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Quasiconformal Analogues of the Hardy-Littlewood Property in Uniformly John Domains*

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Abstract. A result of Hardy-Littlewood relates Hölder continuity of analytic functions over the unit disk to the growth of the derivative. Astala and Gehring extend this result to a quasiconformal analogue in uniform domain in $n-$dimensional space. In this paper, we prove some quasiconformal analogues of Hardy-Littlewood's result in uniformly John domains.

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1. Introduction

Hardy and Littlewood proved the following results in [6].

Theorem 1.1. *Suppose that f is analytic in the unit disk $B = \{z : |z| < 1\}$ and $0 < \alpha \leq 1$. If there exists a constant C_1 such that*

$$|f'(z)| \leq C_1(1 - |z|)^{\alpha-1} = C_1 \text{dist}(z, \partial B)^{\alpha-1} \tag{1}$$

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for all $z \in B$, then f has a continuous extension to $\overline{B} = \{z : |z| \leq 1\}$ and

$$|f(z_1) - f(z_2)| \leq C_2|z_1 - z_2|^\alpha \tag{2}$$

for all $z_1, z_2 \in \overline{B}$ and some constant C_2 which depends only on C_1 and α.

Suppose next that D and D' are domains in $\mathbb{R}^n$ and that $f : D \to D'$ is $K-$quasiconformal with Jacobian J_f. Then $\log J_f$ is integrable over each ball $B \subset D$ and for $x \in D$ we set

$$a_f(x) = \exp\left(\frac{1}{nm(B(x))}\int_{B(x)} \log J_f dm\right), \tag{3}$$

where $B(x) = B(x, \mathrm{dist}(x, \partial D))$, the open ball with center x and radius equal to the distance $\mathrm{dist}(x, \partial D)$ from x to ∂D, $m(B(x))$ denotes the $n-$dimensional Lebesgue measure of $B(x)$. If $n = 2$ and f is conformal in D, then $\log J_f$ is harmonic and hence $a_f(x) = |f'(x)|$. Astala and Gehring observed firstly that for certain distortion properties of quasiconformal mappings the function a_f plays a role exactly analogous to that played by $|f'|$ when $n = 2$ and f is conformal, see[1, 2, 7].

Astala and Gehring [1, 2] use the characteristic of quasiconformal mapping, $a_f(x)$, to study quasiconformal analogues of theorem of Hardy-Littlewood in uniform domains. They obtain the following theorem and other variants of this analogue.

Theorem 1.2. [1] *Suppose that D is a uniform domain in $\mathbb{R}^n$ and that α and m are constants with $0 < \alpha \leq 1$ and $m \geq 0$. If f is $K-$quasiconformal in D with $f(D) \subset \mathbb{R}^n$ and if*

$$a_f(x) \leq m\ \mathrm{dist}(x, \partial D)^{\alpha-1}$$

for $x \in D$, then f has a continuous extension to $\overline{D} \setminus \{\infty\}$ and

$$|f(x_1) - f(x_2)| \leq cm(|x_1 - x_2| + \mathrm{dist}(x_1, \partial D))^\alpha$$

for $x_1, x_2 \in \overline{D} \setminus \{\infty\}$, where c is a constant which depends only on K, n, α and the constants for D.

In this paper, we shall give some quasiconformal analogues of theorem of Hardy-Littlewood in uniformly John domains. The main results are as follows.

Theorem 1.3. *Suppose that D is a uniformly John domain in R^n and that α and m are constants with $0 < \alpha \leq 1$ and $m \geq 0$. If f is $K-$quasiconformal in D with $f(D) \subset \mathbb{R}^n$ and if*

$$a_f(x) \leq m\ \mathrm{dist}(x, \partial D)^{\alpha-1} \tag{4}$$

for $x \in D$, then

$$|f(x_1) - f(x_2)| \leq cm(\rho_D(x_1, x_2) + \text{dist}(x_1, \partial D))^\alpha \tag{5}$$

for $x_1, x_2 \in D$, where c is a constant which depends only on K, n, α and the constants for D, and $\rho_D(x, y) = \inf dia(\gamma)$ for $x, y \in D$, the infimums are taken over all open arcs γ joining x and y in D with diameter $dia(\gamma)$.

Simple examples show that the term $\text{dist}(x_1, \partial D)$ cannot in general be omitted, see [1, Remark 3.12] and [7]. On the other hand, the following alternative quasiconformal analogue of Theorem 1.1 yields a sharper estimate for $|f(x_1) - f(x_2)|$ in the special case where $\alpha \leq K^{1/(1-n)}$.

Theorem 1.4. *Suppose that D is a uniformly John domain in $\mathbb{R}^n$ and that α and m are constants with $0 < \alpha \leq K^{1/(1-n)}$ and $m \geq 0$. If f is $K-$quasiconformal in D with $f(D) \subset \mathbb{R}^n$ and if*

$$a_f(x) \leq m\, \text{dist}(x, \partial D)^{\alpha - 1}$$

for $x \in D$, then

$$|f(x_1) - f(x_2)| \leq cm\rho_D(x_1, x_2)^\alpha \tag{6}$$

for $x_1, x_2 \in D$, where c is a constant which depends only on K, n, α and the constants for D.

2. Notations and Preliminary Results

Throughout this paper, assume that D is a domain in Euclidean $n-$space $\mathbb{R}^n$. We say that D is a *uniformly John domain* if there exist positive constants a and b such that each pair of points $x_1, x_2 \in D$ can be joined by a rectifiable arc $\gamma \subset D$ for which

$$l(\gamma) \leq a\rho_D(x_1, x_2) \tag{7}$$

and

$$\min_{j=1,2} l(\gamma_j) \leq b\ \text{dist}(x, \partial D) \tag{8}$$

for each $x \in \gamma$; here $l(\gamma)$ denotes the euclidean length of γ and γ_1, γ_2 the components of $\gamma \setminus \{x\}$.

A uniformly John domain is a domain intermediate between a uniform domain and a John domain. Balogh and Volberg [3, 4] introduced the uniformly John domain in connection with conformal dynamics.

Given a set D in $\mathbb{R}^n$, we let $Lip_h^\alpha(D)$, $0 < \alpha \leq 1$, denote the *Lipschitz class* of mappings $f : D \to \mathbb{R}^p$ satisfying for some constant $m < \infty$ the inequality

$$|f(x_1) - f(x_2)| \leq mh(x_1, x_2)^\alpha \tag{9}$$

for all x_1 and x_2 in D, where $h(\cdot\,,\cdot)$ is a metric defined in D. If D is a domain in $\mathbb{R}^n$, then $f : D \to \mathbb{R}^p$ is said to belong to the *local Lipschitz class*, $locLip_h^\alpha(D)$, if there is a constant $m < \infty$ such that (9) holds whenever x_1, x_2 lie in any open ball contained in D.

In $Lip_h^\alpha(D)$ and $locLip_h^\alpha(D)$ we shall use seminorms $||f||_{h,\alpha}$ and $||f||_{h,\alpha}^{loc}$ respectively, which mean the infimum of the numbers m for which (9) holds in the corresponding set. We say that a domain $D \subset \mathbb{R}^n$ is a Lip_h^α*−extension domain* if there exists a constant a depending on D and α such that $f \in locLip_h^\alpha(D)$ implies $f \in Lip_h^\alpha(D)$ with

$$||f||_{h,\alpha} \le a||f||_{h,\alpha}^{loc}.$$

When $h(x, y) = |x - y|$, Gehring and Martio [5] prove that uniform domains are Lip_h^α−extension domains. We give an analogous theorem for uniformly John domains as follows.

Lemma 2.1. *Uniformly John domains are* Lip_ρ^α*−extension domains for all* $0 < \alpha \leqslant 1$.

Proof. Let D be a uniformly John domain, fix $x_1, x_2 \in D$. Since D is a uniformly John domain, we can find a rectifiable arc γ joining x_1 and x_2 in D which satisfies (7) and (8). Choose $x_0 \in \gamma$ such that $l(\gamma(x_1, x_0)) = l(\gamma(x_0, x_2))$. Because $\operatorname{dist}(\gamma, \partial D) > 0$, we can choose points $y_0, y_1, \cdots, y_l \in \gamma$ with the following properties

$$\begin{cases} y_0 = x_0, \\ y_{j+1} \in \gamma(y_j, x_1), \\ |y_{j+1} - y_j| = r\ \operatorname{dist}(y_j, \partial D), \\ |y_l - x_1| \le r\ \operatorname{dist}(y_l, \partial D), \\ y_{l+1} = x_1, \end{cases} \tag{10}$$

with $0 < r < 1$. Obviously we may assume that $l \ge 1$. Set

$$u_j = \begin{cases} |y_{j+1} - y_j|, & \text{if} \quad 0 \le j \le l, \\ 0, & \text{if} \quad l + 1 \le j < \infty. \end{cases}$$

If $0 \le k \le l - 1$, then

$$\sum_{j=k}^{\infty} u_j = \sum_{j=k}^{l} |y_{j+1} - y_j| \le l(\gamma(y_k, x_1)) \le l(\gamma(y_k, x_2)),$$

while by (8)

$$l(\gamma(y_k, x_1)) \le b\ \operatorname{dist}(y_k, \partial D) = \frac{b}{r}|y_{k+1} - y_k| = \frac{b}{r}u_k.$$

Hence

$$\sum_{j=k}^{\infty} u_j \leq \frac{b}{r} u_k. \tag{11}$$

If $k \geq l+1$ the inequality (11) is trivially true. For the case of $k = l$, $\rho_D(y_l, x_1) = |y_l - x_1|$ since $|y_l - x_1| \leq r\mathrm{dist}(y_l, \partial D)$, and by (7) we have

$$l(\gamma(y_l, x_1)) \leq a\rho_D(y_l, x_1) = a|y_l - x_1| = au_l,$$

hence

$$\sum_{j=l}^{\infty} u_j \leq au_l.$$

Suppose that $f \in locLip_\rho^\alpha(D)$ with $||f||_{\rho,\alpha}^{loc} = m$. Since in a ball, the metric ρ_D is the same as euclidean metric, we can apply (7), (10) and [1, Lemma 3.1] to obtain

$$\begin{aligned}
|f(x_1) - f(x_0)| &\leq \sum_{j=0}^{l} m|y_{j+1} - y_j|^\alpha \\
&\leq c_0 m|y_1 - y_0|^\alpha \\
&\leq c_0 ml(\gamma(x_1, x_0))^\alpha \\
&= c_0 m(l(\gamma(x_1, x_2))/2)^\alpha \\
&\leq (a/2)^\alpha c_0 m\rho_D(x_1, x_2)^\alpha \\
&= c_1 m\rho_D(x_1, x_2)^\alpha,
\end{aligned}$$

where $c_1 = (\frac{a}{2})^\alpha c_0$ depends only on b, r, α and the constants for D. By the same argument with x_2 in place of x_1,

$$|f(x_2) - f(x_0)| \leq c_1 m\rho_D(x_1, x_2)^\alpha.$$

Hence we obtain

$$|f(x_1) - f(x_2)| \leq 2c_1 m\rho_D(x_1, x_2)^\alpha$$

for any $x_1, x_2 \in D$. ■

Lemma 2.2. *Suppose that D is a uniformly John domain in $\mathbb{R}^n$ and that α, r and m are constants with $0 < \alpha \leq 1$, $0 < r < 1$ and $m \geq 0$. If $g : D \to \mathbb{R}^n$ is an open mapping and if*

$$|g(x_1) - g(x_2)| \leq m|x_1 - x_2|^\alpha \tag{12}$$

for $x_1, x_2 \in D$ with $|x_1 - x_2| = r\ \mathrm{dist}(x_1, \partial D)$, then

$$|g(x_1) - g(x_2)| \leq cm(\rho_D(x_1, x_2) + \mathrm{dist}(x_1, \partial D))^\alpha \tag{13}$$

for $x_1, x_2 \in D$, where c is a constant which depends only on α, r and the constants for D.

Proof. Fix $x_1, x_2 \in D$. Because D is a uniformly John domain, we can find a rectifiable arc γ joining x_1 and x_2 in D which satisfies (7) and (8) with constants a and b which depend only on D. Let x_0 denote the midpoint of γ. We can choose points $y_0, y_1, \cdots, y_l \in \gamma$ with the following properties

$$\begin{cases} y_0 = x_0, \\ y_{j+1} \in \gamma(y_j, x_1), \\ |y_{j+1} - y_j| = r \operatorname{dist}(y_j, \partial D), \\ |y_l - x_1| \le r \operatorname{dist}(y_l, \partial D). \end{cases} \tag{14}$$

Similar to the proof of Theorem 2.1, we can show that

$$|g(y_l) - g(x_0)| \le c_1 m \rho_D(x_1, x_2)^\alpha. \tag{15}$$

By triangle inequality, we have

$$\begin{aligned} \operatorname{dist}(y_l, \partial D) &\le \operatorname{dist}(x_1, \partial D) + |y_l - x_1| \\ &\le \operatorname{dist}(x_1, \partial D) + r\operatorname{dist}(y_l, \partial D), \end{aligned}$$

hence

$$\operatorname{dist}(y_l, \partial D) \le \frac{1}{1-r}\operatorname{dist}(x_1, \partial D).$$

Next, because g is open,

$$\begin{aligned} |g(x_1) - g(y_l)| &\le \sup\{|g(x) - g(y_l)| : |x - y_l| = r\operatorname{dist}(y_l, \partial D)\} \\ &\le m(r\operatorname{dist}(y_l, \partial D))^\alpha \\ &\le c_2 m \operatorname{dist}(x_1, \partial D)^\alpha \end{aligned}$$

where $c_2 = r^\alpha(1-r)^{-\alpha}$, and with (15) we have

$$|g(x_1) - g(x_0)| \le c_3 m(\rho_D(x_1, x_2) + \operatorname{dist}(x_1, \partial D))^\alpha,$$

where $c_3 = \max\{c_1, c_2\}$. By the same argument with x_2 in place of x_1,

$$|g(x_2) - g(x_0)| \le c_3 m(\rho_D(x_1, x_2) + \operatorname{dist}(x_2, \partial D))^\alpha,$$

and since

$$\operatorname{dist}(x_2, \partial D) \le \rho_D(x_1, x_2) + \operatorname{dist}(x_1, \partial D),$$

we obtain (13) with $c = 3c_3$ for $x_1, x_2 \in D$. ∎

3. Proofs of Theorem 1.3 and Theorem 1.4

Proof of Theorem 1.3. Choose $x_1, x_2 \in D$ with

$$\rho_D(x_1, x_2) = \frac{1}{2}\mathrm{dist}(x_1, \partial D).$$

Then by the $n-$dimensional version of Lemma 5.15 in [2] applied to the open ball $B(x_1, \mathrm{dist}(x_1, \partial D))$, we have

$$\begin{aligned}|f(x_1) - f(x_2)| &\leq c_1 a_f(x_1)\mathrm{dist}(x_1, \partial D)^a |x_1 - x_2|^{1-a} \\ &= 2^a c_1 a_f(x_1)|x_1 - x_2|.\end{aligned}$$

where $a = (e/2)||\log J_f||_* \leq c_2$ and $c_j = c_j(K, n)$ for $j = 1, 2$. Next by hypothesis,

$$\begin{aligned}a_f(x_1) &\leq m\,\mathrm{dist}(x_1, \partial D)^{\alpha-1} \\ &= 2^{\alpha-1} m|x_1 - x_2|^{\alpha-1} \\ &\leq m|x_1 - x_2|^{\alpha-1},\end{aligned}$$

and hence

$$|f(x_1) - f(x_2)| \leq 2^{c_2} c_1 m \rho_D(x_1, x_2)^\alpha.$$

The desired conclusion now follows from Lemma 2.2 with $r = 1/2$. ∎

Proof of Theorem 1.4. By Theorem 1.8 in [1] and hypothesis,

$$\begin{aligned}\mathrm{dist}(f(x), \partial f(D)) &\leq c_1 a_f(x)\mathrm{dist}(x, \partial D) \\ &\leq c_1 m\,\mathrm{dist}(x, \partial D)^\alpha,\end{aligned}$$

for $x \in D$ with $c_1 = c_1(K, n)$. Then since $\alpha \leq K^{1/(1-n)}$, Theorem 3.4 in [5] implies that

$$|f(x_1) - f(x_2)| \leq c_2|x_1 - x_2|^\alpha = c_2\rho_D(x_1, x_2)^\alpha$$

whenever $x_1, x_2 \in B \subset D$; here c_2 depends only on c_1 and α. That is $f \in locLip^\alpha_\rho(D)$. Hence by Lemma 2.1, we have

$$|f(x_1) - f(x_2)| \leq c_3 c_2 \rho_D(x_1, x_2)^\alpha$$

for $x_1, x_2 \in D$, where c_3 is a constant depending only on α, and the constants for D. This completes the proof. ∎

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