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d -Koszul Block Modules

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Abstract. Let M be a weakly d-Koszul module and let $\mathbf{G}(M)$ be its associated graded module. We give some relations between the minimal projective resolutions of such M and $\mathbf{G}(M)$. Moreover, the notion of *d-Koszul block module* is introduced. For a perfect graded module M, we show that M is a d-Koszul block module if and only if the Koszul dual $\mathcal{E}(M) = \bigoplus_{i\geq 0} \operatorname{Ext}^i_A(M, A_0)$, is finitely 0-generated as a graded $E(A) = \bigoplus_{i\geq 0} \operatorname{Ext}^i_A(A_0, A_0)$-module.

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1. Introduction

The so-called *weakly d-Koszul module* was first introduced in [4], which is a natural generalization of weakly Koszul module introduced before by Martínez-Villa and Zacharia in [8]. Later, the author of the present paper and Wang revisited weakly d-Koszul modules in [5], and there the following was proved:

• *Let A be a d-Koszul algebra and $M \in \operatorname{gr}(A)$. Then M is weakly d-Koszul if and only if the associated graded module $\mathbf{G}(M)$ is d-Koszul.*

For the definitions of d-Koszul algebra (module) and weakly d-Koszul module, we refer to [2] and [4] for the further details. Now one can ask the following question: For a weakly d-Koszul module M, does there exist some relations between the minimal graded projective resolutions of M and $\mathbf{G}(M)$? In fact, we obtain the following result:

Theorem 1.1. *Let A be a d-Koszul algebra and M a weakly d-Koszul module.*

Let

$$\mathcal{Q} = \cdots \to Q_i \xrightarrow{\partial_i} \cdots \longrightarrow Q_1 \xrightarrow{\partial_1} Q_0 \xrightarrow{\partial_0} M \to 0$$

and

$$\mathcal{P} = \cdots \to P_i \xrightarrow{d_i} \cdots \longrightarrow P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} \mathbf{G}(M) \to 0$$

be the minimal graded projective resolutions of M and $\mathbf{G}(M)$ respectively. Then we have

$$P_i \cong \mathbf{G}(Q_i)[\delta(i)], \quad \forall\, i \geq 0,$$

where

$$\delta(i) = \begin{cases} \dfrac{id}{2}, & \text{if } i \text{ is even} \\ \dfrac{(i-1)d}{2} + 1, & \text{if } i \text{ is odd} \end{cases}$$

and $d \geq 2$ an integer; [ ] the shift functor.

From [4], we know that weakly d-Koszul module is also a natural generalization of d-Koszul module and they have some similar properties. The following can be found in [4].

• *Let M be a weakly d-Koszul module. Then the Koszul dual $\mathcal{E}(M)$ is finitely 0-generated as a graded $E(A)$-module.*

In the case of d-Koszul module, the converse of the above statement is also true. In order to get the equivalent description for weakly d-Koszul module in terms of the Koszul dual $\mathcal{E}(M)$, we introduce the notion of *d-Koszul block module* and we get the following result:

Theorem 1.2. *Let M be a graded perfect module. Then M is a d-Koszul block module if and only if the Koszul dual of M, $\mathcal{E}(M)$ is finitely 0-generated as a graded $E(A)$-module.*

The whole paper is arranged as follows. In Sec. 2, we recall some notations appeared in [4, 5] and give some new definitions and notations. The main purposes of Sec. 3 and 4 are to prove Theorem 1.1 and Theorem 1.2, respectively.

2. Notations and Definitions

Throughout, we will follow the definitions and notations of [4, 5]. For examples, $A = \bigoplus_{i \geq 0} A_i$ denotes a graded $\mathbb{F}$-algebra such that
(a) A_0 is a finite dimensional semi-simple Artin algebra,
(b) A is generated in degrees zero and one; that is, $A_i \cdot A_j = A_{i+j}$ for all $0 \leq i, j < \infty$, and
(c) $\forall i \geq 0$, A_i is a finitely generated $\mathbb{F}$-module.

Let J denote the graded Jacobson radical of A, $\mathrm{gr}(A)$ denote the category of finitely graded module and let $\mathrm{gr}_s(A)$ denote the full subcategory of $\mathrm{gr}(A)$

consisting of graded pure modules. Recall that $\mathcal{WK}^d(A)$ denotes the category of weakly d-Koszul modules...

Definition 2.1. *Let $M \in \mathrm{gr}(A)$. Then we can regrade M as follows:*

$$M/JM \oplus JM/J^2M \oplus J^2M/J^3M \oplus \dots. \tag{1}$$

The module constructed by (1) *is denoted by* $\mathbf{G}(M)$, *which is a graded* $\mathbf{G}(A)$-*module under the grading:*

$$\mathbf{G}(M) = \bigoplus_{i \geq 0} J^iM/J^{i+1}M,$$

where $J^0 = A$. *More precisely,*

$$(\mathbf{G}(M))_0 = M/JM,\ (\mathbf{G}(M))_1 = JM/J^2M,\ \dots.$$

Therefore, $\mathbf{G}$ *can be regarded as a functor from* $\mathrm{gr}(A)$ *to* $\mathrm{gr}_0(A)$ *as follows:*

$$\mathbf{G}:\ \mathrm{gr}(A) \to \mathrm{gr}_0(A)$$

via

$$\mathbf{G}(M) = M/JM \oplus JM/J^2M \oplus J^2M/J^3M \oplus \dots$$

and

$$\mathbf{G}(M \xrightarrow{f} N) = \mathbf{G}(M) \xrightarrow{\mathbf{G}(f)} \mathbf{G}(N),$$

where $M,\ N \in \mathrm{gr}(A)$, $f \in \mathrm{Hom}_A(M,N)$, *and* $\mathbf{G}(f)(x+J^{i+1}M) = f(x)+J^{i+1}N$, *where* $x \in J^iM$.

We will call $\mathbf{G}$ *the associated graded functor* ($\mathbf{G}$-*functor for short*) *and* $\mathbf{G}(M)$ *the associated graded module of* M ($\mathbf{G}$-*module for short*).

Similarly, we can define $\mathbf{G}(A)$ for a graded algebra A.

The following are the basic properties of the functor $\mathbf{G}$.

Proposition 2.2. *Let* $M \in \mathrm{gr}(A)$. *Then*
(a) $\mathbf{G}(A) \cong A$ *as a graded* $\mathbb{F}$-*algebra;*
(b) $\mathbf{G}(M) \in \mathrm{gr}_0(\mathbf{G}(A)) = \mathrm{gr}_0(A)$;
(c) *If* M *is pure, then* $\mathbf{G}(M)[i] \cong M$ *as a graded* A-*module for some* i;
(d) $\mathbf{G}^2 = \mathbf{G}$;
(e) $\mathbf{G}(J^nM) \cong J^n\mathbf{G}(M)$ *for all* $n \geq 0$;
(f) *Let* $M \in \mathrm{Gr}(A)$. *Then* $M \in \mathrm{gr}(A)$ *if and only if* $\mathbf{G}(M) \in \mathrm{gr}(A)$;
(g) *Let* $\{M_i\}_{i=1}^n$ *be a family of finitely generated graded* A-*modules. Then*

$$\mathbf{G}(\bigoplus_{i=1}^n M_i) = \bigoplus_{i=1}^n \mathbf{G}(M_i).$$

i.e., the functor $\mathbf{G}$ *preserves finite direct sums;*

(h) *Let* $P \in \mathrm{gr}(A)$ *be a graded projective module. Then* $\mathbf{G}(P)$ *is a pure graded projective* $\mathbf{G}(A)$*-module;*

(i) *Let* $f \in \mathrm{Hom}_A(M, N)$ *be an epimorphism. Then* $\mathbf{G}(f)$ *is also an epimorphism from* $\mathbf{G}(M)$ *to* $\mathbf{G}(N)$.

Proof. By definition, $\mathbf{G}(A)_i = J_i/J_{i+1} = A_i$ for all $i \geq 0$ since the graded $\mathbb{F}$-algebra $A = A_0 \oplus A_1 \oplus \cdots$ is generated in degrees 0 and 1. Now the first assertion is clear. For the second assertion, we first prove that $\mathbf{G}(M)$ is a graded $\mathbf{G}(A)$-module. Indeed, we can define the module action as follows:

$$\mu : \mathbf{G}(A) \otimes \mathbf{G}(M) \longrightarrow \mathbf{G}(M)$$

via

$$\mu((a + J^i A) \otimes (m + J^j M)) = a \cdot m + J^{i+j-1} M$$

for all $a + J^i A \in \mathbf{G}(A)$ and $m + J^j M \in \mathbf{G}(M)$. It is easy to check that μ is well-defined and under μ, $\mathbf{G}(M)$ is a graded $\mathbf{G}(A)$-module. Note that the generating space of M is $M/JM = \mathbf{G}(M)_0$, now (b) is obvious. The proof of the third assertion is similar to (a) and we omit it. Statements (d), (e) and (f) are clear. For the proof of (g), we only need to show the case of $n = 2$. Note that

$$J(M_1 \oplus M_2) = JM_1 \oplus JM_2$$

and thus

$$\frac{J^i(M_1 \oplus M_2)}{J^{i+1}(M_1 \oplus M_2)} = \frac{J^i M_1 \oplus J^i M_2}{J^{i+1} M_1 \oplus J^{i+1} M_2} \cong \frac{J^i M_1}{J^{i+1} M_1} \oplus \frac{J^i M_2}{J^{i+1} M_2},$$

which follows that

$$\mathbf{G}(M_1 \oplus M_2) = \mathbf{G}(M_1) \oplus \mathbf{G}(M_2).$$

Note that $\mathbf{G}(P)$ is a pure graded $\mathbf{G}(A)$-module and P is a direct summand of some copies of $A[-i]$, where [ ] is the shift functor. Now by (g), (h) is immediate and by the definition of $\mathbf{G}(f)$, (i) is clear. ∎

Definition 2.3. *Let* A *be a* d*-Koszul algebra and* $M \in \mathrm{gr}(A)$*. Let*

$$\cdots \to Q_n \stackrel{f_n}{\to} \cdots \to Q_1 \stackrel{f_1}{\to} Q_0 \stackrel{f_0}{\to} M \to 0$$

be a minimal graded projective resolution of M*. We call* M *a weakly* d*-Koszul module if for* $i, k \geq 0$, $J^k \ker f_i = J^{k+1} Q_i \cap \ker f_i$ *if* i *is even and* $J^k \ker f_i = J^{k+d-1} Q_i \cap \ker f_i$ *if* i *is odd.*

Let $\mathcal{WK}^d(A)$ denote the category of weakly d-Koszul modules.

Definition 2.4 *Let* $M \in \mathrm{gr}(A)$ *then we can find a set* $S^M = \{S_{k_0},\ S_{k_1},\ \cdots,\ S_{k_s}\}$ *of* A_0*-submodules of* M *such that:*

(a) *$k_0 < k_1 < \cdots < k_s$;*

(b) *Each S_{k_i} is concentrated in degree k_i;*

(c) *$M/JM = S_{k_0} \oplus S_{k_1} \oplus \cdots \oplus S_{k_s}$ as graded A_0-modules.*

The submodules $\langle S_{k_i} \rangle$ are called the graded pure submodule (GP-submodules) of M.

If M is the direct sum of its GP-submodules, i.e. $M = \langle S_{k_0} \rangle \oplus \langle S_{k_1} \rangle \oplus \cdots \oplus \langle S_{k_s} \rangle$, we call it *perfect.* A *d-Koszul block module* is a module which is both perfect and weakly d-Koszul. It is easy to see that each pure graded module is perfect and each d-Koszul module is a d-Koszul block module. We will denote the categories of d-Koszul block modules and perfect graded modules by $\mathcal{KB}^d(A)$ and $\mathcal{P}(A)$, respectively.

Now let us discuss some easy properties of perfect graded modules.

Proposition 2.5. *Let $M \in \mathrm{gr}(A)$ and keep the same notations as in Definition 1.4. Then the following statements are equivalent*

(1) *$M \in \mathcal{P}(A)$;*

(2) *$\langle S_{k_0}, S_{k_1}, \cdots, S_{k_i} \rangle \in \mathcal{P}(A)$, where $0 \le i \le s$;*

(3) *$\langle S_{k_i} \rangle = \langle S_{k_j}, S_{k_{j+1}}, \cdots, S_{k_i}, \cdots \rangle / \langle S_{k_j}, S_{k_{j+1}}, \cdots, \widehat{S_{k_i}}, \cdots \rangle$, where $\widehat{S_{k_i}}$ means to omit it.*

Proof. It is immediate from the definition of perfect graded modules. ■

Now we will give examples of d-Koszul block modules.

Example 2.6. Let $M \in \mathrm{gr}(A)$ be generated in degrees $d_0 < d_1 < \cdots < d_p$. Then there exist graded pure A-modules $K_0, K_1, \cdots, K_p$, such that

(i) As A_0-modules, $M \cong \bigoplus_{i=0}^p K_i$;

(ii) $\mathbf{G}(M) \cong \mathbf{G}(\bigoplus_{i=0}^p K_i)$;

(iii) $\mathbf{G}(M) \cong \bigoplus_{i=0}^p K_i[-d_i]$.

Moreover, if $M \in \mathcal{WK}^d(A)$, then for all $0 \le j \le p$, $\bigoplus_{i=0}^j K_i \in \mathcal{KB}^d(A)$.

In fact, let $S_{d_0}, S_{d_1}, \cdots, S_{d_p}$ be the minimal generating spaces of M and each S_{d_i} is an A_0-submodule of M consisting of homogeneous elements of degree d_i, $(0 \le i \le p)$. Let $K_0 = \langle S_{d_0} \rangle$, where $\langle S_{d_0} \rangle$ denotes the graded A-submodule of M generated by S_{d_0}. Let

$$K_1 = \langle (M/\langle S_{d_0} \rangle)_{d_1} \rangle, \ K_2 = \langle ((M/\langle S_{d_0} \rangle)/K_1)_{d_2} \rangle, \ \ldots .$$

Now it is easy to see that each K_i is a graded pure module generated in degree d_i. From the construction of each K_i, the statements (i), (ii) and (iii) are clear. If $M \in \mathcal{WK}^d(A)$, from the approximation chain in [4], one can see that each K_i is a d-Koszul module. Therefore, for all $0 \le j \le p$, $\bigoplus_{i=0}^j K_i \in \mathcal{KB}^d(A)$. That is, we can construct a lot of d-Koszul block modules from a given weakly d-Koszul module.

3. The Proof of Theorem 1.1

Lemma 3.1. *Let* $0 \to K \to M \to N \to 0$ *be an exact sequence in* $\mathrm{gr}(A)$ *and* K, M, N *have the same highest degree* l. *Then for all* $k \geq 0$, *we have* $J^kK = K \cap J^kM$ *if and only if*

$$0 \to \mathbf{G}(K) \to \mathbf{G}(M) \to \mathbf{G}(N) \to 0$$

is an exact sequence in $\mathrm{gr}_0(\mathbf{G}(A)) = \mathrm{gr}_0(A)$.

Proof. ($\Rightarrow$) Consider the following short exact sequence

$$0 \to K \to M \to N \to 0$$

with $J^kK = K \cap J^kM$ for all $k \geq 0$. Obviously, we have the following commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & J^{k+1}K & \longrightarrow & J^{k+1}M & \longrightarrow & J^{k+1}N & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & J^kK & \longrightarrow & J^kM & \longrightarrow & J^kN & \longrightarrow & 0 \end{array}$$

where the vertical arrows are natural embeddings. By the “Snake Lemma”, we can get the following exact sequences

$$0 \to J^kK/J^{k+1}K \to J^kM/J^{k+1}M \to J^kN/J^{k+1}N \to 0$$

for all $k \geq 0$. Applying the exact functor “$\bigoplus$” to the above exact sequences, we have

$$0 \to \bigoplus_{k\geq 0} J^kK/J^{k+1}K \to \bigoplus_{k\geq 0} J^kM/J^{k+1}M \to \bigoplus_{k\geq 0} J^kN/J^{k+1}N \to 0.$$

That is, we have the exact sequence

$$0 \to \mathbf{G}(K) \to \mathbf{G}(M) \to \mathbf{G}(N) \to 0.$$

($\Leftarrow$) Suppose that we have the exact sequence,

$$0 \to \mathbf{G}(K) \to \mathbf{G}(M) \to \mathbf{G}(N) \to 0,$$

which yields an exact sequence

$$0 \to \bigoplus_{k\geq 0} J^kK/J^{k+1}K \to \bigoplus_{k\geq 0} J^kM/J^{k+1}M \to \bigoplus_{k\geq 0} J^kN/J^{k+1}N \to 0.$$

Note that the functor “$\bigoplus$” is exact and $K,\ M,\ N$ have the same highest degree, we have the following exact sequences

$$0 \to J^kK/J^{k+1}K \to J^kM/J^{k+1}M \to J^kN/J^{k+1}N \to 0$$

for all $k \geq 0$, which implies that for all $k \geq 0$, we have $J^k K = K \cap J^k M$.
Therefore, we are done. ■

Now we can prove Theorem 1.1.

Proof. Consider the following exact sequence

$$0 \to \ker f_0 \to Q_0 \to M \to 0.$$

Note that $\mathcal{Q}$ is minimal, which yields the following exact sequence

$$0 \to \ker f_0 \to JQ_0 \to JM \to 0.$$

It is easy to see that the highest degrees of $\ker f_0,\ JQ_0$ and JM are the same. Since $M \in \mathcal{WK}^d(A)$, we have

$$J^k \ker f_0 = \ker f_0 \cap J^{k+1}Q_0 = \ker f_0 \cap J^k(JQ_0), \ \ \forall k \geq 0.$$

Now by Lemma 3.1, we have the following exact sequence

$$0 \to \mathbf{G}(\ker f_0) \to \mathbf{G}(JQ_0) \to \mathbf{G}(JM) \to 0,$$

which yields the following exact sequence

$$0 \to \mathbf{G}(\ker f_0)[1] \to \mathbf{G}(Q_0) \to \mathbf{G}(M) \to 0.$$

Now consider the following exact sequence

$$0 \to \ker f_1 \to Q_1 \to \ker f_0 \to 0.$$

Note that $M \in \mathcal{WK}^d(A)$, we have the following exact sequence

$$0 \to \ker f_1 \to J^{d-1}Q_1 \to J^{d-1} \ker f_0 \to 0.$$

Observe that the highest degrees of $\ker f_1,\ J^{d-1}Q_1$ and $J^{d-1}\ker f_0$ are the same and $M \in \mathcal{WK}^d(A)$, we have

$$J^k \ker f_1 = \ker f_1 \cap J^{k+d-1}Q_1 = \ker f_1 \cap J^k(J^{d-1}Q_1), \ \ \forall k \geq 0.$$

Now by Lemma 3.1, we have the following exact sequence

$$0 \to \mathbf{G}(\ker f_1) \to \mathbf{G}(J^{d-1}Q_1) \to \mathbf{G}(J^{d-1}\ker f_0) \to 0,$$

which yields the following exact sequence

$$0 \to \mathbf{G}(\ker f_1)[d-1] \to \mathbf{G}(Q_1) \to \mathbf{G}(\ker f_0) \to 0,$$

and we have the exact sequence

$$0 \to \mathbf{G}(\ker f_1)[d] \to \mathbf{G}(Q_1)[1] \to \mathbf{G}(\ker f_0)[1] \to 0.$$

Repeat the above argument and by induction, we have a minimal graded projective pure resolutions in gr(A)

$$\cdots \to \mathbf{G}(Q_n)[\delta(n)] \to \cdots \to \mathbf{G}(Q_1)[\delta(1)] \to \mathbf{G}(Q_0) \to \mathbf{G}(M) \to 0.$$

Now consider the following commutative diagram with exact rows

$$\begin{array}{ccccccccccc}
\cdots \longrightarrow & \mathbf{G}(Q_n)[\delta(n)] & \longrightarrow \cdots \longrightarrow & \mathbf{G}(Q_1)[\delta(1)] & \longrightarrow & \mathbf{G}(Q_0)[\delta(0)] & \longrightarrow & \mathbf{G}(M) & \longrightarrow 0 \\
 & \downarrow{\scriptstyle \alpha_n} & & \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \alpha_0} & & \downarrow{\scriptstyle =} & \\
\cdots \longrightarrow & P_n & \longrightarrow \cdots \longrightarrow & P_1 & \longrightarrow & P_0 & \longrightarrow & \mathbf{G}(M) & \longrightarrow 0
\end{array}$$

Note that two rows are pure minimal graded resolutions of $\mathbf{G}(M)$, we see that all the α_i are isomorphisms for $i \geq 0$ and we are done. ∎

Corollary 3.2. *Let A be a d-Koszul algebra and $M \in \mathrm{gr}(A)$. Let*

$$\mathcal{Q} = \cdots \to Q_i \xrightarrow{f_i} \cdots \longrightarrow Q_1 \xrightarrow{f_1} Q_0 \xrightarrow{f_0} M \to 0$$

be a minimal graded projective resolution of M. Then

$$\cdots \to \mathbf{G}(Q_n)[\delta(n)] \to \cdots \to \mathbf{G}(Q_1)[\delta(1)] \to \mathbf{G}(Q_0) \to \mathbf{G}(M) \to 0$$

is a minimal projective resolution if and only if $M \in \mathcal{WK}^d(A)$.

Proof. ($\Rightarrow$) By assumption and the definition of d-Koszul modules, $\mathbf{G}(M)$ is d-Koszul by [5], which is equivalent to that M is weakly d-Koszul.

($\Leftarrow$) Follows from Theorem 1.1. ∎

4. The Proof of Theorem 1.2

Lemma 4.1. *Let $M = \bigoplus_{i \geq k_0} M_i$ be a d-Koszul block module and use the notations in Definition 2.4. Then*

(1) *All the GP-submodules $\langle S_{k_i} \rangle$ are d-Koszul modules, where $0 \leq i \leq t$;*

(2) $\langle S_{k_i} \rangle \cap J^k M = J^k \langle S_{k_i} \rangle$ *for all $k \geq 0$;*

(3) *All the A-modules $\langle S_{k_{i_1}}, S_{k_{i_2}}, \cdots, S_{k_{i_p}} \rangle / \langle S_{k_{j_1}}, S_{k_{j_2}}, \cdots, S_{k_{j_m}} \rangle$ are d-Koszul block modules, where*

$$\{S_{k_{i_1}}, S_{k_{i_2}}, \cdots, S_{k_{i_p}}\} \subseteq \{S_{k_0}, S_{k_1}, \cdots, S_{k_t}\}$$

and

$$\{S_{k_{j_1}}, S_{k_{j_2}}, \cdots, S_{k_{j_m}}\} \subset \{S_{k_{i_1}}, S_{k_{i_2}}, \cdots, S_{k_{i_p}}\}.$$

Proof. It is similar to that of [4, Theorem 2.6] and we omit it. ∎

Lemma 4.2. *Let $M \in \mathrm{gr}(A)$ and use the notations of Definition 2.4. Then M is a d-Koszul block module if and only if*

(1) *All the GP-submodules of M are d-Koszul modules, and*
(2) *M is the direct sum of all its GP-submodules.*

Proof. Suppose M is a d-Koszul block module. By the definition, it is a perfect graded module and M is the direct sum of all its GP-submodules. By Lemma 4.1, all the GP-submodules of M are d-Koszul modules.

Conversely, by condition (2), M is a perfect graded module. To complete the proof of the theorem, we need only show $M \in \mathcal{WK}^d(A)$. We need only consider the case $M = \langle S_{k_0}, S_{k_1} \rangle$ since the other cases can be proved similarly. Note that d-Koszul modules are of course d-Koszul block modules, hence weakly d-Koszul module. Consider the following exact sequence

$$0 \to \langle S_{k_0} \rangle \to \langle S_{k_0}, S_{k_1} \rangle \to \langle S_{k_1} \rangle \to 0,$$

where, by hypothesis, $\langle S_{k_0} \rangle$ and $\langle S_{k_1} \rangle$ are d-Koszul modules and $J\langle S_{k_0} \rangle = \langle S_{k_0} \rangle \cap J\langle S_{k_0}, S_{k_1} \rangle$. From [4], we have $M \in \mathcal{WK}^d(A)$. ■

Proposition 4.3. *Let $M \in \mathcal{P}(A)$ and use the notations in Definition 2.4. Then M is a d-Koszul block module if and only if the Koszul dual of all its GP-submodules, $\mathcal{E}(\langle S_{k_i} \rangle) = \bigoplus_{n \geq 0} \mathrm{Ext}^n_A(\langle S_{k_i} \rangle, A_0)$ are generated in degree 0 as a graded $E(A)$-module.*

Proof. By Lemma 4.1, a perfect graded module is a d-Koszul block module if and only if all its GP-submodules are d-Koszul modules, which is equivalent to the fact that all the Koszul duals, $\mathcal{E}(\langle S_{k_i} \rangle) = \bigoplus_{n \geq 0} \mathrm{Ext}^n_A(\langle S_{k_i} \rangle, A_0)$ are generated in degree 0 as a graded $E(A)$-module (by [2]). ■

Theorem 4.4. *Let $M \in \mathcal{P}(A)$ and use the notations in Definition 2.4. Then M is a d-Koszul block module if and only if the Koszul dual of M, $\mathcal{E}(M) = \bigoplus_{n \geq 0} \mathrm{Ext}^n_A(M, A_0)$ is generated in degree 0 as a graded $E(A)$-module.*

Proof. Assume $M \in \mathcal{P}(A)$ is a d-Koszul block module, then

$$\begin{aligned}
\mathcal{E}(M) &= \bigoplus_{n \geq 0} \mathrm{Ext}^n_A(\langle S_{k_0}, S_{k_1}, \cdots, S_{k_t} \rangle, A_0) \\
&= \bigoplus_{n \geq 0} \mathrm{Ext}^n_A(\langle S_{k_0} \rangle \oplus \langle S_{k_1} \rangle \oplus \cdots \oplus \langle S_{k_t} \rangle, A_0) \\
&= \bigoplus_{n \geq 0} (\mathrm{Ext}^n_A(\langle S_{k_0} \rangle, A_0) \oplus \mathrm{Ext}^n_A(\langle S_{k_1} \rangle, A_0) \oplus \cdots \oplus \mathrm{Ext}^n_A(\langle S_{k_t} \rangle, A_0)) \\
&= (\bigoplus_{n \geq 0} (\mathrm{Ext}^n_A(\langle S_{k_0} \rangle, A_0)) \oplus (\bigoplus_{n \geq 0} (\mathrm{Ext}^n_A(\langle S_{k_1} \rangle, A_0)) \\
&\oplus \cdots \oplus (\bigoplus_{n \geq 0} (\mathrm{Ext}^n_A(\langle S_{k_t} \rangle, A_0))
\end{aligned}$$

is generated in degree 0 as a graded $E(A)$-module.

Conversely, since $M \in \mathcal{P}(A)$ and $\mathcal{E}(M) = \bigoplus_{n\geq 0} \mathrm{Ext}_A^n(M, A_0)$ is generated in degree 0 as a graded $E(A)$-module, then each $\bigoplus_{n\geq 0} \mathrm{Ext}_A^n(\langle S_{k_i}\rangle, A_0)$ is a direct summand of $\mathcal{E}(M)$. Hence all $\mathcal{E}(\langle S_{k_i}\rangle) = \bigoplus_{n\geq 0} \mathrm{Ext}_A^n(\langle S_{k_i}\rangle, A_0)$ are generated in degree 0 as a graded $E(A)$-module. By Proposition 4.3, we finish the proof.

■

Now put Theorem 4.4 and [5, Theorem 4.3] together, we complete the proof of Theorem 1.2.

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