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Stochastic Differential-algebraic Equations of Index 1

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Abstract. In this paper we investigate nonautonomous linear stochastic differential algebraic equations (SDAE). We give a rigorous definition of solutions of such kind of equations. In an analogue with the deterministic case of differential algebraic equations we define the class of index 1 SDAE and prove a theorem on existence and uniqueness of solution for this class.

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1. Introduction

In science and practical applications there are numerous problems such as the problem of description of dynamic systems, electric circuit or problems in cybernetics etc ... requiring investigation of solutions of differential equations of the type

$$A(t)x'(t) + B(t)x(t) = f(t), \quad t \in J := [t_0, T], \tag{1}$$

where $A, B \in C(J, \mathbb{R}^{n.n})$, $f \in C(J, \mathbb{R}^n)$ and the matrix $A(t)$ is singular for every $t \in J$; such equations are called differential algebraic equations (DAE). Without lost of generality, we assume $t_0 = 0$. Investigation of DAE was carried out intensively by many researchers around the world (see [4, 5, 7] and the references therein).

Recently, there has been some incipient work (see [2, 8]) on stochastic differential algebraic equations (SDAE)

$$Adx_t = f(t, x_t)dt + G(t, x_t)dW_t, \quad t \in J, \tag{2}$$

where A is a constant matrix and $\det A = 0$. Here x_t is an $\mathbb{R}^n$-valued stochastic process defined on J, and W denotes an m-dimensional Wiener process given on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $(\mathcal{F}_t)_{t\in J}$. This kind of equation can be considered as a generalization of (1) to include possible random influence of the environment on the system.

Since the focus in [2] and [8] is on numerical computation of solutions and the particular applications (only the case of constant A is considered), some interesting basic theoretical questions (definition of solutions etc.) have been left aside in these papers. As far as we know, up-to-now the most basic notion—formal definition of solution for (2), is still unavailable.

A natural tool in investigation of (2) is Ito stochastic calculus. However, due to the singularity of A, like the case of DAE, one should take care of choosing appropriate definition of solution as well as definition of various classes of SDAE.

In this paper we investigate SDAE (2) with nonautonomous A. We will give a rigorous definition of solution. In an analogue with the DAE case we will define the class of index 1 SDAE and prove a theorem on existence and uniqueness of solution for this class.

In the sequel, we will use the following notations:
The superscript $\top$ stands for transposition,
$|x|$ stands for the norm of $x \in \mathbb{R}^n$ defined by $|x|^2 = \sum_{i=1}^n x_i^2 = x^\top x$,
$|A|$ stands for the norm of a matrix A defined by $|A|^2 = \sum_{i,j=1}^n a_{ij}^2 = trAA^\top$,
$\|f\|_\infty = \max_{t\in[0,T]} |f(t)|$ with the continuous function $f \in C([0,T], \mathbb{R})$.

2. Preliminaries on DAE and SDE

In this section we briefly introduce two topics: differential algebraic equation (DAE) of index-1 and stochastic differential equation (SDE). An expanded introduction on the first topic can be found in [4, 5], while the basic theory of stochastic differential equation can be found in [1, 3, 6].

2.1. Differential algebraic equations of index-1

In this subsection, we consider DAE

$$A(t)x'(t) + B(t)x(t) = f(t), \quad t \in J := [t_0, T], \tag{3}$$

where A, B are assumed to belong to $C(J, L(\mathbb{R}^n))$, $A(t)$ is singular with nullspace $\ker A(t), t \in J$ which is supposed to depend smoothly on t, i.e., there is a projector function $Q \in C^1(J, L(\mathbb{R}^n))$ such that $Q(t)^2 = Q(t)$, $\operatorname{im} Q(t) = \ker A(t)$. Set $P := I - Q$. From the obvious relations

$$AQ \equiv 0, AP \equiv A$$

it follows that

$$Ax' = APx' = A\{(Px)' - P'x\}.$$

Therefore, for (3) it is not necessary to require differentiability of x: differentiability of Px suffices for determination of the terms in (3). Thus, we introduce the following function space which will serve as domain of definition of solutions of (3)

$$C_A^1(J) := \{x \in C(J, \mathbb{R}^n) : Px \in C^1(J, \mathbb{R}^n)\}.$$

Note that C_A^1 does not depend on the choice the C^1-smooth projector Q on $\ker A$.

Definition 2.1. Assume that $\ker A(t)$ is C^1-smooth with Q being a C^1-smooth projector on $\ker A$. A functions $x \in C_A^1(J, \mathbb{R}^n)$ is said to be *a solution of* (3) *on* J if the identity

$$A[(Px)' - P'x] + Bx + f(t) = 0$$

hold for all $t \in J$.

Definition 2.2. DAE (3) is called *tractable with index-1* (or, for short, *of index 1*) if $A_1 := A + B_0 Q$ is nonsingular on J, where $B_0 := B - AP'$.

In case (3) is of index-1, we decouple it into the system

$$\begin{cases} (Px)' &= (P' - PA_1^{-1}B)Px + PA_1^{-1}f(t), \\ Qx &= -QA_1^{-1}BPx + QA_1^{-1}f(t). \end{cases} \tag{4}$$

System (4) shows how to state an initial condition, namely

$$P(0)x(0) = P(0)x^0, \quad x^0 \in \mathbb{R}^n, \tag{5}$$

i.e., the initial condition should fix the free integration constants of the inherent in (3) regular ODE for the component $u := Px$

$$u' = (P' - PA_1^{-1}B)u + PA_1^{-1}f(t). \tag{6}$$

The subspace $\operatorname{im} P(t)$ is easily checked to be invariant for the regular ODE (6), that is, $u(0) \in \operatorname{im} P(0)$ implies $Q(t)u(t) \equiv 0$.

We introduce notations $Q_{can} := QA_1^{-1}B$, $P_{can} := I - Q_{can}$. Then Q_{can} represents again projector onto $\ker A$ along $S := \{x \in \mathbb{R}^n : B_0 x \in \operatorname{im} A\}$; Q_{can} is called the *canonical projector* of (3) when (3) is of index 1. Note that, in general, Q_{can} is only continuous but not C^1-smooth as we require for the projector Q. However, the solutions of (3) with the initial condition (5) can be represented by

$$\begin{aligned}x &= Px + Qx\\ &= u - QA_1^{-1}Bu + QA_1^{-1}f(t)\\ &= (I - QA_1^{-1}B)u + QA_1^{-1}f(t)\\ &= P_{can}u + QA_1^{-1}f(t),\end{aligned}$$

where $u \in C^1$ solves the inherent regular ODE (6) with the initial condition (5). Obviously, the consistent initial value is

$$x_0 := x(0) = P_{can}(0)x^0 + Q(0)A_1^{-1}(0)f(0).$$

We have $P(0)x_0 = P(0)x^0$, but not $x_0 = x^0$, in general.

2.2. Stochastic differential equations

Let W_t denote an m-dimensional Wiener process with independent components given on a probability space $(\Omega,\ \mathcal{F},\ \mathbb{P})$. Denote by $(\mathcal{F}_t)_{t\in J}$ the natural filtration of W_t.

Definition 2.3 ([1,3,6]). A stochastic differential equation is an equation of the form

$$dx_t = f(t,x)dt + G(t,x)dW_t, \quad t \in J, \tag{7}$$

or, in integral form

$$x_t - x^0 = \int_0^t f(s,(x(s))ds + \int_0^t G(s,x(s))dW_s, \quad t \in J, \tag{8}$$

where x^0 is $\mathbb{R}^n$-valued random variables independent of W_t. A solution of (7) (or (8)) on J is a process $x(\cdot) = (x(t))_{t\in J}$ with continuous sample paths that fulfils the following conditions:
(i) $x(\cdot)$ is adapted to the filtration $(\mathcal{F}_t)_{t\in J}$,
(ii) With probability 1, we have

$$\int_0^T |f(s,x(s))|ds < \infty \quad \text{and} \quad \int_0^T |G(s,x(s))|^2 ds < \infty,$$

(iii) (8) holds for every $t \in J$ with probability 1.

Theorem 2.4 ([6]). *Suppose that the SDE* (7) *satisfies the conditions: there exists a constant $K > 0$ such that*
(i) (*Lipschitz condition*) *for all $t \in J$, $x, y \in \mathbb{R}^n$,*

$$|f(t,x) - f(t,y)| + |G(t,x) - G(t,y)| \le K|x-y|;$$

(ii) (*Restriction on growth*) *For all $t \in J$ and $x \in \mathbb{R}^n$,*

$$|f(t,x)|^2 + |G(t,x)|^2 \le K^2(1 + |x|^2).$$

Then, with every random variable x^0 which is independent of W_t, the equation (7) has on J a unique solution $x(t)$, which is continuous with probability 1, that satisfies the initial condition x^0, that is, if $x(t)$ and $y(t)$ are continuous solutions of (7) with the same initial value x^0, then

$$\mathbb{P}[\sup_{t\in J}|x(t)-y(t)|>0]=0.$$

If, additionally, $\mathbb{E}|x^0|^{2n}<\infty$, where n is a positive integer, then

$$\mathbb{E}|x(t)|^{2n}\leq(1+\mathbb{E}|x^0|^{2n})e^{Ct}$$

and

$$\mathbb{E}|x(t)-x^0|^{2n}\leq D(1+\mathbb{E}|x^0|^{2n})t^n e^{Ct},$$

where $C=2n(2n+1)K^2$ and D is a positive constant depending only on n, K and T.

Definition 2.5 ([1]). An *Ito process* is a stochastic process $\{x_t, t\in J\}$ which has Ito stochastic differential

$$dx_t=A_t^{(1)}dt+A_t^{(2)}dW_t,\quad t\in J, \tag{9}$$

or equivalently, x_t satisfies the stochastic integral equation

$$x_t-x_{t_0}=\int_{t_0}^t A_s^{(1)}ds+\int_{t_0}^t A_s^{(2)}dW_s,\quad t\in J, \tag{10}$$

where $A_t^{(1)}$ and $A_t^{(2)}$ are stochastic process of appropriate dimension, adapted to the filtration $(\mathcal{F}_t)_{t\in J}$ and such that the integrals in (10) are well defined Lebesgue and Ito integrals.

Note that in the conditions of Theorem 2.4 the solution of (7) is an Ito process.

3. Stochastic Differential-algebraic Equations of Index 1

Let us consider the linear stochastic differential-algebraic equations (SDAE)

$$A(t)dx+(B(t)x+f(t))dt+G(t,x)dW_t=0,\quad t\in J, \tag{11}$$

where $A,B:J\to L(\mathbb{R}^n,\mathbb{R}^n)$ are continuous $n\times n$-matrix functions, rank $A(t)=r$, r is a fixed integer, $r<n$, $f:J\to\mathbb{R}^n$, $G:J\times\mathbb{R}^n\to\mathbb{R}^{n\times m}$ are continuous functions. For simplicity of notation we set $J=[0,T]$. In this section we present our main results, namely we give a rigorous definition of solutions of (11) and discuss its correctness. We also give definition of index 1 of SDAE, and a theorem on existence and uniqueness of solution of (11) in case of index 1.

First, let us have a look at a simple two-dimensional example, which shows that an appropriate approach is needed for definition of solution of (11).

Example 3.1.

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} d \begin{pmatrix} x \\ y \end{pmatrix} = \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} f_1(t) \\ f_2(t) \end{pmatrix} \right) dt + \begin{pmatrix} 1 \\ a \end{pmatrix} dW_t, \tag{12}$$

where $a \in \mathbb{R}$, $f_1(t), f_2(t)$ are continuous on J. The integral form of (12) is

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x(t) - x(0) \\ y(t) - y(0) \end{pmatrix} = \int_0^t (\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}) ds + \int_0^t \begin{pmatrix} 1 \\ a \end{pmatrix} dW_s.$$

We write this two-dimensional integral equation in a system of two scalar equations

$$\begin{cases} x(t) - x(0) = \int_0^t (x(s) + f_1(s)) ds + \int_0^t dW_s, \\ 0 = \int_0^t (y(s) + f_2(s)) ds + \int_0^t a dW_s, \end{cases}$$

or, equivalently

$$\begin{cases} x(t) = x(0) + \int_0^t (x(s) + f_1(s)) ds + \int_0^t dW_s, \\ \int_0^t (y(s) + f_2(s)) ds = -aW_t. \end{cases}$$

If we consider a solution of this system as an usual continuous stochastic process that satisfies this equation then a has to be equal to zero and now $y(t) = -f_2(t)$ a.s. Therefore, if $f_2(t) \notin C^1(J)$ then $y(t)$ is not an Ito process (clearly $x(t)$ is an Ito process). This example shows that in the case of SDAE not all the coordinates of the solutions can be required to be Ito processes.

Recall from Sec. 2.1 that the solution space $C_A^1(J)$ of a deterministic DAE is a space on continuous functions with differentiable part of coordinates. By considering Ito differential as a stochastic analogue of ordinary differential we shall naturally look for solutions of (11) from the space

$$C_N^1(J, \Omega) := \{x : J \times \Omega \mapsto \mathbb{R}^n \text{ is a continuous stochastic process, } Px \text{ is an Ito process}\}.$$

We will show that this is an appropriate choice of solution space for (11). Let us denote by $N(t) := \ker A(t)$. We assume $N(t) \in C^1$. Let $Q(t)$ be a C^1-projector onto $N(t)$, $P(t) := I - Q(t)$. For simplicity of notation, we omit the argument t here and in the following if no confusion can arise. We call equation

$$A(t) dx_t + (B(t) x_t + f(t)) dt = 0, \quad t \in J, \tag{13}$$

the *deterministic part* of (11).

Lemma 3.2. *The space $C_N^1(J, \Omega)$ does not depend on the choice of the projector P.*

Proof. Let $\tilde{Q}$ be any C^1-projector onto $\ker A$ and $\widetilde{P} := I - \widetilde{Q}$. It is easily seen that $\widetilde{P}P = \widetilde{P}$. Let $x \in C^1_N(J, \Omega)$ be arbitrary, i.e. x is continuous and Px is an Ito process. Since $\widetilde{P}$ is C^1-smooth, by Ito formula $\widetilde{P}(Px)$ is also an Ito process. Therefore, $\widetilde{P}x = \widetilde{P}(Px)$ is also an Ito process. Consequently, whether x belongs to $C^1_N(J, \Omega)$ does not depend on the choice of the projector P. ∎

Now, in an analogue with the deterministic DAE, we note that from the obvious equalities $AQ = 0, AP = A$ it follows that

$$Adx = APdx = A(dPx - P'xdt). \tag{14}$$

Here, we use the equality $dPx = Pdx + P'xdt$ which holds identically if x is an Ito process. Using the arguments similar to that of deterministic DAE we shall use (14) for definition of the term Adx in the SDAE (11). Thus in order to determine Adx we need to require x only to belong to $C^1_N(J, \Omega)$ to enable us to compute dPx. We will prove that this is actually an appropriate approach to the SDAE. First, we show in the following lemma that the use of (14) for definition of the term Adx is correct in the sense that it is independent of the choice of the projector P.

Lemma 3.3. *If $x \in C^1_N(J, \Omega)$ then $A(dPx - P'xdt)$ does not depend on the choice of the C^1-smooth projector $Q = I - P$ onto* $\ker A$.

Proof. Let $Q = I - P$ and $\widetilde{Q} = I - \widetilde{P}$ be two C^1-smooth projectors onto $\ker A$. Since $P = P\widetilde{P}$, by Ito formula, we have

$$d(Px) = dP\widetilde{P}x = P'\widetilde{P}xdt + Pd(\widetilde{P}x).$$

Using the identity $P' = (P\widetilde{P})' = P'\widetilde{P} + P\widetilde{P}'$, we obtain

$$\begin{aligned} A(dPx - P'xdt) &= A(P'\widetilde{P}xdt + Pd\widetilde{P}x - P'xdt) \\ &= A(P'xdt - P\widetilde{P}'xdt + Pd\widetilde{P}x - P'xdt) \\ &= A(Pd\widetilde{P}x - P\widetilde{P}'xdt) \\ &= AP(d\widetilde{P}x - \widetilde{P}'xdt) \\ &= A(d\widetilde{P}x - \widetilde{P}'xdt). \end{aligned}$$

∎

To summarize, we shall understand (11) as

$$AdPx + ((B - AP')x + f)dt + G(t, x)dW_t = 0, \quad t \in J. \tag{15}$$

Like the deterministic case of DAE, we use the notation $B_0 := B - AP'$. Now we come to our definition of solution of SDAE (11).

Definition 3.4. A stochastic process $x \in C^1_N(J, \Omega)$ is said to be a solution of the SDAE (11) if with probability 1 we have

$$\int_{t_0}^{t} A(s)dPx + \int_{t_0}^{t} (B_0x(s) + f(s))ds + \int_{t_0}^{t} G(s, x(s))dW_s = 0, \quad t \in J. \quad (16)$$

Proposition 3.5. *Definition 3.4 does not depend on the choice of the C^1-smooth projector $Q = I - P$ onto* $\ker A$.

Proof. The proposition follows immediately from Lemma 3.3. ■

Remark 3.6. Like the case of deterministic DAE, the Definition 3.4 can be generalized to nonlinear SDAE as well.

Theorem 3.7. *Suppose that $x(t) \in C^1_N(J, \Omega)$ and Px has Ito differential presented in the form*

$$dPx = a(t)dt + b(t)dW_t, \quad (17)$$

where a and b are stochastic processes adapted to the natural filtration of W_t. Then x is a solution of (11) if and only if

$$\begin{cases} A(t)a(t,\omega) + B_0x(t) + f(t) = 0 & a.s.\ for\ almost\ all\ t \in J, \\ A(t)b(t,\omega) + G(x,t) = 0 & a.s.\ for\ almost\ all\ t \in J. \end{cases} \quad (18)$$

Proof. Suppose that $x(t)$ is a solution of (11) with Px having Ito differential presented in (17). By Definition 3.4, we have

$$\int_{t_0}^{t} (A(s)a(s) + B_0x(s) + f(s))ds + \int_{t_0}^{t} (A(s)b(s) + G(x(s), s))dW_s = 0, \quad t \in J. \quad (19)$$

From the theory of stochastic Ito integral (see [1]) it is known that (18) is equivalent to (19). The theorem is proved. ■

As we have seen in Example 3.1 above, in general a solution of a SDAE can not be an usual continuous stochastic process. Hence we need to impose a restriction on the system in order to be able to solve it in the class of usual continuous stochastic processes. A natural restriction is the so-called condition that the noise sources do not appear in the constraints, or equivalently a requirement that the solution process is not directly affected by white noise (see Chein and Denk [2] and Winkler [8]). We will see below that when the deterministic part (13) of (11) is tractable with index-1 DAE, the noise sources appear in constraints via the term $QA_1^{-1}G(t,x)$, hence the requirement that the noise sources do not appear in constraints means that $QA_1^{-1}G(t,x) \equiv 0$, i.e, $\operatorname{im} G(t,x) \subset \operatorname{im} A(t)$ for all $(t,x) \in J \times \mathbb{R}^n$. Let us look back at Example 3.1, we see that the condition $a = 0$ is needed to ensure solution being usual continuous stochastic process. Note that, condition $a = 0$ is equivalent to above condition $\operatorname{im} G(t,x) \subset \operatorname{im} A(t)$. With these reasonings we arrive at the following definition of index 1 for SDAEs.

Definition 3.8. The SDAE (11) is called *tractable with index 1* (or, for short, *of index-1*) if

(i) The deterministic part (13) of (11) is a deterministic DAE which is tractable with index-1,
(ii) $\operatorname{im} G(t,x) \subset \operatorname{im} A(t)$ for all $(t,x) \in J \times \mathbb{R}^n$.

Remark 3.9. (i) Like the deterministic case, tractability with index 1 remains invariant under scaling of (11) by a matrix function $E \in C(J, L(\mathbb{R}^m))$ and transformations $x =: Fy$ with $F \in C^1(J, L(\mathbb{R}^m))$, where $E(t)$ and $F(t)$ are nonsingular on J.

(ii) The notion of index 1 can also be generalized to nonlinear SDAE; note that we should use transferability of the deterministic DAE part instead of the tractable with index-1 (see [4, 5]).

Now, we deal with the problem of existence and uniqueness of solution of (11) in the case of index 1 in a similar way as the deterministic case. First, we make some transformations and decomposition. Multiplying (18) by A_1^{-1} (recall that $A_1 := A + B_0 Q$ is nonsingular since (13) is of index 1), we get

$$\begin{cases} Pa(t) + A_1^{-1}BPx + Qx + A_1^{-1}f(t)) = 0, \quad a.s., \quad t \in J, \\ Pb(t) + A_1^{-1}G(t,x) = 0, \quad a.s., \quad t \in J. \end{cases} \tag{20}$$

By multiplying (20) by P, Q, resp., we decouple it into the system

$$\begin{cases} Pa(t) + PA_1^{-1}BPx + PA_1^{-1}f(t)) = 0, \\ QA_1^{-1}BPx + Qx + QA_1^{-1}f(t) = 0, \\ Pb(t) + PA_1^{-1}G(t,x) = 0, \\ QA_1^{-1}G(t,x) = 0. \end{cases} \tag{21}$$

Since the SDAE (11) is of index 1, we have $\operatorname{im} G(t,x) \subset \operatorname{im} A(t)$, hence

$$QA_1^{-1}G(t,x) = 0, \quad PA_1^{-1}G(t,x) = A_1^{-1}G(t,x).$$

Consequently, (20) is equivalent to

$$\begin{cases} Pa(t) + PA_1^{-1}BPx + PA_1^{-1}f(t) = 0, \\ Qx = -QA_1^{-1}BPx - QA_1^{-1}f(t), \\ Pb(t) + A_1^{-1}G(t,x) = 0. \end{cases} \tag{22}$$

Taking into account the identity $P = PP$, from Ito formula it follows that

$$dPx = dPPx = P'Pxdt + PdPx. \tag{23}$$

This, together with (17) implies

$$\begin{cases} a(t) = P'Px + Pa(t), \\ b(t) = Pb(t). \end{cases} \tag{24}$$

From (22) and (24), we obtain

$$\begin{cases} a(t) = P'Px - PA_1^{-1}BPx - PA_1^{-1}f(t), \\ b(t) = -A_1^{-1}G(t,x), \\ Qx = -QA_1^{-1}BPx - QA_1^{-1}f(t). \end{cases} \tag{25}$$

Thus,

$$\begin{cases} dPx = ((P' - PA_1^{-1}B)Px - PA_1^{-1}f(t))dt - A_1^{-1}G(t,x)dW_t, \\ Qx = -QA_1^{-1}BPx - QA_1^{-1}f(t). \end{cases} \tag{26}$$

We introduce the notations $u := Px, v := Qx$. Then $x = u + v$, and we obtain the expression for v via u

$$v = -QA_1^{-1}Bu - QA_1^{-1}f, \tag{27}$$

and a (classical) Ito stochastic differential equation for u

$$\begin{aligned} du = \{&(P' - PA_1^{-1}B)u - PA_1^{-1}f(t)\}dt \\ &-\{A_1^{-1}G(t,(I - QA_1^{-1}B)u - QA_1^{-1}f)\}dW_t. \end{aligned} \tag{28}$$

Definition 3.10. Equation (28) is called *an inherent regular SDE* (under P) of the SDAE (11).

Remark 3.11. If the SDAE (11) is linear (homogeneous, homogeneous autonomous, resp.), then so is the inherent equation (28).

Remark 3.12. $\operatorname{im} P(t)$ is an invariant subspace of the inherent regular SDE (28) in the sense with probability one:

$$\text{if} \quad u(0) \in \operatorname{im} P(0) \quad \text{then} \quad u(t) = P(t)u(t) \quad \text{for all} \quad t \in J.$$

Indeed, for $z(t) := Q(t)u(t)$ according to Ito formula, using the identities $Q' = -P'$, $QP = 0$ and (28), we have

$$\begin{aligned} dz &= Q'udt + Qdu \\ &= Q'udt + Q\{(P' - PA_1^{-1}B)u - PA_1^{-1}f(t))dt - PA_1^{-1}G(u+v,t)dW_t\} \\ &= (-P'u + QP'u)dt \\ &= -PP'udt \\ &= -P'Qudt = -P'zdt. \end{aligned}$$

This is a homogeneous linear explicit differential equation for $z(t)$. Since the initial condition $z(0) = Q(0)u(0) = 0$ we get $z(t) \equiv 0$ a.s., hence $u(t) = (P + Q)u(t) = P(t)u(t)$ for all $t \in J$.

Furthermore, we note that the equation (27) leads to $v(t) = Qv(t)$ for all $t \in J$. Clearly, initial value problems for (11) may become solvable only for arbitrary $u_0 \in \operatorname{im} P(0)$ and $v_0 = -Q(0)A_1^{-1}(0)B_0(0)u_0 - Q(0)A^{-1}(0)f(0)$, i.e. v_0 is not arbitrary but is computable via u_0. Inspired by the above decoupling procedure, we state the consistent initial conditions for the SDAE (11) of index 1 as follows

$$\begin{cases} A(0)(x(0) - x^0) = 0 \quad \text{a.s.,} \\ x^0 \text{ is such an } \mathbb{R}^n\text{-valued random variable that } A(0)x^0 \\ \text{ is } \mathcal{F}_0\text{- measurable, independent of the Wiener process } W_t. \end{cases} \tag{29}$$

As in the case of deterministic DAE, we have $u(0) = P(0)x(0) = P(0)x^0$ a.s. In general, unless $Q(0)x^0 = Q(0)A_1^{-1}B_0(0)u^0 + Q(0)A_1^{-1}f(0)$ a.s., the consistent initial value $x(0)$ will differ from the given x^0. Thus, solving (28) with the initial condition (29) and using (27), we get an expression for the solution of SDAE (11) as follows

$$x(t) = (I - QA_1^{-1}B)u(t) - QA_1^{-1}f(t). \tag{30}$$

Remark 3.13. If we use canonical projector Q_{can} then the formulas (27), (28), (30) can be rewritten as follows

$$v(t,u) = -Q_{can}u(t) - QA_1^{-1}f(t), \quad \text{a.s.}, t \in J, \tag{31}$$

$$\begin{aligned} du = \{&(P' - PA_1^{-1}B)u - PA_1^{-1}f(t)\}dt \\ &-\{A_1^{-1}G(t, P_{can}u - QA_1^{-1}f(t))\}dW_t, \end{aligned} \tag{32}$$

and

$$x(t) = P_{can}u(t) - QA_1^{-1}f(t). \tag{33}$$

Now we are able to prove our main theorem on the existence and uniqueness of solution of SDAE of index 1.

Theorem 3.14. *Suppose that* (11) *is an SDAE of index 1 with* A, B, f, G *being continuous and* G *being Lipschitz-continuous with respect to* x*, then the initial value problem of* (11) *with initial condition* (29) *has a solution process* $x(\cdot)$ *on* J*, that is path-wise unique and is given by the formula*

$$x(t) = (I - QA_1^{-1}B)u(t) - QA_1^{-1}f(t),$$

where $u(t)$ *is a solution of regular SDE* (28) *with initial condition* $u(0) = P(0)x^0$*. Moreover, if* $\mathbb{E}|A(0)x^0|^{2n} < \infty$*, where* n *is a positive integer then the following inequalities hold*

$$\mathbb{E}|x(t)|^{2n} \le C_0(t) + C_1(1 + \mathbb{E}|P(0)x^0|^{2n})e^{Ct},$$
$$\mathbb{E}|x(t) - x(0)|^{2n} \le C_2(1 + \mathbb{E}|P(0)x^0|^{2n})t^n e^{Ct} + C_3(t),$$

where $C_0(\cdot), C_3(\cdot)$ *are continuous functions,* $C_3(0) = 0$*, and* C, C_1, C_2 *are positive constants.*

Proof. We shall prove that under the hypothesis of Theorem 3.14, the regular inherent SDE (28) has on J a unique solution, which is continuous with probability 1. To this end we show that the conditions of Theorem 2.4 are satisfied for (28).

(i) *Lipschitz condition.* Put $\widehat{f}(t,u) := (P'(t) - PA_1^{-1}B(t))u - PA_1^{-1}f(t)$. Since A_1^{-1} is continuous, so is $\widehat{f}(t,u)$. We have

$$\begin{aligned}|\widehat{f}(t,u) - \widehat{f}(t,\bar{u})| &\le |(PA_1^{-1}B(t) - P'(t))(u-\bar{u})| \\ &\le \|PA_1^{-1}B - P'\|_\infty |u-\bar{u}|.\end{aligned}$$

Since $J=[0,T]$ is compact $\|PA_1^{-1}B - P'\|_\infty = \max_{t\in J}|PA_1^{-1}B(t) - P'(t)|$ is finite, hence $\widehat{f}$ is Lipschitz with respect to u.

Now we put $\widehat{G}(t,u) := -A_1^{-1}G(t,u+v)$. Note that $v(t,u) = -QA_1^{-1}B(t)u - QA_1^{-1}f(t)$ is continuous with respect to t and Lipschitz with respect to u with a constant $L_v := \|Q_{can}\|_\infty$. Since $G(t,x)$ is Lipschitz with respect to x with a constant L_G we have

$$\begin{aligned}&|\widehat{G}(t,u) - \widehat{G}(t,\bar{u})| = \\ &= |A_1^{-1}(t)G(t,u+v(t,u)) - A_1^{-1}(t)G(t,\bar{u}+v(t,\bar{u}))| \\ &\le |A_1^{-1}(t)|L_G|(u+v(t,u)) - (\bar{u}+v(t,\bar{u}))| \\ &\le \|A_1^{-1}\|_\infty L_G|(u-\bar{u}) + (v(t,u)-v(t,\bar{u}))| \\ &\le \|A_1^{-1}\|_\infty L_G\{|u-\bar{u}| + L_v|u-\bar{u}|\} = \|A_1^{-1}\|_\infty L_G(1+L_v)|u-\bar{u}|.\end{aligned}$$

Hence, $\widehat{G}(t,u)$ is Lipschitz with respect to u.

(ii) *Restriction on growth.* We note that, for a continuous function $g(t,x)$ on compact time-intervals J, the Lipschitz condition with respect to x implies the usual growth condition, indeed, for all $(t,x)\in J\times\mathbb{R}^n$ we have

$$|g(t,x)| \le (|g(t,x) - g(t,0)| + |g(t,0)|) \le \max(\|g(\cdot,0)\|_\infty, L_g)(1+|x|),$$

where L_g denotes the Lipschitz constant of g with respect to the variable x.

(iii) *Initial condition.* We have $P(0)x^0 = A_1^{-1}(0)A(0)x^0$ so that $u(0) := P(0)x^0$ is $\mathcal{F}_0$ - measurable and independent of the Wiener process W_t.

Now Theorem 2.4 is applicable to the inherent regular SDE (28) and entails: the inherent regular SDE (28) has a path-wise unique continuous solution process $u(t)$ with the initial condition $u(0) = P(0)x^0$. Consequently, $x = (I - QA_1^{-1}B)u(t) - QA_1^{-1}f(t) = P_{can}(t)u(t) - QA_1^{-1}f(t)$ is a solution of (11).

Next, we will prove that it is also the unique solution of (11). Indeed, suppose that $\tilde{x} = P_{can}\tilde{u}(t) - \tilde{Q}\tilde{A}_1^{-1}f(t)$ is a solution of (11), where $\tilde{u}$ is the unique solution of the inherent regular SDE under $\tilde{P}$ of the SDAE (11) with the initial condition $\tilde{u}(0) = \tilde{P}(0)x^0$.

It is easy to check that $z(t) := \tilde{P}u(t)$ is a solution of the inherent regular SDE under $\tilde{P}$ of the SDAE (11) satisfying the initial condition

$$z(0) = \tilde{P}(0)u(0) = \tilde{P}(0)P(0)x^0 = \tilde{P}(0)x^0.$$

From the uniqueness of solutions of the inherent regular SDE under $\tilde{P}$ of the SDAE (11) it follows that $z(t) \equiv \tilde{u}(t)$, hence $\tilde{P}u(t) \equiv \tilde{u}(t)$. Consequently, $P_{can}u(t) \equiv P_{can}\tilde{u}(t)$.

Notice that, $QA_1^{-1}f$ does not depend on the choice of the projector Q onto $\ker A$. This implies that

$$x(t) = P_{can}u(t) - QA_1^{-1}f = P_{can}\tilde{u}(t) - \tilde{Q}\tilde{A}_1^{-1}f = \tilde{x}(t).$$

The uniqueness of solutions of (11) is proved.

Now, if $\mathbb{E}|A(0)x^0|^{2n} < \infty$ then

$$\mathbb{E}|u(0)|^{2n} = \mathbb{E}|A_1^{-1}(0)A(0)x^0|^{2n} \le |A_1^{-1}(0)|^{2n}\mathbb{E}|A(0)x^0|^{2n} < \infty.$$

In this case Theorem 2.4 asserts that

$$\mathbb{E}(|u(t)|^{2n}) \le D(1 + \mathbb{E}(|u(0)|^{2n})e^{Ct},$$

$$\mathbb{E}(|u(t) - u(0)|^{2n}) \le D(1 + \mathbb{E}(|u(0)|^{2n})t^n e^{Ct},$$

where $t \in J$, $C := 2n(2n+1)K^2$ and D is a positive constant depending only on n, K, T. Since $x(t) = u(t) + v(t,u)$, applying the elementary inequality $(a+b)^n \le 2^n(a^n + b^n)$ we get

$$|x(t)|^{2n} \le 2^{2n}(\|P_{can}\|_\infty^{2n}|u(t)|^{2n} + |QA_1^{-1}f(t)|^{2n}).$$

Consequently,

$$\begin{aligned}\mathbb{E}|x(t)|^{2n} &\le 2^{2n}\{\|P_{can}\|_\infty^{2n}\mathbb{E}|u(t)|^{2n} + |QA_1^{-1}f(t)|^{2n}\} \\ &\le 2^{2n}\|P_{can}\|_\infty^{2n}(1 + \mathbb{E}|u(0)|^{2n})e^{Ct} + 2^{2n}|QA_1^{-1}f(t)|^{2n} \\ &= C_0(t) + C_1(1 + \mathbb{E}|u(0)|^{2n})e^{Ct},\end{aligned}$$

where

$$C_0(t) := 2^{2n}|QA_1^{-1}f(t)|^{2n}, C_1 := 2^{2n}\|P_{can}\|_\infty^{2n}.$$

Now, we have

$$\begin{aligned}|x(t) - x(0)| &= |P_{can}(t)u(t) - QA_1^{-1}f(t)) - (P_{can}(0)u(0) - QA_1^{-1}f(0)| \\ &\le |P_{can}(t)||(u(t) - u(0)| + |P_{can}(t) - P_{can}(0)||u(0)| + \\ &\qquad + |QA_1^{-1}f(t) - QA_1^{-1}f(0)|.\end{aligned}$$

Applying the elementary inequality $(a+b+c)^{2n} \le 3^{2n-1}(a^{2n} + b^{2n} + c^{2n})$, we get

$$\begin{aligned}
&\mathbb{E}|x(t)-x(0)|^{2n} \leq \\
&\leq 3^{2n-1}\mathbb{E}\{|P_{can}(t)|^{2n}|u(t)-u(0)|^{2n} + |P_{can}(t)-P_{can}(0)|^{2n}|u(0)|^{2n}+ \\
&\qquad\quad + |QA_1^{-1}f(t)-QA_1^{-1}f(0)|^{2n}\} \\
&\leq 3^{2n-1}\{\|P_{can}\|_\infty^{2n}\mathbb{E}|u(t)-u(0)|^{2n} + |P_{can}(t)-P_{can}(0)|^{2n}\mathbb{E}|u(0)|^{2n}+ \\
&\qquad\quad + |QA_1^{-1}f(t)-QA_1^{-1}f(0)|^{2n}\} \\
&\leq 3^{2n-1}\{\|P_{can}\|_\infty^{2n}D(1+\mathbb{E}|u(0)|^{2n})t^n e^{Ct} \\
&\qquad\quad + |P_{can}(t)-P_{can}(0)|^{2n}\mathbb{E}|u(0)|^{2n} + |QA_1^{-1}f(t)-QA_1^{-1}f(0)|^{2n}\} \\
&= C_2(1+\mathbb{E}|u(0)|^{2n})t^n e^{Ct} + C_3(t),
\end{aligned}$$

where

$$C_2 = 3^{2n-1}D\|P_{can}\|_\infty^{2n},$$
$$C_3(t) = 3^{2n-1}\{|P_{can}(t)-P_{can}(0)|^{2n}\mathbb{E}|u(0)|^{2n} + |QA_1^{-1}f(t)-QA_1^{-1}f(0)|^{2n}\}.$$

Clearly, $C_0(\cdot)$ and $C_3(\cdot)$ are continuous, and $C_3(0)=0$. The theorem is proved. ∎

Remark 3.15. (i) If $A(t)$ is nonsingular for all $t \in J$ then, by multiplying with A^{-1}, (11) becomes a (classical) Ito SDE $dx + A^{-1}(B(t)x + f(t))dt + A^{-1}G(t,x)dW_t = 0$, $t \in J$ and our results reduce to the well known results for Ito SDE.
(ii) If $G(x,t) \equiv 0$ then (11) becomes a deterministic DAE. In this case, our results reduce to the well known results for deterministic DAE (see [4, 5]).

Now we give an example to illustrate our results above.

Example 3.16. Let us consider a SDAE on $\mathbb{R}^+$

$$\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} dx + (\begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} x + \begin{pmatrix} 0 \\ f_2 \end{pmatrix})dt + \begin{pmatrix} -e^{-t} \\ e^{-t} \end{pmatrix} dW_t = 0, \qquad (34)$$

where f_2 is continuous but nondifferentiable function.

For this SDAE we have $\ker A(t) = \{(x,y) \in \mathbb{R}^2 : x = y\}$. Choose

$$Q = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, P = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix},$$

then $A_1 = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$. Clearly, the matrix A_1 is nonsingular and $\operatorname{im} G(t,x) \subset \operatorname{im} A(t)$ for all $t \geq 0$, hence, the SDAE (34) is of index 1. Furthermore, we have $A_1^{-1} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$, $Q_{can} = QA_1^{-1}B = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} = Q$, and the inherent SDE is

$$du = \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix} udt + \begin{pmatrix} e^{-t} \\ 0 \end{pmatrix} dW_t.$$

We can solve the inherent SDE for u with the initial condition

$$u(0) = P(0)x^0 = \begin{pmatrix} x_1^0 - x_2^0 \\ 0 \end{pmatrix},$$

where x^0 is a random variable that $A(0)x^0$ is independent of the Wiener process W_t, and we get the solution

$$u = \begin{pmatrix} e^{-t}(x_1^0 - x_2^0 + W_t) \\ 0 \end{pmatrix}.$$

Obviously, $u = Px$ is an Ito process as it must be since it is a solution of an Ito SDE. Now, having solved the inherent SDAE we can easily obtain solution of the SDAE (34), namely,

$$x = P_{can}u - QA_1^{-1}f = \begin{pmatrix} e^{-t}(x_1^0 - x_2^0 + W_t) - f_2 \\ -f_2 \end{pmatrix}.$$

In this Example, we have $Qx = -QA_1^{-1}f = \begin{pmatrix} -f_2 \\ -f_2 \end{pmatrix}$. Clearly, Qx is not an Ito process since f_2 is not differentiable.

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