are modifications of conditions considered in [8].

A property P defined relative to a real-valued function on a topological space is a $c-$*property* provided that any constant function has property P and provided that the sum of a function with property P and any continuous function also has property P. If P_1 and P_2 are $c-$properties, the following terminology is used: (i) A space X has the *weak* $c-$*insertion property* for (P_1, P_2) if and only if for any functions g and f on X such that $g \leq f$, g has property P_1 and f has property P_2, then there exists a continuous function h such that $g \leq h \leq f$. (ii) A space X has the $c-$*insertion property* for (P_1, P_2) if and only if for any functions g and f on X such that $g < f$, g has property P_1 and f has property P_2, then there exists a continuous function h such that $g < h < f$. (iii) A space X has the *weakly* $c-$*insertion property* for (P_1, P_2) if and only if for any functions g and f on X such that $g < f$, g has property P_1, f has property P_2 and $f - g$ has property P_2, then there exists a continuous function h such that $g < h < f$.

In this paper, a sufficient condition for the weak $c-$insertion property is given. Also for a space with the weak $c-$insertion property, we give a necessary and sufficient condition for the space to have the $c-$insertion property. Several insertion theorems are obtained as corollaries of these results.

2. The Main Result

Before giving a sufficient condition for insertability of a continuous function, the necessary definitions and terminology are stated.

Let (X, τ) be a topological space, the family of all $\alpha-$open, $\alpha-$closed, $C-$open and $C-$closed will be denoted by $\alpha O(X, \tau)$, $\alpha C(X, \tau)$, $CO(X, \tau)$ and $CC(X, \tau)$, respectively.

Definition 2.1. Let A be a subset of a topological space (X, τ). Respectively, we define the $\alpha-$*closure,* $\alpha-$*interior,* $C-$*closure* and $C-$*interior* of a set A, denoted by $\alpha\mathrm{Cl}(A), \alpha\mathrm{Int}(A), C\mathrm{Cl}(A)$ and $C\mathrm{Int}(A)$ as follows:

$\alpha\mathrm{Cl}(A) = \cap\{F : F \supseteq A, F \in \alpha C(X, \tau)\}$,

$\alpha\mathrm{Int}(A) = \cup\{O : O \subseteq A, O \in \alpha O(X, \tau)\}$,

$C\mathrm{Cl}(A) = \cap\{F : F \supseteq A, F \in CC(X, \tau)\}$ and

$C\mathrm{Int}(A) = \cup\{O : O \subseteq A, O \in CO(X, \tau)\}$.

Respectively, we have $\alpha\mathrm{Cl}(A), C\mathrm{Cl}(A)$ are $\alpha-$closed, semi-preclosed and $\alpha\mathrm{Int}(A), C\mathrm{Int}(A)$ are $\alpha-$open, semi-preopen.

The following first two definitions are modifications of conditions considered in [6, 7].

Definition 2.2. If ρ is a binary relation in a set S then $\bar{\rho}$ is defined as follows: $x \ \bar{\rho} \ y$ if and only if $y \ \rho \ v$ implies $x \ \rho \ v$ and $u \ \rho \ x$ implies $u \ \rho \ y$ for any u and v in S.

Definition 2.3. A binary relation ρ in the power set $P(X)$ of a topological space X is called a *strong binary relation* in $P(X)$ in case ρ satisfies each of the following conditions:

1) If $A_i \ \rho \ B_j$ for any $i \in \{1, \ldots, m\}$ and for any $j \in \{1, \ldots, n\}$, then there exists a set C in $P(X)$ such that $A_i \ \rho \ C$ and $C \ \rho \ B_j$ for any $i \in \{1, \ldots, m\}$ and any $j \in \{1, \ldots, n\}$.

2) If $A \subseteq B$, then $A \ \bar{\rho} \ B$.

3) If $A \ \rho \ B$, then $\mathrm{Cl}(A) \subseteq B$ and $A \subseteq \mathrm{Int}(B)$.

The concept of a lower indefinite cut set for a real-valued function was defined by Brooks [1] as follows:

Definition 2.4. If f is a real-valued function defined on a space X and if $\{x \in X : f(x) < \ell\} \subseteq A(f, \ell) \subseteq \{x \in X : f(x) \leq \ell\}$ for a real number ℓ, then $A(f, \ell)$ is called a *lower indefinite cut set* in the domain of f at the level ℓ.

We now give the following main result:

Theorem 2.5. *Let g and f be real-valued functions on a topological space X with $g \leq f$. If there exists a strong binary relation ρ on the power set of X and if there exist lower indefinite cut sets $A(f,t)$ and $A(g,t)$ in the domain of f and g at the level t for each rational number t such that if $t_1 < t_2$ then $A(f,t_1)\ \rho\ A(g,t_2)$, then there exists a continuous function h defined on X such that $g \leq h \leq f$.*

Proof. Let g and f be real-valued functions defined on X such that $g \leq f$. By hypothesis there exists a strong binary relation ρ on the power set of X and there exist lower indefinite cut sets $A(f,t)$ and $A(g,t)$ in the domain of f and g at the level t for each rational number t such that if $t_1 < t_2$ then $A(f,t_1)\ \rho\ A(g,t_2)$.

Define functions F and G mapping the rational numbers $\mathbb{Q}$ into the power set of X by $F(t) = A(f,t)$ and $G(t) = A(g,t)$. If t_1 and t_2 are any elements of $\mathbb{Q}$ with $t_1 < t_2$, then $F(t_1)\ \bar{\rho}\ F(t_2), G(t_1)\ \bar{\rho}\ G(t_2)$, and $F(t_1)\ \rho\ G(t_2)$. By Lemmas 1 and 2 of [7] it follows that there exists a function H mapping $\mathbb{Q}$ into the power set of X such that if t_1 and t_2 are any rational numbers with $t_1 < t_2$, then $F(t_1)\ \rho\ H(t_2), H(t_1)\ \rho\ H(t_2)$ and $H(t_1)\ \rho\ G(t_2)$.

For any x in X, let $h(x) = \inf\{t \in \mathbb{Q} : x \in H(t)\}$.

We first verify that $g \leq h \leq f$: If x is in $H(t)$ then x is in $G(t')$ for any $t' > t$; since x is in $G(t') = A(g,t')$ implies that $g(x) \leq t'$, it follows that $g(x) \leq t$. Hence $g \leq h$. If x is not in $H(t)$, then x is not in $F(t')$ for any $t' < t$; since x is not in $F(t') = A(f,t')$ implies that $f(x) > t'$, it follows that $f(x) \geq t$. Hence $h \leq f$.

Also, for any rational numbers t_1 and t_2 with $t_1 < t_2$, we have $h^{-1}(t_1,t_2) = \mathrm{Int}(H(t_2)) \setminus \mathrm{Cl}(H(t_1))$. Hence $h^{-1}(t_1,t_2)$ is an open subset of X, i.e., h is a continuous function on X. ■

The above proof used the technique of proof of Theorem 1 of [6].

Theorem 2.6. *Let P_1 and P_2 be $c-$properties and X be a space that satisfies the weak $c-$insertion property for (P_1,P_2). Also assume that g and f are functions on X such that $g < f$, g has property P_1 and f has property P_2. The space X has the $c-$insertion property for (P_1,P_2) if and only if there exist lower cut sets $A(f-g, 3^{-n+1})$ and there exists a decreasing sequence $\{D_n\}$ of subsets of X with empty intersection and such that for each n, $X \setminus D_n$ and $A(f-g, 3^{-n+1})$ are completely separated by continuous functions.*

Proof. Theorem 2.1 of [9]. ■

3. Applications

The abbreviations αc and Cc are used for $\alpha-$continuous and $C-$continuous, respectively.

Corollary 3.1. *If for each pair of disjoint $\alpha-$closed (resp. $C-$closed) sets F_1, F_2 of X, there exist open sets G_1 and G_2 of X such that $F_1 \subseteq G_1$, $F_2 \subseteq G_2$ and*

$G_1 \cap G_2 = \varnothing$ then X has the weak $c-$insertion property for $(\alpha c, \alpha c)$ (resp. (Cc, Cc)).

Proof. Let g and f be real-valued functions defined on X, such that f and g are αc (resp. Cc), and $g \leq f$. If a binary relation ρ is defined by $A\ \rho\ B$ in case $\alpha\mathrm{Cl}(A) \subseteq \alpha\mathrm{Int}(B)$ (resp. $C\mathrm{Cl}(A) \subseteq C\mathrm{Int}(B)$), then by hypothesis ρ is a strong binary relation in the power set of X. If t_1 and t_2 are any elements of $\mathbb{Q}$ with $t_1 < t_2$, then

$$A(f, t_1) \subseteq \{x \in X : f(x) \leq t_1\} \subseteq \{x \in X : g(x) < t_2\} \subseteq A(g, t_2);$$

since $\{x \in X : f(x) \leq t_1\}$ is an $\alpha-$closed (resp. $C-$closed) set and since $\{x \in X : g(x) < t_2\}$ is an $\alpha-$open (resp. $C-$open) set, it follows that $\alpha\mathrm{Cl}(A(f, t_1)) \subseteq \alpha\mathrm{Int}(A(g, t_2))$ (resp. $C\mathrm{Cl}(A(f, t_1)) \subseteq C\mathrm{Int}(A(g, t_2))$). Hence $t_1 < t_2$ implies that $A(f, t_1)\ \rho\ A(g, t_2)$. The proof follows from Theorem 2.5. ■

Corollary 3.2. *If for each pair of disjoint $\alpha-$closed (resp. $C-$closed) sets F_1, F_2, there exist open sets G_1 and G_2 such that $F_1 \subseteq G_1$, $F_2 \subseteq G_2$ and $G_1 \cap G_2 = \varnothing$ then every $\alpha-$continuous (resp. $C-$continuous) function is continuous.*

Proof. Let f be a real-valued $\alpha-$continuous (resp. $C-$continuous) function defined on X. Set $g = f$, then by Corollary 3.1, there exists a continuous function h such that $g = h = f$. ■

Corollary 3.3. *If for each pair of disjoint $\alpha-$closed (resp. $C-$closed) sets F_1, F_2 of X, there exist open sets G_1 and G_2 of X such that $F_1 \subseteq G_1$, $F_2 \subseteq G_2$ and $G_1 \cap G_2 = \varnothing$ then X has the strong $c-$insertion property for $(\alpha c, \alpha c)$ (resp. (Cc, Cc)).*

Proof. Let g and f be real-valued functions defined on the X, such that f and g are αc (resp. Cc), and $g \leq f$. Set $h = (f + g)/2$, thus $g \leq h \leq f$ and if $g(x) < f(x)$ for any x in X, then $g(x) < h(x) < f(x)$. Also, by Corollary 3.2, since g and f are continuous functions hence h is a continuous function. ■

Corollary 3.4. *If for each pair of disjoint subsets F_1, F_2 of X, such that F_1 is $\alpha-$closed and F_2 is $C-$closed, there exist open subsets G_1 and G_2 of X such that $F_1 \subseteq G_1$, $F_2 \subseteq G_2$ and $G_1 \cap G_2 = \varnothing$ then X have the weak $c-$insertion property for $(\alpha c, Cc)$ and $(Cc, \alpha c)$.*

Proof. Let g and f be real-valued functions defined on X, such that g is αc (resp. Cc) and f is Cc (resp. αc), with $g \leq f$. If a binary relation ρ is defined by $A\ \rho\ B$ in case $C\mathrm{Cl}(A) \subseteq \alpha\mathrm{Int}(B)$ (resp. $\alpha\mathrm{Cl}(A) \subseteq C\mathrm{Int}(B)$), then by hypothesis ρ is a strong binary relation in the power set of X. If t_1 and t_2 are any elements of $\mathbb{Q}$ with $t_1 < t_2$, then

$$A(f, t_1) \subseteq \{x \in X : f(x) \leq t_1\} \subseteq \{x \in X : g(x) < t_2\} \subseteq A(g, t_2);$$

since $\{x \in X : f(x) \leq t_1\}$ is a $C-$closed (resp. $\alpha-$closed) set and since $\{x \in X : g(x) < t_2\}$ is an $\alpha-$open (resp. $C-$open) set, it follows that $C\mathrm{Cl}(A(f,t_1)) \subseteq \alpha\mathrm{Int}(A(g,t_2))$ (resp. $\alpha\mathrm{Cl}(A(f,t_1)) \subseteq C\mathrm{Int}(A(g,t_2))$). Hence $t_1 < t_2$ implies that $A(f,t_1)\ \rho\ A(g,t_2)$. The proof follows from Theorem 2.5. ■

Before stating consequences of Theorem 2.6, we state and prove some necessary lemmas.

Lemma 3.5. *The following conditions on the space X are equivalent:*

(i) *For each pair of disjoint subsets F_1, F_2 of X, such that F_1 is $\alpha-$closed and F_2 is $C-$closed, there exist open subsets G_1, G_2 of X such that $F_1 \subseteq G_1, F_2 \subseteq G_2$ and $G_1 \cap G_2 = \varnothing$.*

(ii) *If F is a $C-$closed (resp. $\alpha-$closed) subset of X which is contained in an $\alpha-$open (resp. $C-$open) subset G of X, then there exists an open subset H of X such that $F \subseteq H \subseteq \mathrm{Cl}(H) \subseteq G$.*

Proof. (i) $\Rightarrow$ (ii) Suppose that $F \subseteq G$, where F and G are $C-$closed (resp. $\alpha-$closed) and $\alpha-$open (resp. $C-$open) subsets of X, respectively. Hence, G^c is an $\alpha-$closed (resp. $C-$closed) and $F \cap G^c = \varnothing$.

By (i) there exist two disjoint open subsets G_1, G_2 of X such that $F \subseteq G_1$ and $G^c \subseteq G_2$. But

$$G^c \subseteq G_2 \Rightarrow G_2^c \subseteq G,$$

and

$$G_1 \cap G_2 = \varnothing \Rightarrow G_1 \subseteq G_2^c$$

hence

$$F \subseteq G_1 \subseteq G_2^c \subseteq G$$

and since G_2^c is a closed set containing G_1 we conclude that $\mathrm{Cl}(G_1) \subseteq G_2^c$, i.e.,

$$F \subseteq G_1 \subseteq \mathrm{Cl}(G_1) \subseteq G.$$

By setting $H = G_1$, condition (ii) holds.

(ii) $\Rightarrow$ (i) Suppose that F_1, F_2 are two disjoint subsets of X, such that F_1 is $\alpha-$closed and F_2 is $C-$closed.

This implies that $F_2 \subseteq F_1^c$ and F_1^c is an $\alpha-$open subset of X. Hence by (ii) there exists an open set H such that $F_2 \subseteq H \subseteq \mathrm{Cl}(H) \subseteq F_1^c$.
But

$$H \subseteq \mathrm{Cl}(H) \Rightarrow H \cap (\mathrm{Cl}(H))^c = \varnothing$$

and

$$\mathrm{Cl}(H) \subseteq F_1^c \Rightarrow F_1 \subseteq (\mathrm{Cl}(H))^c.$$

Furthermore, $(\mathrm{Cl}(H))^c$ is an open set of X. Hence $F_2 \subseteq H$, $F_1 \subseteq (\mathrm{Cl}(H))^c$ and $H \cap (\mathrm{Cl}(H))^c = \varnothing$. This means that condition (i) holds. ■

Lemma 3.6. *Suppose that X is a topological space. If each pair of disjoint subsets F_1, F_2 of X, where F_1 is $\alpha-$closed and F_2 is $C-$closed, can be separated by* *open subsets of X then there exists a continuous function $h : X \to [0,1]$ such that $h(F_1) = \{0\}$ and $h(F_2) = \{1\}$.*

Proof. Suppose F_1 and F_2 are two disjoint subsets of X, where F_1 is $\alpha-$closed and F_2 is $C-$closed. Since $F_1 \cap F_2 = \varnothing$, hence $F_2 \subseteq F_1^c$. In particular, since F_1^c is an $\alpha-$open subset of X containing the $C-$closed subset F_2 of X, by Lemma 3.5, there exists an open subset $H_{1/2}$ of X such that

$$F_2 \subseteq H_{1/2} \subseteq \mathrm{Cl}(H_{1/2}) \subseteq F_1^c.$$

Note that $H_{1/2}$ is also an $\alpha-$open subset of X and contains F_2, and F_1^c is an $\alpha-$open subset of X and contains the $C-$closed subset $\mathrm{Cl}(H_{1/2})$ of X. Hence, by Lemma 3.5, there exists open subsets $H_{1/4}$ and $H_{3/4}$ such that

$$F_2 \subseteq H_{1/4} \subseteq \mathrm{Cl}(H_{1/4}) \subseteq H_{1/2} \subseteq \mathrm{Cl}(H_{1/2}) \subseteq H_{3/4} \subseteq \mathrm{Cl}(H_{3/4}) \subseteq F_1^c.$$

By continuing this method for every $t \in D$, where $D \subseteq [0,1]$ is the set of rational numbers that their denominators are exponents of 2, we obtain open subsets H_t of X with the property that if $t_1, t_2 \in D$ and $t_1 < t_2$, then $H_{t_1} \subseteq H_{t_2}$. We define the function h on X by $h(x) = \inf\{t : x \in H_t\}$ for $x \notin F_1$ and $h(x) = 1$ for $x \in F_1$.

Note that for every $x \in X$, $0 \leq h(x) \leq 1$, i.e., h maps X into [0, 1]. Also, we note that for any $t \in D, F_2 \subseteq H_t$; hence $h(F_2) = \{0\}$. Furthermore, by definition, $h(F_1) = \{1\}$. It remains only to prove that h is a continuous function on X. For every $\beta \in \mathbb{R}$, we have if $\beta \leq 0$ then $\{x \in X : h(x) < \beta\} = \varnothing$ and if $0 < \beta$ then $\{x \in X : h(x) < \beta\} = \cup\{H_t : t < \beta\}$, hence, they are open subsets of X. Similarly, if $\beta < 0$ then $\{x \in X : h(x) > \beta\} = X$ and if $0 \leq \beta$ then $\{x \in X : h(x) > \beta\} = \cup\{(\mathrm{Cl}(H_t))^c : t > \beta\}$ hence, each of them is an open subset of X. Consequently h is a continuous function. ■

Lemma 3.7. *Suppose that X is a topological space such that every two disjoint $C-$closed and $\alpha-$closed subsets of X can be separated by open subsets of X. The following conditions are equivalent:*

(i) *Every countable covering of $C-$open (resp. $\alpha-$open) subsets of X has a refinement consisting of $\alpha-$open (resp. $C-$open) subsets of X such that for every $x \in X$, there exists an open subset of X containing x such that it intersects only finitely many members of the refinement.*

(ii) *Corresponding to every decreasing sequence $\{F_n\}$ of $C-$closed (resp. $\alpha-$closed) subsets of X with empty intersection there exists a decreasing sequence $\{G_n\}$ of $\alpha-$open (resp. $C-$open) subsets of X such that $\bigcap_{n=1}^{\infty} G_n = \varnothing$ and for every $n \in \mathbb{N}, F_n \subseteq G_n$.*

Proof. (i) $\Rightarrow$ (ii) Suppose that $\{F_n\}$ is a decreasing sequence of $C-$closed (resp. $\alpha-$closed) subsets of X with empty intersection. Then $\{F_n^c : n \in \mathbb{N}\}$ is a countable covering of $C-$open (resp. $\alpha-$open) subsets of X . By hypothesis (i) and Lemma 3.5, this covering has a refinement $\{V_n : n \in \mathbb{N}\}$ such that every V_n is an open subset of X and $\mathrm{Cl}(V_n) \subseteq F_n^c$. By setting $G_n = (\mathrm{Cl}(V_n))^c$, we obtain a decreasing sequence of open subsets of X with the required properties.

(ii) $\Rightarrow$ (i) Now if $\{H_n : n \in \mathbb{N}\}$ is a countable covering of $C-$open (resp. $\alpha-$open) subsets of X, we set for $n \in \mathbb{N}$, $F_n = (\bigcup_{i=1}^n H_i)^c$. Then $\{F_n\}$ is a decreasing sequence of $C-$closed (resp. $\alpha-$closed) subsets of X with empty intersection. By (ii) there exists a decreasing sequence $\{G_n\}$ consisting of $\alpha-$open (resp. $C-$open) subsets of X such that $\bigcap_{n=1}^\infty G_n = \varnothing$ and for every $n \in \mathbb{N}, F_n \subseteq G_n$. Now we define the subsets W_n of X in the following manner:

W_1 is an open subset of X such that $G_1^c \subseteq W_1$ and $\mathrm{Cl}(W_1) \cap F_1 = \varnothing$.

W_2 is an open subset of X such that $\mathrm{Cl}(W_1) \cup G_2^c \subseteq W_2$ and $\mathrm{Cl}(W_2) \cap F_2 = \varnothing$, and so on. (By Lemma 3.5, W_n exists).

Then since $\{G_n^c : n \in \mathbb{N}\}$ is a covering for X, hence $\{W_n : n \in \mathbb{N}\}$ is a covering for X consisting of open subsets of X. Moreover, we have

(i) $\mathrm{Cl}(W_n) \subseteq W_{n+1}$

(ii) $G_n^c \subseteq W_n$

(iii) $W_n \subseteq \bigcup_{i=1}^n H_i$.

Now set $S_1 = W_1$ and for $n \geq 2$, we set $S_n = W_{n+1} \setminus \mathrm{Cl}(W_{n-1})$.

Then since $\mathrm{Cl}(W_{n-1}) \subseteq W_n$ and $S_n \supseteq W_{n+1} \setminus W_n$, it follows that $\{S_n : n \in \mathbb{N}\}$ consists of open subsets of X and covers X. Furthermore, $S_i \cap S_j \neq \varnothing$ if and only if $|i - j| \leq 1$. Finally, consider the following sets:

$$\begin{array}{llllll}
S_1 \cap H_1, & S_1 \cap H_2 & & & & \\
S_2 \cap H_1, & S_2 \cap H_2, & S_2 \cap H_3 & & & \\
S_3 \cap H_1, & S_3 \cap H_2, & S_3 \cap H_3, & S_3 \cap H_4 & & \\
\vdots & & & & & \\
S_i \cap H_1, & S_i \cap H_2, & S_i \cap H_3, & S_i \cap H_4, & \cdots, & S_i \cap H_{i+1} \\
\vdots & & & & &
\end{array}$$

These sets are open subsets of X, cover X and refine $\{H_n : n \in \mathbb{N}\}$. In addition, $S_i \cap H_j$ can intersect at most the sets in its row, and in the immediately above, or immediately below row.

Hence if $x \in X$ and $x \in S_n \cap H_m$, then $S_n \cap H_m$ is an open subset of X containing x that intersects at most finitely many of sets $S_i \cap H_j$. Consequently, $\{S_i \cap H_j : i \in \mathbb{N}, j = 1, \ldots, i+1\}$ refines $\{H_n : n \in \mathbb{N}\}$ such that its elements are open subsets of X, and for every point in X we can find an open subset of X containing the point that intersects only finitely many elements of that refinement. ■

Corollary 3.8. *If every two disjoint $C-$closed and $\alpha-$closed subsets of X can be separated by open subsets of X, and in addition, every countable covering of $C-$open (resp. $\alpha-$open) subsets of X has a refinement that consists of $\alpha-$open (resp. $C-$open) subsets of X such that for every point of X we can find an* *open subset containing that point such that it intersects only a finite number of refining members then X has the weakly $c-$insertion property for $(\alpha c, Cc)$ (resp. $(Cc, \alpha c)$).*

Proof. Since every two disjoint $C-$closed and $\alpha-$closed sets can be separated by open subsets of X, therefore by Corollary 3.4, X has the weak $c-$insertion property for $(\alpha c, Cc)$ and $(Cc, \alpha c)$. Now suppose that f and g are real-valued functions on X with $g < f$, such that g is αc (resp. Cc), f is Cc (resp. αc) and $f - g$ is Cc (resp. αc). For every $n \in \mathbb{N}$, set

$$A(f-g, 3^{-n+1}) = \{x \in X : (f-g)(x) \leq 3^{-n+1}\}.$$

Since $f - g$ is Cc (resp. αc), hence $A(f - g, 3^{-n+1})$ is a $C-$closed (resp. $\alpha-$closed) subset of X. Consequently, $\{A(f-g, 3^{-n+1})\}$ is a decreasing sequence of $C-$closed (resp. $\alpha-$closed) subsets of X and furthermore since $0 < f - g$, it follows that $\bigcap_{n=1}^{\infty} A(f - g, 3^{-n+1}) = \varnothing$. Now by Lemma 3.7, there exists a decreasing sequence $\{D_n\}$ of $\alpha-$open (resp. $C-$open) subsets of X such that $A(f - g, 3^{-n+1}) \subseteq D_n$ and $\bigcap_{n=1}^{\infty} D_n = \varnothing$. But by Lemma 3.6, the pair $A(f - g, 3^{-n+1})$ and $X \setminus D_n$ of $C-$closed (resp. $\alpha-$closed) and $\alpha-$closed (resp. $C-$closed) subsets of X can be completely separated by continuous functions. Hence by Theorem 2.6, there exists a continuous function h defined on X such that $g < h < f$, i.e., X has the weakly $c-$insertion property for $(\alpha c, Cc)$ (resp. $(Cc, \alpha c)$). ■

References

1. F. Brooks, Indefinite cut sets for real functions, *Amer. Math. Monthly* **78** (1971), 1007-1010.
2. J. Dontchev, The characterization of some peculiar topological space via $\alpha-$ and $\beta-$sets, *Acta Math. Hungar.* **69** (1-2) (1995), 67-71.
3. J. Dontchev, Between $\alpha-$ and $\beta-$sets, *Math. Balkanica* (N.S) **12** (3-4) (1998), 295-302.
4. M. Ganster and I. Reilly, A decomposition of continuity, *Acta Math. Hungar.* **56** (3-4) (1990), 299-301.
5. E. Hatir, T. Noiri, and S. Yksel, A decomposition of continuity, *Acta Math. Hungar.* **70** (1-2) (1996), 145-150.
6. M. Katětov, On real-valued functions in topological spaces, *Fund. Math.* **38** (1951), 85-91.
7. M. Katětov, Correction to, On real-valued functions in topological spaces, *Fund. Math.* **40** (1953), 203-205.
8. E. Lane, Insertion of a continuous function, *Pacific J. Math.* **66** (1976), 181-190.
9. M. Mirmiran, Insertion of a function belonging to a certain subclass of $\mathbb{R}^X$, *Bull. Iran. Math. Soc.* **28** (2) (2002), 19-27.
10. M. Przemski, A decomposition of continuity and $\alpha-$continuity, *Acta Math. Hungar.* **61** (1-2) (1993), 93-98.