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On Oscillation, Convergence and Boundedness of Solutions of Some Nonlinear Difference Equations with Multiple Delay*

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Abstract. The oscillation, convergence and boundedness of the solutions of some nonlinear difference equations with multiple delay of the form

$$x_{n+1} = G(x_n, x_{n-m_1}, \dots, x_{n-m_r}), \quad n = 0, 1, \dots$$

are investigated, where $m_i \in \mathbb{N}_0, \forall i = 1, \dots, r$ and G is a function mapping $\mathbb{R}^{m+1}$ to $\mathbb{R}$.

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1. Introduction

It is well-known that the difference equation

$$x_{n+1} = G(x_n, x_{n-m_1}, \dots, x_{n-m_r})$$

includes difference equations

$$x_{n+1} = \lambda_n x_n + \sum_{i=1}^{r} \alpha_i(n) F(x_{n-m_i}), \tag{1}$$

and

$$x_{n+1} = G(x_n, x_{n-1}, \dots, x_{n-m}). \tag{2}$$

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In [2, 5], the oscillation of solutions of the difference equation (1) was discussed. In addition, the convergence and the boundedness of solutions of (1) were also investigated in [5]. The authors in [1, 4] studied the global attractivity, the local stability and the existence of positive periodic solutions of some equations, which are special cases of (2).

Equation (2) includes many discrete versions of the most celebrated delay differential equation for single species. For example, the Mackey-Glass hematopoiesis model, the Lasota-Wazewska red blood model, and the Nicholson's blowflies model.

Studying the population dynamics has attracted much attention from both mathematicians and mathematical biologists recently. Many authors have investigated the extinction, persistence, global stability and the existence of positive periodic solutions for several population models; see for example [3, 6–9] and the references therein.

Motivated by the work above, in the present paper, we aim to study the oscillation as well as nonoscillation of (1) and investigating the convergence, boundedness of (2).

2. The Oscillation

Consider the difference equation (1), for $n \in \mathbb{N}, n \geqslant a$ for some $a \in \mathbb{N}$, where $r, m_i \geqslant 1$, $1 \leqslant i \leqslant r$ are fixed positive integers ;$\{\lambda_n\}_n, \{\alpha_i(n)\}_n$ are sequences of numbers and the function F is defined on $\mathbb{R}$. Recall that, the solution $\{x_n\}_{n\geqslant a}$ of (1) is called *oscillatory* if for any $n_1 \geqslant a$ there exists $n_2 \geqslant n_1$ such that $x_{n_2}x_{n_2+1} \leqslant 0$. The difference equation (1) is said to be oscillatory if all its solutions are oscillatory. The solution $\{x_n\}_{n\geqslant a}$ of (1) is called *nonoscillatory* if it is eventually positive or negative, i.e. there exists a $n_1 \geqslant a$ such that $x_n x_{n+1} > 0$ for all $n \geqslant n_1$.

Theorem 1. *Assume that* $\lambda_n = 1,\ \forall n \in \mathbb{N}$*;* $\alpha_i(n) \geqslant 0, n \in \mathbb{N}, 1 \leqslant i \leqslant r$*;* $xF(x) < 0,\ \forall x \neq 0$*;* $\sup\limits_{x\neq 0} \dfrac{-F(x)}{x} = M > 0$*. Then,* (1) *has a nonoscillatory solution if the following holds*

$$\sup_n \sum_{i=1}^r \alpha_i(n) \leqslant \frac{1}{M}\frac{m_*^{m^*}}{(m_*+1)^{m^*+1}}$$

where $m_* = \min\{m_1, m_2, \ldots, m_r\}$*,* $m^* = \max\{m_1, m_2, \ldots, m_r\}$*.*

Proof. Setting $v_n = \dfrac{x_n}{x_{n+1}}$ and dividing (1) by x_n, we obtain

$$\frac{1}{v_n} = 1 + \Big[\sum_{i=1}^r \alpha_i(n)\frac{F(x_{n-m_i})}{x_{n-m_i}}\prod_{\ell=1}^{m_i} v_{n-\ell}\Big], \quad n \in \mathbb{N},$$

or

$$v_n^{-1} = 1 - \sum_{i=1}^r \alpha_i(n)\Big[-\frac{F(x_{n-m_i})}{x_{n-m_i}}\Big]\prod_{\ell=1}^{m_i} v_{n-\ell}, \quad n \in \mathbb{N}. \tag{3}$$

We shall prove that the equation (3) has a positive solution. Indeed, we define

$$v_{-1} = v_{-2} = \cdots = v_{-m^*} = \frac{m_* + 1}{m_*}.$$

We have

$$v_0 = \Big\{1 - \sum_{i=1}^{r} \alpha_i(0)\Big[-\frac{F(x_{-m_i})}{x_{-m_i}}\Big]\prod_{\ell=1}^{m_i} v_{-\ell}\Big\}^{-1}.$$

Since

$$\begin{aligned}
0 \leqslant \sum_{i=1}^{r} \alpha_i(0)\Big[-\frac{F(x_{-m_i})}{x_{-m_i}}\Big]\prod_{\ell=1}^{m_i} v_{-\ell} &\leqslant M\sum_{i=1}^{r} \alpha_i(0)\prod_{\ell=1}^{m_i} v_{-\ell}\\
&= M\sum_{i=1}^{r} \alpha_i(0)\Big(\frac{m_*+1}{m_*}\Big)^{m_i}\\
&\leqslant M\sum_{i=1}^{r} \alpha_i(0)\Big(\frac{m_*+1}{m_*}\Big)^{m^*}\\
&\leqslant M\cdot\frac{1}{M}\frac{m_*^{m^*}}{(m_*+1)^{m^*+1}}\Big(\frac{m_*+1}{m_*}\Big)^{m^*}\\
&= \frac{1}{m_*+1} < 1,
\end{aligned}$$

we obtain $1 > 1 - \sum_{i=1}^{r} \alpha_i(0)\Big[-\frac{F(x_{-m_i})}{x_{-m_i}}\Big]\prod_{\ell=1}^{m_i} v_{-\ell} > 0$ and therefore $v_0 > 1$. We can check that $v_0 \leqslant \frac{m_*+1}{m_*}$. So, by (3) we have

$$v_1 = \Big\{1 - \sum_{i=1}^{r} \alpha_i(1)\Big[-\frac{F(x_{1-m_i})}{x_{1-m_i}}\Big]\prod_{\ell=1}^{m_i} v_{1-\ell}\Big\}^{-1} \leqslant \frac{m_*+1}{m_*}$$

and now by induction $1 < v_n \leqslant \frac{m_*+1}{m_*}$ for all $n = 2, 3, \ldots$ so that $\{v_n\}$ is a positive solution of (3). Next, we define

$$x_{i-m^*} = \Big(\frac{m_*+1}{m_*}\Big)^{m^*-i}, \quad 0 \leqslant i \leqslant m^*, \quad x_n = \frac{x_{n-1}}{v_{n-1}}, \quad n = 1, 2, \ldots,$$

it follows that $\{x_n\}$ is a nonoscillatory solution of (1). ■

Example 1. Consider the difference equation

$$x_{n+1} = x_n + \frac{m^m}{(m+1)^{m+1}}(-x_{n-m}). \tag{4}$$

It is clear that this equation is a particular case of (1), where $\lambda_n = 1$, $\alpha_i(n) = \frac{1}{r}\frac{m^m}{(m+1)^{m+1}}, \forall n \in \mathbb{N}, 1 \leqslant i \leqslant r$, $m_i = m,\ 1 \leqslant i \leqslant r$ and $F(x) \equiv -x$. It is easy to check that the assumptions of the Theorem 1 are satisfied. If we put $x_{i-m} = \Big(\frac{m^m}{(m+1)^{m+1}}\Big)^{i-m}\beta, \quad 1 \leqslant i \leqslant m, \quad \beta \neq 0$, the solution of (4) is

$$x_n = \Big(\frac{m^m}{(m+1)^{m+1}}\Big)^n \beta, \quad n \in \mathbb{N},$$

which is nonoscillatory.

Theorem 2. *Assume that* $\lambda_n = 1$, $\forall n \in \mathbb{N}$; $\alpha_i(n) \geqslant 0, n \in \mathbb{N}, 1 \leqslant i \leqslant r$; $xF(x) < 0$, $-F(x) \geqslant x, \forall x \neq 0$. *Then,* (1) *is oscillatory if the following inequality holds*

$$\liminf_{n\to\infty} \frac{1}{m^*} \sum_{\ell=n-m^*}^{n-1} \alpha_{m^*}(\ell) > \frac{{m^*}^{m^*}}{(m^*+1)^{m^*+1}}$$

where $m^* = \max\{m_1, m_2, \dots, m_r\}$.

Proof. The proof of the Theorem 2 can be obtained similarly as the proof of Theorem 3, in [2], so we omit it here. ■

Theorem 3. *Assume that* $\lambda_n = \lambda \geqslant 1, \forall n \in \mathbb{N}$; $\alpha_i(n) \geqslant 0, n \in \mathbb{N}, 1 \leqslant i \leqslant r$; $xF(x) < 0$, $\forall x \neq 0$ *and there exists* $i_0 \in \{1, 2, \dots, r\}$ *such that*

$$\sum_{\ell \in \mathbb{N}} \frac{1}{\lambda^\ell} \alpha_{i_0}(\ell) = \infty. \tag{5}$$

Suppose further that, if $|u| \geqslant c$ *then* $|F(u)| \geqslant c_1$ *where* c *and* c_1 *are positive constants. Then, every solution* $\{x_n\}_n$ *of* (1) *is either oscillatory or*

$$\lim_{n\to\infty} \frac{x_n}{\lambda^n} = 0.$$

Proof. Let $\{x_n\}_n$ be a nonoscillatory solution of (1). Suppose that $\{x_n\}_n$ is an eventually positive solution. Then there is $n_1 \in \mathbb{N}$ such that $x_n > 0$ and $x_{n-m_i} > 0$ for all $n \geqslant n_1$ and $i = 1, 2, \dots, r$. Since

$$\frac{x_{n+1}}{\lambda^{n+1}} - \frac{x_n}{\lambda^n} = \frac{1}{\lambda^{n+1}}(x_{n+1} - \lambda x_n) \leqslant 0, \quad \forall n \geqslant n_1,$$

we have $\{\frac{x_n}{\lambda^n}\}_{n \geqslant n_1}$ is nonincreasing for all $n \geqslant n_1$. Therefore, there exists $\lim\limits_{n\to\infty} \frac{x_n}{\lambda^n}$. Putting $\beta = \lim\limits_{n\to\infty} \frac{x_n}{\lambda^n}$, we shall show $\beta = 0$. Suppose $\beta > 0$, then there exists $n_2 > n_1$ such that

$$x_n \geqslant \beta\lambda^n, \quad \forall n \geqslant n_2.$$

Putting $n_3 = n_2 + m_{i_0}$, where $i_0 \in \{1, 2, \dots r\}$, we get

$$x_{n-m_{i_0}} \geqslant \beta\lambda^{n-m_{i_0}} \geqslant \beta, \quad \forall n \geqslant n_3$$

and by hypotheses, there exists a positive constant β_1 such that

$$|F(x_{n-m_{i_0}})| = -F(x_{n-m_{i_0}}) \geqslant \beta_1, \quad \forall n \geqslant n_3.$$

This implies

$$\frac{1}{\lambda^{n+1}}\alpha_{i_0}(n)F(x_{n-m_{i_0}}) \leqslant -\beta_1 \frac{1}{\lambda^{n+1}}\alpha_{i_0}(n).$$

On the other hand, from (1) we have

$$\frac{x_{n+1}}{\lambda^{n+1}} - \frac{x_n}{\lambda^n} \leqslant \frac{1}{\lambda^{n+1}}\alpha_{i_0}(n)F(x_{n-m_{i_0}}).$$

Hence

$$\frac{x_{n+1}}{\lambda^{n+1}} \leqslant \frac{x_n}{\lambda^n} - \beta_1 \frac{1}{\lambda^{n+1}}\alpha_{i_0}(n) = \frac{x_n}{\lambda^n} - \frac{\beta_1}{\lambda}\frac{1}{\lambda^n}\alpha_{i_0}(n), \quad \forall n \geqslant n_3$$

or

$$\frac{x_n}{\lambda^n} \leqslant \frac{x_{n_3}}{\lambda^{n_3}} - \frac{\beta_1}{\lambda}\sum_{\ell=n_3}^{n-1}\frac{1}{\lambda^\ell}\alpha_{i_0}(\ell), \quad \forall n \geqslant n_3.$$

But, in view of (5) this leads to a contradiction to our assumption that $x_n > 0$ eventually. The case $x_n < 0$ eventually can be considered similarly. ■

Theorem 4. *If the given hypothesis on the parameter λ in Theorem 3 is replaced by $0 < \lambda < 1$, then every solution $\{x_n\}_n$ of (1) is either oscillatory or*

$$\lim_{n\to\infty}\frac{x_n}{n} = 0.$$

Proof. Let $\{x_n\}_n$ be a nonoscillatory solution of (1). Suppose that $\{x_n\}_n$ is an eventually positive solution. We have

$$\frac{x_{n+1}}{\lambda^{n+1}} \leqslant \frac{x_n}{\lambda^n}, \quad \forall n \geq n_1.$$

This yields

$$x_{n+1} \leqslant \frac{x_n}{\lambda^n}\lambda^{n+1} = \lambda x_n < x_n \text{ because } \lambda \in (0,1)$$

and

$$\frac{x_{n+1}}{n+1} < \frac{x_n}{n+1} < \frac{x_n}{n}, \quad \forall n \geqslant n_1.$$

Hence, $\{\frac{x_n}{n}\}_n$ is nonincreasing for all $n \geqslant n_1$ and therefore $\lim\limits_{n\to\infty}\frac{x_n}{n} = \beta$ exists. We can prove $\beta = 0$ similarly as in the proof of Theorem 3. Thus Theorem 4 is proved. ■

Theorem 5. *Assume that $\lambda_n = \lambda \geqslant 1$, $\forall n \in \mathbb{N}$; $\alpha_i(n) \geqslant 0, n \in \mathbb{N}, 1 \leqslant i \leqslant r$; $xF(x) < 0$, $\forall x \neq 0$ and there exist $i_0 \in \{1, 2, \ldots, r\}$ and $L > 0$ such that*

$$|F(u)| \geqslant L, \quad \forall u \in \mathbb{R} \text{ and } L\alpha_{i_0}(n)\frac{1}{\lambda^{m_{i_0}+1}} \geqslant 1, \quad \forall n \in \mathbb{N}.$$

Then (1) *is oscillatory.*

Proof. Let $\{x_n\}_n$ be as in Theorem 3 so that $\{\frac{x_n}{\lambda^n}\}_{n\geqslant n_1}$ is nonincreasing for all $n \geqslant n_1$. Thus, for all $n \geqslant n_1$ we have

$$x_{n-m_{i_0}} \geqslant x_n \lambda^{-m_{i_0}}$$

and

$$\begin{aligned} x_{n+1} &= \lambda x_n + \sum_{i=1}^{r} \alpha_i(n) F(x_{n-m_i}) \\ &\leqslant \lambda x_n + \alpha_{i_0}(n) F(x_{n-m_{i_0}}), \\ &\leqslant \lambda x_n - \alpha_{i_0}(n) L x_{n-m_{i_0}}, \\ &\leqslant \lambda x_n - \alpha_{i_0}(n) L x_n \lambda^{-m_{i_0}}, \\ &= \lambda x_n [1 - \alpha_{i_0}(n) L \lambda^{-m_{i_0}-1}], \\ &= \lambda x_n [1 - L\alpha_{i_0}(n) \frac{1}{\lambda^{m_{i_0}+1}}] \leqslant 0. \end{aligned}$$

This contradicts our assumption. ∎

3. Convergence and Boundedness

Consider the difference equation (2), where $n \in \mathbb{N}, x_{-m}, x_{-m+1}, \ldots, x_0$ are positive initial values and the function

$$G(z_0, z_1, \ldots, z_m) : \mathbb{R}^+ \times \ldots \times \mathbb{R}^+ \to \mathbb{R}^+.$$

We give conditions under which every solution of this equation is convergent or bounded. First of all we have

Lemma 1. *If* $\lambda_0 + \lambda_1 + \lambda_2 + \cdots + \lambda_m < 1$ *then there exists a number* $s > 1$ *such that*

$$\lambda_0 s + \lambda_1 s^2 + \lambda_2 s^3 + \cdots + \lambda_m s^{m+1} < 1.$$

Lemma 2. *Let* $\{\beta_n\}_n$ *be a sequence which satisfies the following relations:*

$$\beta_0 = \beta_{-1} = \cdots = \beta_{-m} = 1,$$

$$\beta_{n+1} = \lambda_0 \beta_n + \lambda_1 \beta_{n-1} + \cdots + \lambda_m \beta_{n-m}.$$

If $P := \lambda_0 + \lambda_1 + \lambda_2 + \cdots + \lambda_m > 1$ *where* $\lambda_i \geq 0$, *then* $\beta_n > 1$, $\forall n \in \mathbb{N}_0$ *and* β_n *is monotone increasing for* $n \in \mathbb{N}_0$.

Theorem 6. *Assume that* $G(z_0, z_1, \ldots, z_m) \leqslant \sum_{i=0}^{m} \lambda_i z_i$ *and* $\sum_{i=0}^{m} \lambda_i < 1$. *Then every solution of* (2) *converges to zero.*

Proof. Since $G(z_0, z_1, \ldots, z_m) \leqslant \sum_{i=0}^{m} \lambda_i z_i$, for a positive number $a > 1$ we get

$$a^{x_{n+1}} = a^{G(x_n,\ldots,x_{n-m})} \leqslant a^{\lambda_0 x_n} a^{\lambda_1 x_{n-1}} \ldots a^{\lambda_m x_{n-m}}.$$

Put $y_n = a^{x_n}$. Clearly

$$y_{n+1} \leqslant [y_n]^{\lambda_0}.\ [y_{n-1}]^{\lambda_1} \ldots [y_{n-m}]^{\lambda_m}$$

and $y_n \geq 1$. Hence, $\eta = \max\{y_{-m}, y_{-m+1}, \ldots, y_0\} \geq 1$. Using Lemma 1, we can prove the following estimations by induction

$$y_{n+1} \leqslant \eta^{s^{-n}}, \quad n \in \mathbb{N}_0, \tag{6}$$

where s was given in Lemma 1. For $n = 0$, we have

$$y_1 \leqslant [y_0]^{\lambda_0},\ [y_{-1}]^{\lambda_1} \ldots [y_{-m}]^{\lambda_m} \leqslant \eta^{\lambda_0+\lambda_1+\ldots+\lambda_m} < \eta^1 = \eta^{s^{-0}}.$$

Assume that (6) holds for the steps $1, 2, \ldots, n$, we estimate the solution at step $n + 1$ as follows

$$\begin{aligned} y_{n+1} &\leqslant [y_n]^{\lambda_0} \cdot [y_{n-1}]^{\lambda_1} \ldots [y_{n-m}]^{\lambda_m} \\ &\leqslant \eta^{s^{-(n-1)}.\lambda_0}.\eta^{s^{-(n-2)}.\lambda_1} \ldots \eta^{s^{-(n-m+1)}.\lambda_m} \\ &= \eta^{s^{-n}.(\lambda_0 s+\lambda_1 s^2+\lambda_2 s^3+\ldots+\lambda_m s^{m+1})} \\ &\leqslant \eta^{s^{-n}}. \end{aligned}$$

This implies $\lim_{n\to\infty} y_n \leqslant \eta^0 = 1$. Since $y_n \geq 1$ for all n, we have $\lim_{n\to\infty} y_n = 1$, hence $\lim_{n\to\infty} x_n = 0$. The proof is complete. ∎

Assume that equation (2) has a unique positive equilibrium $\overline{x}$. We have a sufficient condition for convergence to $\overline{x}$.

Corollary 1. *If $G(z_0, z_1, \ldots, z_m)$ satisfies Lipschitz condition in every variable z_i with Lipschitz factors L_i which satisfy $\sum_{i=0}^{m} L_i < 1$, then every solution of (2) is convergent to the positive equilibrium $\overline{x}$.*

Proof. We have

$$\begin{aligned} |x_{n+1} - \overline{x}| &= |G(x_n, x_{n-1}, \ldots, x_{n-m}) - G(\overline{x}, \overline{x}, \ldots, \overline{x})| \\ &\leqslant |G(x_n, x_{n-1}, \ldots, x_{n-m}) - G(\overline{x}, x_{n-1}, \ldots, x_{n-m})| \\ &+ |G(\overline{x}, x_{n-1}, \ldots, x_{n-m}) - G(\overline{x}, \overline{x}, x_{n-2}, \ldots, x_{n-m})| \\ &\ldots \\ &+ |G(\overline{x}, \overline{x}, \ldots, \overline{x}, x_{n-m}) - G(\overline{x}, \overline{x}, \ldots, \overline{x})| \\ &\leqslant L_0|x_n - \overline{x}| + L_1|x_{n-1} - \overline{x}| + \ldots + L_m|x_{n-m} - \overline{x}|. \end{aligned}$$

Putting $y_n = |x_n - \overline{x}|$, we have

$$y_{n+1} \leqslant L_0 y_n + L_1 y_{n-1} + \ldots + L_m y_{n-m}.$$

By Theorem 6, the proof is complete. ∎

Remark 1. In the case of

$$G(x_n, x_{n-1}, \ldots, x_{n-m}) = \lambda_n x_n + \sum_{i=1}^{m} \alpha_i(n) F(x_{n-i}),$$

where $\alpha_i(n) \geq 0, \sum_{i=1}^{m} \alpha_i(n) = 1, \forall n \in \mathbb{N}$ and $F : [0, \infty) \to [0, \infty)$ is a continuous function, applying Theorem 6 to equation (2), we obtain some convergence results presented in [5, 6].

Under converse conditions, the following theorem gives a sufficient condition for the non-convergence to zero of the solutions of (2).

Theorem 7. *Assume that* $G(z_0, z_1, \ldots, z_m) \geq \sum_{i=0}^{m} \lambda_i z_i$ *and* $\sum_{i=0}^{m} \lambda_i > 1$. *Then, every solution* $\{x_n\}$ *of* (2) *satisfies*

$$\liminf_{n\to\infty} x_n > 0.$$

Proof. As in the proof of Theorem 6, we put $y_n = a^{x_n}$. Then we have

$$y_{n+1} \geq [y_n]^{\lambda_0} \ldots [y_{n-1}]^{\lambda_1} \ldots [y_{n-m}]^{\lambda_m}$$

and $\theta = \min\{y_0, y_{-1}, \ldots, y_{-m}\} > 1$. We prove $y_n \geq \theta^{\beta_n}$ by induction.

Clearly, $y_1 \geq [y_0]^{\lambda_0} \dot{[}y_{-1}]^{\lambda_1} \ldots [y_{-m}]^{\lambda_m} \geq \theta^{\lambda_0+\lambda_1+\lambda_2+\ldots+\lambda_m} = \theta^{\beta_1}$. Assuming that $y_n \geq \theta^{\beta_n}$ for the steps $1, 2, \ldots, n$, we have

$$\begin{aligned} y_{n+1} &\geq [y_n]^{\lambda_0} . [y_{n-1}]^{\lambda_1} \ldots [y_{n-m}]^{\lambda_m} \\ &\geq \theta^{\lambda_0 \beta_n} . \theta^{\lambda_1 \beta_{n-1}} \ldots \theta^{\lambda_m \beta_{n-m}} \\ &= \theta^{\lambda_0 \beta_n + \lambda_1 \beta_{n-1} + \ldots + \lambda_m \beta_{n-m}} \\ &= \theta^{\beta_{n+1}}. \end{aligned}$$

By Lemma 2, we get $y_{n+1} \geqslant \theta^{\beta_{n+1}} \geqslant \theta^{\beta_1} = \theta^P, \quad \forall n \in \mathbb{N}_0$. This yields $x_{n+1} \geqslant P. \log_a \theta > 0$. Hence, $\liminf_{n\to\infty} x_n \geqslant P. \log_a \theta > 0$. ∎

Definition 1. *A solution* $\{x_n\}_n$ *of* (2) *is called persistent if*

$$0 < \liminf_{n\to\infty} x_n \leqslant \limsup_{n\to\infty} x_n < \infty.$$

The following theorem gives a sufficient condition for the persistence of (2).

Theorem 8. *Assume that*

$$G(x_0, x_1, \dots, x_m) = H(x_0, x_1, \dots, x_m; x_0, x_1, \dots, x_m)$$

where

$$H(x_0, x_1, \dots, x_m; y_0, y_1, \dots, y_m) : [0, \infty)^{2(m+1)} \to [0, \infty)$$

is a continuous function, increasing in x_i and decreasing in y_i and

$$H(x_0, x_1, \dots, x_m; y_0, y_1, \dots, y_m) > 0$$

if $x_i, y_i > 0$. Suppose further that

$$\limsup_{x_i, y_i \to \infty} \frac{H(x_0, x_1, \dots, x_m; y_0, y_1, \dots, y_m)}{x_0 + x_1 + \ldots + x_m} < \frac{1}{m+1},$$

$$\liminf_{x_i, y_i \to 0^+} \frac{H(x_0, x_1, \dots, x_m; y_0, y_1, \dots, y_m)}{x_0 + x_1 + \ldots + x_m} > \frac{1}{m+1}.$$

Then every solution $\{x_n\}_{n=-m}^{\infty}$ of (2) is persistent.

Proof. The proof of this theorem can be obtained similarly as the proof of Theorem 2 in [6]. ■

4. Conclusion

New results for oscillation or nonoscillation of the difference equation (1) and the extensive results for convergence and boundedness of a class of general difference equations (2) are given in this paper. Note that, some results in [5, 6] are particular cases of Theorem 6 and Theorem 8.

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