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Hadamard Gap Theorem and Overconvergence for Faber-Erokhin Expansions

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Abstract. We extend the Hadamard-Fabry gap theorem for power series to Faber-Erokhin ones.

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1. A Short Survey on Faber-Erokhin Basis

Let $\Omega \subset \mathbb{C}$ be a simply connected domain, $K \subset \Omega$ a compact set such that $\Omega \setminus K$ is doubly connected. Under these hypothesis, we know that (up to a rotation) there exists a biholomorphic mapping

$$\Phi \ : \ \Omega \setminus K \longrightarrow C(0\,;1,R) = \{z \in \mathbb{C} : \ 1 < |z| < R\},$$

where $R > 1$ is the modulus of the condensor $\mathcal{C} = (\Omega, K)$. Let

$$h_{\Omega,K}(z) := \sup\{u(z) : \ u \in \mathrm{SH}(\Omega) : \ u \leq 1, \ u_{/K} \leq 0\}$$

be the relative extremal function and let $\Omega_\alpha = \{z \in \Omega \ : \ h_{\alpha,K}(z) < \alpha\}$ be its level sets $(0 < \alpha < 1)$; we have

$$\Omega_\alpha = \Phi^{-1}(D(0, R^\alpha) = \{z \in \mathbb{C} : \ |z| < R^\alpha\}), \quad \forall\, \alpha \in]0,1[.$$

• Let $f \in \mathcal{O}(\Omega)$, then $f \circ \Phi^{-1}$ is holomorphic on the annulus $C(0;1,R)$, we have by the Laurent expansion

$$f \circ \Phi^{-1}(\xi) = \sum_{-\infty}^{+\infty} c_n \xi^n,\ 1 < |\xi| < R, \tag{1}$$

where

$$c_n = \frac{1}{2i\pi} \int_{|\zeta|=\rho} \frac{f \circ \Phi^{-1}(\zeta)}{\zeta^{n+1}} d\zeta,\ 1 < \rho < R,\ n \in \mathbb{Z}, \tag{2}$$

and the series converges normally on compact sets of the annulus. Changing $\xi \in C(0;1,R)$ by $\Phi(z) \in \Omega \setminus K$, the formula (1) becomes

$$f(z) = \sum_{-\infty}^{+\infty} c_n \Phi(z)^n, \quad z \in \Omega \setminus K$$

with normal convergence on compact sets of $\Omega \setminus K$.

But now, unlike $f \circ \Phi$, the function f is holomorphic on the whole Ω and by Cauchy formula we have for all $\alpha \in]0,1[$ and $z \in \Omega_\alpha$

$$f(z) = \frac{1}{2i\pi} \int_{\partial\Omega_\alpha} \frac{f(t)}{t-z} dt = \sum_{-\infty}^{+\infty} c_n \cdot \frac{1}{2i\pi} \int_{\partial\Omega_\alpha} \frac{\Phi(t)^n}{t-z} dt.$$

So

$$f(z) = \sum_{-\infty}^{+\infty} c_n E_n(z), \quad \forall\, z \in \Omega \tag{3}$$

and

$$E_n(z) = \frac{1}{2i\pi} \int_{\partial\Omega_\alpha} \frac{\Phi(t)^n}{t-z} dt, \tag{4}$$

where $\alpha \in]0,1[$ and $z \in \Omega_\alpha$.

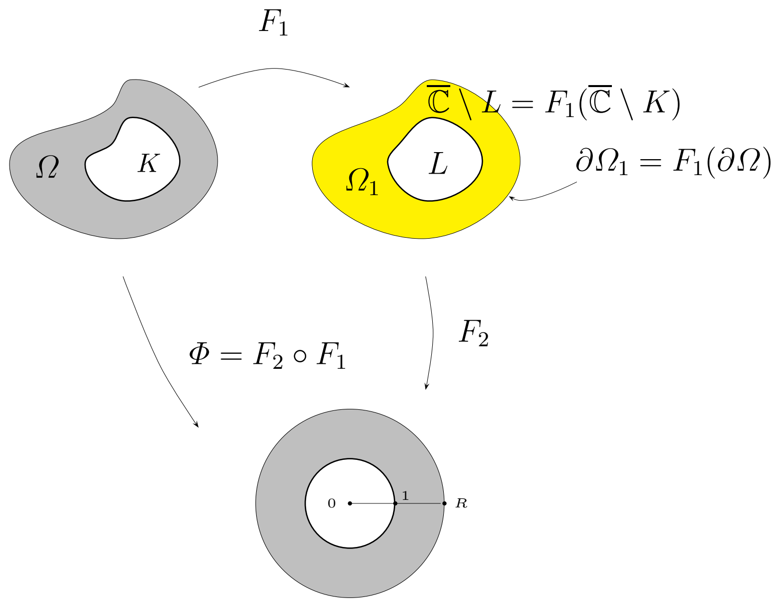
- In the exceptional case where Φ extends to a conformal mapping of $\overline{\mathbb{C}} \setminus K$ with $\Phi(\infty) = \infty$, then $E_n = 0,\ \forall\, n < 0$. With (4) it is easy to see that $E_n,\ (n \geq 0)$ is a polynomial of degree n, they are the classical Faber polynomials [5]. The Faber polynomial sequence $(E_n)_0^\infty$ is a basis of $\mathcal{O}(U)$ for all open level set U of the Green function $G_K = G(\cdot, \overline{\mathbb{C}} \setminus K, \infty)$ associated to K.
- The pioneer work of Erokhin [2, 5] extends the notion of Faber polynomial to a regular condensor (Ω, K), where $\Omega \setminus K$ is a doubly connected domain. His work is built on a "fundamental lemma" about the decomposition of a conformal map onto an annulus:

Erokhin's Fundamental Lemma 1 *Every conformal map Φ from a doubly connected domain $\Omega \setminus K$ onto an annulus $C(0,1,R) = \{ w \in \mathbb{C} \ :\ 1 < |w| < R\}$ can be decomposed into $\Phi = F_2 \circ F_1$ where F_1 and F_2 are conformal maps between simply connected domains, precisely:*

1. *F_1 maps conformly the simply connected domain $\overline{\mathbb{C}} \setminus K$ onto a simply connected domain $\overline{\mathbb{C}} \setminus L$ where L is compact in $\mathbb{C}$. The image by F_1 of the boundary of Ω : $F_1(\partial\Omega)$ defines a simply connected domain Ω_1 which contains L.*

2. F_2 is the biholomorphic map $F_2 \;:\; \Omega_1 \to D(0,R)$ such that $F_2(\partial L) = C(0,1)$.

So we are in the following situation:



• **The Faber-Erokhin basis:** With this decomposition, the Faber-Erokhin basis is defined by analogy with the Faber one by formula (4) with $n \in \mathbb{N}$ only

$$E_n(z) = \frac{1}{2i\pi}\int_{\partial\Omega_\alpha} \frac{\Phi(t)^n}{t-z}dt, \quad \forall\, \alpha \in]0,1[\text{ and } z \in \Omega_\alpha.$$

Erokhin shows that the sequence $(E_n)_{n\geq 0}$ is a common basis for the spaces $\mathcal{O}(\Omega)$, $\mathcal{O}(\Omega_\alpha)$, $(0 < \alpha < 1)$ but generally $E_n \not\equiv 0$ when $n < 0$. The trivial expansion (3) is then tranformed in

$$f(z) = \sum_0^{+\infty} a_n E_n(z), \quad z \in \Omega,$$

where the a_n are in general new coefficients given by an integral formula usually more complicated than (2). Precisely, we have for all $f \in \mathcal{O}(\Omega_\alpha)$, $0 < \rho < \alpha < 1$:

$$a_n = \frac{1}{2i\pi}\int_{|\zeta|=\rho} \frac{\varphi_f(\zeta)}{\zeta^{n+1}}d\zeta$$

with for all $|\zeta| < R^\rho$

$$\varphi_f(\zeta) = \sum_0^{+\infty} a_n\zeta^n = \frac{1}{2i\pi}\int_{|\tau|=R^\rho} \frac{f(\Phi^{-1}(\tau))(F_2^{-1})'(\tau)}{F_2^{-1}(\tau) - F_2^{-1}(\zeta)}d\tau. \tag{5}$$

2. Hadamard Type Results for Faber-Erokhin Expansions

Let f be a holomorphic function on the level set Ω_α such that $f \notin \mathcal{O}(\Omega_\gamma)$, for all $\alpha < \gamma < 1$. Let $f = \sum_{n\geq 0} a_n E_n$ be its expansion in the Faber-Erokhin basis, so the power series

$$\varphi_f(\zeta) := \sum_0^\infty a_n \zeta^n$$

has R^α as radius of convergence. Moreover, (5) implies that for all $0 < \beta < \alpha$ and $|\zeta| < R^\beta$:

$$\varphi_f(\zeta) = \sum_0^{+\infty} a_n \zeta^n = \frac{1}{2i\pi} \int_{|\tau|=R^\beta} \frac{f(\Phi^{-1}(\tau))(F_2^{-1})'(\tau)}{F_2^{-1}(\tau) - F_2^{-1}(\zeta)} d\tau. \qquad (6)$$

Theorem 2.1. *f extends holomorphically across a point $z_0 \in \partial\Omega_\alpha$ if and only if φ_f extends holomorphically across the point $\zeta_0 := \Phi(z_0) \in C(0, R^\alpha)$.*

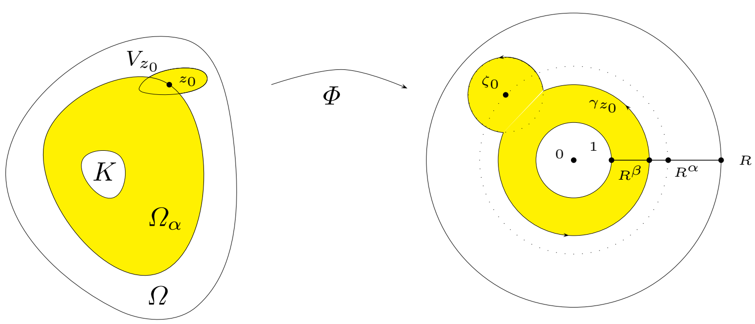
Proof. • *Necessary condition.* Suppose that there exists a neighborhood $V_{z_0} \subset \Omega \setminus K$ of z_0 such that f extends holomorphically on $\Omega_\alpha \cup V_{z_0}$. Let $r > 0$ be such that

$$D(\zeta_0, r) \subset\subset \Phi(V_{z_0}) \subset C(0; 1, R),$$

and choose $0 < \beta < \alpha$ sufficiently close to α so that

$$D(\zeta_0, r) \cap D(0, R^\beta) \neq \emptyset.$$

Now, consider the oriented path γ_{z_0} below



Then the function defined by the formula

$$\psi(\zeta) = \frac{1}{2i\pi} \int_{\gamma_{z_0}} \frac{f(\Phi^{-1}(\tau))(F_2^{-1})'(\tau)}{F_2^{-1}(\tau) - F_2^{-1}(\zeta)} d\tau, \quad \zeta \in D(\zeta_0, r) \cup D(0, R^\beta). \qquad (7)$$

is clearly holomorphic on $D(\zeta_0, r) \cup D(0, R^\beta)$.

On the other hand by the Cauchy formula

$$\frac{1}{2i\pi}\int_{C(\zeta_0,r)^+}\frac{f(\Phi^{-1}(\tau))(F_2^{-1})'(\tau)}{F_2^{-1}(\tau)-F_2^{-1}(\zeta)}d\tau=0,\quad \forall\,\zeta\in D(\zeta_0,r). \tag{8}$$

Formula (8) combined with (6) and (7) assures that

$$\psi=\varphi_f \quad \text{on} \quad D(0,R^\beta)\cap D(\zeta_0,r)\neq\emptyset,$$

so we succed to extends holomorphically φ_f across ζ_0.

• *Sufficient condition.* The proof is the same; it is built on the dual formula of (5)

$$(5')\qquad f(z)=\sum_0^{+\infty}a_nE_n(z)=\frac{1}{2i\pi}\int_{\partial\Omega_\beta}\frac{\varphi_f(\Phi(t))}{t-z}dt,\quad \forall\,z\in\Omega_\beta.$$

■

Applications. By contradiction, we have the following property: $f\in\mathcal{O}(\Omega_\alpha)$ *has* Ω_α *as domain of holomorphy if and only if* φ_f *has the disc* $D(0,R^\alpha)$ *as domain of holomorphy.*

So we are able to extend for expansions following the Faber-Erokhin basis some theorems on the boundary behaviour of a power series. For example, we have

• (*Hadamard*)*: Let* $f(z)=\sum_0^{+\infty}a_{n_k}E_{n_k}(z)\in\mathcal{O}(D_\alpha)$ *be such that* $f\notin\mathcal{O}(D_\beta),\ \forall\,\beta>\alpha$. *If there exists a constant* $c>0$ *such that* $n_{k+1}-n_k>c\cdot n_k,\ \forall\,k\in\mathbb{N}$, *then* D_α *is the domain of holomorphy of* f.

Or in a stronger form, we have

• (*Fabry-Pólya*)*: Let* $f(z)=\sum_0^{+\infty}n_kE_{n_k}(z)\in\mathcal{O}(D_\alpha)$ *be such that* $f\notin\mathcal{O}(D_\beta),\ \forall\,\beta>\alpha$. *If* $\lim_k\frac{n_k}{k}=\infty$ *then* Ω_α *is the domain of holomorphy of* f. *Conversely* (*Pólya*), *every increasing sequence of integers* $n_0<n_1<\dots$ *such that every series* $\sum_0^{+\infty}a_{n_k}E_{n_k}$ *has* Ω_α *as domain of holomorphy, satisfies* $\lim_k\frac{n_k}{k}=\infty$.

For example, the function $f(z)=\sum_0^{+\infty}R^{-2^n\alpha}E_{2^n}(z)$ (Hadamard) or $g(z)=\sum_0^{+\infty}R^{-n^2\alpha}E_{n^2}(z)$ (Fabry) admits Ω_α as domain of holomorphy but this is not the cases for $h(z)=\sum_0^{+\infty}R^{-n\alpha}E_n(z)$ which presents a unique singular point (which of course is $\Phi^{-1}(1)$) on the boundary $\partial\Omega_\alpha$.

3. The Case of an Arbitrary Common Basis.

With the same hypothesis on the pair (K,Ω) let us consider now an arbitrary common basis $(\varphi_n)_n$ for the spaces $\mathcal{O}(K),\ \mathcal{O}(\Omega)$. It extends as a common basis of the intermediate spaces $\mathcal{O}(\Omega_\alpha),\ (0<\alpha<1)$. This is not difficult to see that the preceeding results are no longer true for any common basis $(\varphi_n)_n$: consider the simple example where $K = \overline{D(0,1/2)} \subset \Omega = D(0,2)$. This condensor admits as level sets the discs $\Omega_\alpha = D(0, 2^{\frac{3\alpha}{2}+\frac{1}{2}})$. Consider the common basis

$$\varphi_n(z) = z^{\pi(n)}, \quad n \in \mathbb{N}$$

where $\pi : \mathbb{N} \to \mathbb{N}$ is a bijection such that $\pi(2^n) = 2n$. Then the function $f(z) = \sum_0^{+\infty} \varphi_{2^n}(z)$ satisfies the Hadamard lacunary condition but

$$f(z) = \sum_0^{+\infty} \varphi_{2^n}(z) = \sum_0^{+\infty} z^{2n} = \frac{1}{1-z^2}$$

holomorphic on $D(0,1) = \Omega_{1/3}$ admits $\mathbb{C} \setminus \{\pm 1\}$ as domain of holomorphy.

Remark 3.1. [1], J. A. Adepoju proved the Fabry-type gap theorem for Faber polynomials, his proof followed the classical one for entire series and is rather complicated.

In [4] we extend Fatou-type theorems to all common bases of the pair $(\mathcal{O}(K), \mathcal{O}(\Omega))$ in a more general situation.

4. Overconvergence

In the spirit of the proof of Theorem 2.1, the formulas (5) and (5') lead us to transport overconvergence phenomena to Faber-Erokhin series. Let $f = \sum_0^{+\infty} a_n E_n \in \mathcal{O}(\Omega_\alpha)$. If f is not holomorphic on larger level sets $\Omega_\beta,\ \alpha < \beta$, then we will say that the series $\sum_0^{+\infty} a_n E_n$ is *overconvergent* if there exists a subsequence $(m_k)_k$ such that the corresponding partial sums

$$s_{m_k}(f, z) := \sum_{\nu=0}^{m_k} a_\nu E_\nu(z),$$

converge compactly in a domain that contains properly Ω_α.

The unicity of coefficients in the Faber-Erokhin expansion and formula (5) give

$$s_{m_k}(\varphi_f, \zeta) := \sum_{\nu=0}^{m_k} a_\nu z^\nu = \frac{1}{2i\pi} \int_{|\tau|=R^\beta} \frac{s_{m_k}(f, \Phi^{-1}(\tau))(F_2^{-1})'(\tau)}{F_2^{-1}(\tau) - F_2^{-1}(\zeta)} d\tau. \qquad (9)$$

Suppose now that the sequence $(s_{m_k}(f,\cdot))_k$ converges uniformly on a neighborhood V_{z_0} of a boundary point $z_0 \in \partial\Omega_\alpha$, then as in Theorem 2.1, we have

$$\begin{aligned}
&\sup_{\zeta \in D(\zeta_0, r)} |s_{m_k}(\varphi_f, \zeta) - s_{m_{k'}}(\varphi_f, \zeta)| \\
&\leq \sup_{\zeta \in V_{z_0}} |s_{m_k}(f, z) - s_{m_{k'}}(f, z)| \times \int_{\gamma_{z_0}} \frac{|F_2^{-1})'(\tau)| \cdot |d\tau|}{|F_2^{-1}(\tau) - F_2^{-1}(\zeta)|} \\
&\leq C \cdot \sup_{\zeta \in V_{z_0}} |s_{m_k}(f, z) - s_{m_{k'}}(f, z)|
\end{aligned}$$

where, as before, $\zeta_0 = \Phi(z_0), D(z_0, r) \subset \Phi(V_{z_0})$. This implies that $(s_{m_k}(\varphi_f, \cdot))_k$ is a uniformly convergent Cauchy sequence on the disc $D(\zeta_0, r)$: the series $\sum_0^{+\infty} a_k z^k$ is overconvergent. By duality, the overconvergence of $\sum_0^{+\infty} a_k z^k$ implies the one for $\sum_0^{+\infty} a_k E_k$.

As an application, we have the following Ostrowski Theorem ([6]) for Faber-Erokhin expansions: let $f = \sum_{n \geq 0} a_n E_n \in \mathcal{O}(\Omega_\alpha)$ be such that f is not holomorphic on larger level sets $\Omega_\beta,\ \alpha < \beta$; suppose that there is an infinite number of gaps in the sequence of coefficients as follows: there exist $\nu > 0$, sequences of integers $(p_k)_k,\ (q_k)_k$ such that $a_n = 0$ for $p_k < a_n < q_k$ and $q_k \geq (1+\nu)p_k$ for all k. Then, the sequence of partial sums $(\sum_{j=0}^{p_k} a_j E_j(z))_k$ is uniformly convergent on compact sets of a domain which contains all the regular points of f on the boundary of Ω_α.

References

1. J. A. Adepoju, Fabry-type gap theorem for Faber series, *Demonstration Math.* **21** (1988), 573–588.
2. V.D. Erokhin, Best linear approximation of functions analytically continuable from a given continuum into a given region, *Uspehi Math. Nauk* **23** (1968), 93–135.
3. J. P. Kahane, A. Melas, and V. Nestoridis, Sur les séries de Taylor universelles, *C.R. Acad. Sci. Paris, Série I*, **330** (2000), 1003–1006.
4. P. Lassère and T. V. Nguyen, Gaps and Fatou Theorem for series in Schauder basis of holomorphic functions, *Complex Variables and Elliptic Equations* **51** (2006), 161–164.
5. P. K. Suetin, *Series of Faber Polynomials*, Gordon and Breach Science Publishers,1998.
6. E. C. Titchmarsh, *Theory of Functions*, Oxford University Press, 1976.