Vietnam Journal of Mathematics **37**:1(2009) 43-48

Vietnam Journal
of
MATHEMATICS

The Normality of Cubic Cayley Graphs for Dihedral Groups

Mehdi Alaeiyan, M. Ghasemi, and G. R. Safakish

Department of Mathematics,
Iran University of Science and Technology,
Narmak, Tehran 16844, Iran

Received April 25, 2007
Revised November 25, 2008

Abstract. A Cayley graph $\Gamma = \mathrm{Cay}(G, S)$ is called normal for G, if G_R, the right regular representation of G, is a normal subgroup of the full automorphism group $\mathrm{Aut}(\Gamma)$ of Γ. In this paper we determine the normality of connected and undirected Cayley graphs of valency three for dihedral groups.

2000 Mathematics Subject Classification: 05C25, 20B25.

Key words: Cayley graph, normal Cayley graph, dihedral group, automorphism group.

1. Introduction

Let G be a finite group, and S a subset of G such that $1_G \notin S$. The *Cayley digraph* $\Gamma = \mathrm{Cay}(G, S)$ of G with respect to S is defined to have vertex set $V(\Gamma) = G$ and arc set $E(\Gamma) = \{(g, sg) | g \in G, s \in S\}$. From the definition the following obvious facts are basic for Cayley digraphs:

(1) The automorphism group $\mathrm{Aut}(\Gamma)$ of Γ contains G_R, the right regular representation of G, as a subgroup;

(2) Γ is connected if and only if S generates the group G;

(3) Γ is undirected if and only if $S = S^{-1}$. A Cayley graph $\Gamma = \mathrm{Cay}(G, S)$ is called *normal* if the right regular representation of G is a normal subgroup of the automorphism group of Γ.

The study of the normality of Cayley graphs is important in many cases, for example for CI-subsets (see [10]). For abelian groups, Baik et al. [2, 3] classified the Cayley graphs of valency 3, 4 and 5, which are not normal, and Fang et al. [7] proved that, for most finite simple groups G, connected cubic Cayley graphs of G are all normal. Also Wang et al. [9] proved that every finite group G has a normal Cayley graph unless $G \cong Z_4 \times Z_2$ or $G \cong Q_8 \times Z_2^r$ $(r \geq 0)$ and that every finite group has a normal Cayley digraph. In general, it is known to be difficult to determine the normality of Cayley graphs. The only groups for which the complete information about the normality of Cayley graphs is available, are the cyclic groups of prime order [1] and the groups of order twice a prime [4]. For notation and terminology on permutation groups we refer the reader to [8].

Throughout this paper, we suppose that $D_{2n} = \langle a, b | a^n = b^2 = 1, bab^{-1} = a^{-1} \rangle$ is a dihedral group, and $\Gamma = \mathrm{Cay}(D_{2n}, S)$ is connected and undirected cubic Cayley graph. The main result of this paper is the following theorem:

Theorem 1.1. *Let G be the dihedral group D_{2n}, and let $\Gamma = \mathrm{Cay}(G, S)$ be a connected and undirected cubic Cayley graph. Then Γ is normal except one of the following cases happens:*

(1) $n = 4$, $S = \{b, ab, a^2b\}$, $\Gamma \cong K_{4,4} - 4K_2$;

(2) $n = 8$, $S = \{b, ab, a^3b\}$, $\Gamma \cong P(8, 3)$, *the generalized Petersen graph;*

(3) $n = 3$, $S = \{b, ab, a^2b\}$, $\Gamma \cong K_{3,3}$;

(4) $n = 7$, $S = \{b, ab, a^3b\}$, $\Gamma \cong S(7)$, *the Heawood's graph.*

2. Preliminaries

In this section we give some basic facts on Cayley graphs, which will be useful for our purpose. Now we have the first lemma from [10].

Lemma 2.1. [10, Proposition 1.5] *Let $\Gamma = \mathrm{Cay}(G, S)$, and $A = \mathrm{Aut}(\Gamma)$. Then Γ is normal if and only if $A_1 = \mathrm{Aut}G, S)$, where A_1 is the stabilizer of 1 in A.*

Lemma 2.2. [6, Lemma 4.4] *All 1-regular cubic Cayley graphs on the dihedral group D_{2n} are normal.*

Lemma 2.3. [5, Lemma 3.2] *Let Γ be a connected cubic graph on dihedral group D_{2n}, and let B_1 and B_2 be two orbits of $C = \langle a \rangle$. Also let G^* be the subgroup of G fixing setwise B_1 and B_2, respectively. If G^* acts unfaithfully on one of B_1 and B_2, then $\Gamma \cong K_{3,3}$.*

Let C_G be the core of $C = \langle a \rangle$ in $G = D_{2n}$. By assuming the hypothesis in the above lemma we have the following results:

Lemma 2.4. [5, Lemma 3.5] *If C_G is a proper subgroup of C, then Γ is isomorphic to $\mathrm{Cay}(D_{14}, \{b, ab, a^3b\})$ or $\mathrm{Cay}(D_{16}, \{b, ab, a^3b\})$.*

Lemma 2.5. [5, Lemma 3.6] *If* $C_G = C$, *then* Γ *is isomorphic to* $\mathrm{Cay}(D_{2n}, \{b, ab, a^k b\})$, *where* $k^2 - k + 1 = 0 \,(\mathrm{mod}\, n)$, *and* $n \geq 13$.

Let $G = D_{2n}$. Then the elements of G are a^i and $a^i b$, where $i = 0, 1, ..., n-1$. All $a^i b$ are involutions, and a^i is an involution if and only if n is even and $i = \frac{n}{2}$. Finally in this section we obtain a preliminary result restricting S for a cubic Cayley graphs of $\mathrm{Cay}(D_{2n}, S)$. We can easily prove the following lemma:

Lemma 2.6. *Let* $\Gamma = \mathrm{Cay}(G, S)$ *be a Cayley graphs of* $G = D_{2n}$. *Then* Γ *is cubic, connected and undirected if and only if one of the following conditions holds:*

(1) *When* n *is odd, we have:*

$$S_{o_1} = \{a^i b, a^j b, a^k b\},\ 0 \leq i < j < k < n,$$
$$S_{o_2} = \{a^i, a^{-i}, a^j b\},\ 0 < i < \frac{n}{2},\ 0 \leq j < n;$$

(2) *When* n *is even, we have:*

$$S_{e_1} = \{a^i b, a^j b, a^k b\},\ 0 \leq i < j < k < n,$$
$$S_{e_2} = \{a^i, a^{-i}, a^j b\},\ 0 < i < \frac{n}{2}, 0 \leq j < n,$$
$$S_{e_3} = \{a^{n/2}, a^i, a^{-i}\},\ 0 < i < \frac{n}{2},$$
$$S_{e_4} = \{a^{n/2}, a^i b, a^j b\},\ 0 \leq i < j < n.$$

Let X and Y be two graphs. The *direct product* $X \times Y$ is defined as the graph with vertex set $V(X \times Y) = V(X) \times V(Y)$ such that for any two vertices $u = [x_1, y_1]$ and $v = [x_2, y_2]$ in $V(X \times Y)$, $[u, v]$ is an edge in $X \times Y$ whenever $x_1 = x_2$ and $[y_1, y_2] \in E(Y)$ or $y_1 = y_2$ and $[x_1, x_2] \in E(X)$. Two graphs are called relatively prime if they have no nontrivial common direct factor. The *lexicographic product* $X[Y]$ is defined as the graph with vertex set $V(X[Y]) = V(X) \times V(Y)$ such that for any two vertices $u = [x_1, y_1]$ and $v = [x_2, y_2]$ in $V(X[Y])$, $[u, v]$ is an edge in $X[Y]$ whenever $[x_1, x_2] \in E(X)$ or $x_1 = x_2$ and $[y_1, y_2] \in E(Y)$. Let $V(Y) = \{y_1, y_2, ..., y_n\}$. Then there is a natural embedding nX in $X[Y]$, where for $1 \leq i \leq n$, the ith copy of X is the subgraph induced on the vertex subset $\{(x, y_i) | x \in V(X)\}$ in $X[Y]$. The *deleted lexicographic product* $X[Y] - nX$ is the graph obtained by deleting all the edges of (this natural embedding of) nX from $X[Y]$.

3. Proof of Theorem 1.1

In order to prove Theorem 1.1, we must determine all non-normal connected undirected cubic Cayley graphs for dihedral group D_{2n}. If $n = 2$, then dihedral group D_4 is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$, and so it is easy to show that the cubic

Cayley graph $\mathrm{Cay}(D_4, S)$ is normal. So from now we assume that $n \geq 3$. Also, since $\mathrm{Cay}(D_{2n}, S)$ when $S = S_{e_3}$ is disconnected, thus we do not consider this case for the proof of the main theorem. Moreover $\Gamma_2(1)$ is the subgraph of Γ with vertex set $1 \cup S \cup S^2$ and edge set $\{[1, s], [s, ts] | s, t \in S\}$. First we prove the following lemma which will be used in the proof of Theorem 1.1:

Lemma 3.1. *Let G be the dihedral group D_{2n} with $n \geq 3$ and $\Gamma = \mathrm{Cay}(G, S)$ be a cubic Cayley graph. Then*

(a) *If S is S_{e_4} and Γ is connected, then $S \cap (S^2 - \{1\}) = \emptyset$ holds;*

(b) *If S is S_{o_2} or S_{e_2} and Γ is connected, then $S \cap (S^2 - \{1\}) = \emptyset$ if $n > 3$. For $n = 3$ and S is S_{o_2} or S_{e_2}, we have $S \cap (S^2 - \{1\}) \neq \emptyset$ and $\Gamma = \mathrm{Cay}(D_6, S)$ is connected and normal;*

(c) *If S is S_{o_1} or S_{e_1}, then $S \cap (S^2 - \{1\}) = \emptyset$ always holds.*

Proof. (a) Suppose first that $S = S_{e_4} = \{a^{n/2}, a^i b, a^j b\}$. Then

$$S^2 - \{1\} = \{a^{n/2+i} b, a^{n/2+j} b, a^{i-j}, a^{j-i}\}.$$

We show that $S \cap (S^2 - \{1\}) = \emptyset$. Suppose to the contrary that $S \cap (S^2 - \{1\}) \neq \emptyset$. We may suppose that $a^{i-j} = a^{j-i} = a^{n/2}$. Now we have $\Gamma = nK_1[Y]$, where $Y = K_4$. Hence Γ is not connected, which is a contradiction.

(b) Now suppose that $S = S_{o_2}$ or $S = S_{e_2}$, that is, $S = \{a^i, a^{-i}, a^j b\}$. For $n > 3$, we have $S^2 - \{1\} = \{a^{2i}, a^{-2i}, a^{i+j} b, a^{j-i} b\}$. We claim that $S \cap (S^2 - \{1\}) = \emptyset$. Suppose to the contrary that $S \cap (S^2 - \{1\}) \neq \emptyset$. We may suppose that $a^{2i} = a^{-i}$. Then $\Gamma = mK_1[Y]$, where $Y = \mathrm{Cay}(S, \langle S \rangle)$ and $|D_{2n} : \langle S \rangle| = m$. So Γ is not connected, which is a contradiction. Now let $n = 3$. Then $S = \{a, a^{-1}, b\}$, $\{a, a^{-1}, ab\}$, or $\{a, a^{-1}, a^2 b\}$, respectively. Therefore $S^2 - 1 = \{a^2, ab, a^2 b, a\}$, $\{a^2, a^2 b, a, b\}$, or $\{a^2, b, a, ab\}$, respectively. Obviously Γ is connected, and $G \cong D_6$. Also we have $S \cap (S^2 - \{1\}) \neq \emptyset$, and $\mathrm{Cay}(D_6, \{a, a^{-1}, b\}) \cong \; Cay(D_6, \{a, a^{-1}, ab\}) \cong \mathrm{Cay}(D_6, \{a, a^{-1}, a^2 b\})$. Let σ be an automorphism of $\Gamma = \mathrm{Cay}(D_6, \{a, a^{-1}, b\})$, which fixes 1 and all elements of S. Since $a^\sigma = a$, and $(a^2)^\sigma = a^2$, we have $\{1, a^2, a^2 b\}^\sigma = \{1, a^2, a^2 b\}$ and $\{1, a, ab\}^\sigma = \{1, a, ab\}$. Therefore $(ab)^\sigma = ab$, and $(a^2 b)^\sigma = a^2 b$, and hence σ fixes all elements of S^2. Thus $\sigma = 1$, and A_1 acts faithfully on S. So we may view A_1 as a permutation group on S. Now let α be an arbitrary element of A_1. Since $1^\alpha = 1$, we have $\{a, a^2, b\}^\alpha = \{a, a^2, b\}$. If $b^\alpha = a$ or $b^\alpha = a^2$, then $\{1, ab, a^2 b\}^\alpha = \{1, a^2 b, a^2\}$ or $\{1, ab, a^2 b\}^\alpha = \{1, ab, a\}$, which is a contradiction. Thus $b^\alpha = b$, and A_1 is generated by the permutation (a, a^2). So $|A_1| = 2$. On the other hand, $\beta : a^t b^l \to a^{2t} b^l$ is an element of $\mathrm{Aut}(G, S)$. Therefore $|A_1| = |\mathrm{Aut}(G, S)| = 2$, and hence by Lemma 2.1, Γ is normal.

(c) Finally, suppose that $S = S_{o_1}$ or $S = S_{e_1}$, that is, $S = \{a^i b, a^j b, a^k b\}$. Then $S^2 - \{1\} = \{a^{i-j}, a^{j-i}, a^{i-k}, a^{k-i}, a^{j-k}, a^{k-j}\}$. Clearly $S \cap (S^2 - \{1\}) = \emptyset$. The results now follow. ■

Now we complete the proof of Theorem 1.1. First assume that $S = S_{e_4}$. Since Γ is connected, by Lemma 3.1 (a), $S \cap (S^2 - \{1\}) = \emptyset$. Now consider the graph

$\Gamma_2(1)$, and let σ be an automorphism of $\Gamma = \mathrm{Cay}(D_{2n}, \{a^{n/2}, a^i b, a^j b\})$, which fixes 1 and all elements of S. Since $(a^{n/2})^\sigma = a^{n/2}$, $(a^i b)^\sigma = a^i b$, and $(a^j b)^\sigma = a^j b$, we have $\{1, a^{n/2+i}b, a^{n/2+j}b\}^\sigma = \{1, a^{n/2+i}b, a^{n/2+j}b\}$, $\{1, a^{n/2+i}b, a^{j-i}\}^\sigma = \{1, a^{n/2+i}b, a^{j-i}\}$, and $\{1, a^{n/2+j}b, a^{i-j}\}^\sigma = \{1, a^{n/2+j}b, a^{i-j}\}$, respectively. Therefore $(a^{n/2+i}b)^\sigma = a^{n/2+i}b$, $(a^{n/2+j}b)^\sigma = a^{n/2+j}b$, $(a^{j-i})^\sigma = a^{j-i}$, and $(a^{i-j})^\sigma = a^{i-j}$, and hence σ fixes all elements of S^2. Because of the connectivity of Γ, this automorphism is the identity in $\mathrm{Aut}(\Gamma)$. Therefore A_1 acts faithfully on S. So we may view A_1 as a permutation group on S. Now let α be an arbitrary element of A_1. Since $1^\alpha = 1$, we have $\{a^{n/2}, a^i b, a^j b\}^\alpha = \{a^{n/2}, a^i b, a^j b\}$. If $(a^{n/2})^\alpha = a^i b$, or $(a^{n/2})^\alpha = a^j b$, then $\{1, a^{n/2+i}b, a^{n/2+j}b\}^\alpha = \{1, a^{n/2+i}b, a^{j-i}\}$, or $\{1, a^{n/2+i}b, a^{n/2+j}b\}^\alpha = \{1, a^{n/2+j}b, a^{i-j}\}$, respectively. Now again we consider $\Gamma_2(1)$. In this subgraph $a^{n/2+i}b$, and $a^{n/2+j}b$ have valency 2, and a^{i-j}, a^{j-i} have valency 1. This implies a contradiction. Thus $(a^{n/2})^\alpha = a^{n/2}$, and A_1 is generated by the permutation $(a^i b, a^j b)$. So $|A_1| = 2$. On the other hand, $\beta : a^t b^l \longrightarrow a^{-t}(a^{i+j}b)^l$ is an element of $\mathrm{Aut}(G, S)$. Therefore $|A_1| = |\mathrm{Aut}(G, S)| = 2$, and hence by Lemma 2.1, Γ is normal.

Now assume that $S = S_{e_2} = \{a^i, a^{-i}, a^j b\}$, or $S = S_{o_2} = \{a^i, a^{-i}, a^j b\}$. If $n = 3$, then by Lemma 3.1 (b), $\Gamma = \mathrm{Cay}(D_6, S)$, and Γ is normal. Now if $n > 3$, then again by Lemma 3.1 (b), $S \cap (S^2 - \{1\}) = \emptyset$. Considering the graph $\Gamma_2(1)$, with the same reason as before if an automorphism of Γ fixes 1 and all elements of S, then it also fixes all elements of S^2. Because of the connectivity of Γ, this automorphism is the identity in $\mathrm{Aut}(\Gamma)$. Therefore A_1 acts faithfully on S. So we may view A_1 as a permutation group on S. We can easily see that A_1 is generated by the permutation (a^i, a^{-i}). So $|A_1| = 2$. On the other hand, $\sigma : a^t b^l \longrightarrow a^{-t}(a^{2j}b)^l$ is an element of $\mathrm{Aut}(G, S)$. Therefore $|A_1| = |\mathrm{Aut}(G, S)| = 2$, and hence by Lemma 2.1, Γ is normal.

Finally assume that $S = S_{e_1} = \{a^i b, a^j b, a^k b\}$, or $S = S_{o_1} = \{a^i b, a^j b, a^k b\}$. Up to graph isomorphism $S = \{b, a^j b, a^k b\}$, where $< j, k >= Z_n^*$. In this case, Γ is a bipartite graph with the partition $B = B_1 \cup B_2$, where B_1 and B_2 are just two orbits of $C = \langle a \rangle$ and we assume the block B_1 contains 1. Let G^* be the subgroup of G fixing setwise B_1 and B_2, respectively. If G^* acts unfaithfully on one of B_1 and B_2, then by Lemma 2.3, $\Gamma \cong K_{3,3}$, and $\sigma = (b, ab)$ is not in $\mathrm{Aut}(G, S)$ but in A_1 and so Γ is not normal. Let G^* acts faithfully on B_1 and B_2. Then $n \neq 3$. If $n = 4$, then Γ is isomorphic to $K_{4,4} - 4K_2$, and $\sigma = (b, ab)(a^2, a^3)$ is not in $\mathrm{Aut}(G, S)$ but in A_1 and so Γ is not normal. From now on we assume $n \geq 5$. Now suppose that C_G, the core of C in G, is a proper subgroup of C. Then by Lemma 2.4, $\Gamma \cong \mathrm{Cay}(D_{14}, \{b, ab, a^3 b\})$ or $\Gamma \cong \mathrm{Cay}(D_{16}, \{b, ab, a^3 b\})$. For the first case $\sigma = (a, a^2, a^3, a^6)(a^4, a^5)(a^2 b, a^6 b, a^5 b, a^4 b)(ab, b)$ is not in $\mathrm{Aut}(G, S)$ but in A_1 and so Γ is not normal. For the second case $\sigma = (a, a^7, a^6)(a^2, a^5, a^3)(b, ab, a^3 b)(a^4 b, a^5 b, a^7 b)$ is not in $\mathrm{Aut}(G, S)$ but in A_1 and so Γ is not normal. Finally we suppose that $C_G = C$. Then by Lemma 2.5, Γ is isomorphic to the $\mathrm{Cay}(D_{2n}, \{b, ab, a^k b\})$, where $k^2 - k + 1 \equiv 0 (\mod n)$ and $n \geq 13$. The Cayley graph Γ is 1-regular and by Lemma 2.2, Γ is normal. The result now follows.

References

1. B. Alspach, Point-symmetric graphs and digraphs of prime order and transitive permutation groups of prime degree, *J. Combin. Theory* **15** (1973), 12–17.
2. Y. G. Baik, Y. Q. Feng, H. S. Sim, and M. Y. Xu, On the normality of Cayley graphs of abelian groups, *Algebra Colloq.,* **5** (3) (1998), 279–304.
3. Y. G. Baik, Y. Q. Feng, and H. S. Sim, The normality of Cayley graphs of abelian groups with valency 5, *J. Systems Science and Mathematical Sciences* **13** (4) (2000), 420–431.
4. S. F. Du, R. J. Wang, and M. Y. Xu, On the normality of Cayley digraphs of order twice a prime, *Australasian Journal of Combinatorics* **18** (1998), 227-234.
5. S. F. Du, Y. Q. Feng, J. H. Kwak, and M. Y. Xu, Cubic Cayley graphs on dihedral groups, *Mathematical Analysis and Applications* **7** (2004), 224-234.
6. Y. Q. Feng, J. H. Kwak, and M. Y. Xu, s-regular cubic Cayley graphs on abelian or dihedral groups, *Institute of Mathematics Peking University, research report*, **53**, 2000.
7. X. G. Fang, C. H. Li, R. J. Wang, and M. Y. Xu, On cubic normal Cayley graphs of finite simple groups, *Discrete Math.* **244** (2002), 67–75.
8. H. Wielandt, *Finite Permutation Groups*, Academic press, New York, 1964.
9. C. Q. Wang, D. J. Wang, and M. Y. Xu, On normal Cayley graphs of finite groups, *Science in China* (A) **28** (1998), 131–139.
10. M. Y. Xu, Automorphism groups and isomorphisms of Cayley digraphs, *Discrete Math.* **182** (1998), 309–319.