On the Fekete-Szegö Problem for Certain Subclasses of Analytic Functions

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Abstract. In this present investigation, the authors obtain Fekete-Szegö’s inequality for certain normalized analytic functions \( f(z) \) defined on the open unit disk for which
\[
(1 - \alpha) \left( \frac{f(z)}{z} \right)^\beta + \alpha f'(z) \left( \frac{1}{f(z)} \right)^{\beta - 1}, \quad (\beta \geq 0, \ 0 \leq \alpha < 1)
\]
lies in a region starlike with respect to 1 and is symmetric with respect to the real axis. Also certain applications of the main result for a class of functions defined by convolution are given. As a special case of this result, Fekete-Szegö’s inequality for a class of functions defined through fractional derivatives is obtained. The motivation of this paper is to give a generalization of the Fekete-Szegö inequalities obtained by Srivastava and Mishra.

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1. Introduction

Let \( \mathcal{A} \) denote the class of all analytic functions \( f(z) \) of the form
\[
f(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad (z \in \Delta := \{ z \in \mathbb{C} \mid |z| < 1 \})
\]
and \( \mathcal{S} \) be the subclass of \( \mathcal{A} \) consisting of univalent functions. Let \( \phi(z) \) be an analytic function with positive real part on \( \Delta \) with \( \phi(0) = 1, \ \phi'(0) > 0 \) which
maps the unit disk $\Delta$ onto a region starlike with respect to 1 which is symmetric with respect to the real axis. Let $S^*(\phi)$ be the class of functions in $f \in S$ for which

$$\frac{zf'(z)}{f(z)} \prec \phi(z), \quad (z \in \Delta),$$

and $C(\phi)$ be the class of functions in $f \in S$ for which

$$1 + \frac{zf''(z)}{f'(z)} \prec \phi(z), \quad (z \in \Delta),$$

where $\prec$ denotes the subordination between analytic functions. These classes were introduced and studied by Ma and Minda [3]. They have obtained the Fekete-Szegö inequality for the functions in the class $C(\phi)$. Since $f \in C(\phi)$ if and only if $zf'(z) \in S^*(\phi)$, we get the Fekete-Szegö inequality for functions in the class $S^*(\phi)$. Recently, Shanmugam and Sivasubramanian [7] obtained Fekete-Szegö inequalities for the class of functions $zf'(z) + \alpha z^2f''(z) \prec \phi(z) \quad (\alpha \geq 0)$.

For a brief history of Fekete-Szegö problem for the class of starlike, convex and close-to-convex functions, see the recent paper by Srivastava et al. [9].

In the present paper, we obtain the Fekete-Szegö inequality for functions in a more general class $M_{\alpha,\beta}(\phi)$ of functions which we define below. Also we give applications of our results to certain functions defined through convolution (or Hadamard product) and in particular we consider a class $M_{\lambda,\alpha,\beta}(\phi)$ of functions defined by fractional derivatives. The motivation of this paper is to give a generalization of the Fekete-Szegö inequalities of Srivastava and Mishra [8].

**Definition 1.1.** Let $\phi(z)$ be a univalent starlike function with respect to 1 which maps the unit disk $\Delta$ onto a region in the right half plane which is symmetric with respect to the real axis, $\phi(0) = 1$ and $\phi'(0) > 0$. A function $f \in A$ is in the class $M_{\alpha,\beta}(\phi)$ if

$$(1 - \alpha) \left( \frac{f(z)}{z} \right)^\beta + \alpha f'(z) \left( \frac{z}{f(z)} \right)^{\beta - 1} \prec \phi(z), \quad (\beta \geq 0, 0 \leq \alpha < 1).$$

For fixed $g \in A$, we define the class $M_{\alpha,\beta}(\phi)$ to be the class of functions $f \in A$ for which $(f \ast g) \in M_{\alpha,\beta}(\phi)$.

To prove our main result, we need the following.

**Lemma 1.2.** [3] If $p_1(z) = 1 + c_1 z + c_2 z^2 + \cdots$ is an analytic function with positive real part in $\Delta$, then
On the Fekete-Szegő Problem for Certain Subclasses of Analytic Functions

\[ |c_2 - vc_1^2| \leq \begin{cases} -4v + 2 & \text{if } v \leq 0 \\ 2 & \text{if } 0 \leq v \leq 1 \\ 4v - 2 & \text{if } v \geq 1 \end{cases} \]

When \( v < 0 \) or \( v > 1 \), the equality holds if and only if \( p_1(z) = (1 + z)/(1 - z) \) or one of its rotations. If \( 0 < v < 1 \), then the equality holds if and only if \( p_1(z) = (1 + z^2)/(1 - z^2) \) or one of its rotations. If \( v = 0 \), the equality holds if and only if

\[
p_1(z) = \left( \frac{1}{2} + \frac{1}{2} \lambda \right) \frac{1 + z}{1 - z} + \left( \frac{1}{2} - \frac{1}{2} \lambda \right) \frac{1 - z}{1 + z} \quad (0 \leq \lambda \leq 1)
\]

or one of its rotations. If \( v = 1 \), the equality holds if and only if \( p_1 \) is the reciprocal of one of the functions such that the equality holds in the case of \( v = 0 \).

Also the above upper bound is sharp, and it can be improved as follows when \( 0 < v < 1 \):

\[
|c_2 - vc_1^2| + v|c_1|^2 \leq 2 \quad (0 < v \leq 1/2)
\]

and

\[
|c_2 - vc_1^2| + (1 - v)|c_1|^2 \leq 2 \quad (1/2 < v \leq 1).
\]

2. Fekete-Szegő Problem

Our main result is the following.

**Theorem 2.1.** Let \( \phi(z) = 1 + B_1z + B_2z^2 + B_3z^3 + \cdots \). If \( f(z) \) given by (1.1) belongs to \( M_{\alpha,\beta}(\phi) \), then

\[
|a_3 - \mu a_2^2| \leq \begin{cases} \frac{B_2}{(\beta + 2\alpha)} - \frac{\mu}{(\beta + \alpha)^2}B_1^2 + \frac{1 - \beta}{2(\beta + \alpha)^2}B_1^2 & \text{if } \mu \leq \sigma_1 \\ \frac{B_2}{(\beta + 2\alpha)} - \frac{\mu}{(\beta + \alpha)^2}B_1^2 - \frac{1 - \beta}{2(\beta + \alpha)^2}B_1^2 & \text{if } \sigma_1 \leq \mu \leq \sigma_2 \\ - \frac{B_2}{(\beta + 2\alpha)} + \frac{\mu}{(\beta + \alpha)^2}B_1^2 & \text{if } \mu \geq \sigma_2 \end{cases}
\]

where

\[
\sigma_1 := \frac{2(\beta + \alpha)^2(B_2 - B_1) + (1 - \beta)(\beta + 2\alpha)B_1^2}{2(\beta + 2\alpha)B_1^2},
\]

\[
\sigma_2 := \frac{2(\beta + \alpha)^2(B_2 + B_1) + (\beta + 2\alpha)(1 - \lambda)B_1^2}{2(\beta + 2\alpha)B_1^2}.
\]

The result is sharp.
Proof. For \( f(z) \in M_{\alpha,\beta}(\phi) \), let
\[
p(z) := (1 - \alpha) \left( \frac{f(z)}{z} \right)^{\beta} + \alpha f'(z) \left( \frac{z}{f(z)} \right)^{\beta-1} = 1 + b_1 z + b_2 z^2 + \cdots \tag{2.1}
\]
From (2.1), we obtain
\[
(\beta + \alpha) a_2 = b_1 \quad \text{and} \quad (\beta + 2\alpha) a_3 = b_2 - \frac{(\beta - 1)(\beta + 2\alpha)}{2} a_2.
\]
Since \( \phi(z) \) is univalent and \( p < \phi \), the function
\[
p_1(z) = \frac{1 + \phi^{-1}(p(z))}{1 - \phi^{-1}(p(z))} = 1 + c_1 z + c_2 z^2 + \cdots
\]
is analytic and has positive real part in \( \Delta \). Also we have
\[
p(z) = \phi \left( \frac{p_1(z) - 1}{p_1(z) + 1} \right) \tag{2.2}
\]
and from this equation (2.2), we obtain
\[
\begin{align*}
b_1 &= \frac{1}{2} B_1 c_1 \\
b_2 &= \frac{1}{2} B_1 (c_2 - \frac{1}{2} c_1^2) + \frac{1}{4} B_2 c_1^2.
\end{align*}
\]
Therefore we have
\[
a_3 - \mu a_2^2 = \frac{B_1}{2(\beta + 2\alpha)} (c_2 - v c_1^2) \tag{2.3}
\]
where
\[
v := \frac{1}{2} \left[ 1 - \frac{B_2}{B_1} + \frac{(\beta - 1 + 2\mu)(2\alpha + \beta)}{2(\beta + \alpha)^2} B_1 \right].
\]
Our result now follows by an application of Lemma 1.2. To show that the bounds are sharp, we define the functions \( K^{\phi_n} \ (n = 2, 3, \ldots) \) by
\[
(1 - \alpha) \left( \frac{K^{\phi_n}}{z} \right)^{\beta} + \alpha [K^{\phi_n}]'(z) \left( \frac{z}{K^{\phi_n}} \right)^{\beta-1} = \phi(z^{n-1}),
\]
\[
K^{\phi_n}(0) = 0 = [K^{\phi_n}]'(0) - 1
\]
and the function \( F^\lambda \) and \( G^\lambda \ (0 \leq \lambda \leq 1) \) by
\[
(1 - \alpha) \left( \frac{F^\lambda(z)}{z} \right)^{\beta} + \alpha [F^\lambda]'(z) \left( \frac{z}{F^\lambda(z)} \right)^{\beta-1} = \phi \left( \frac{z(z + \lambda)}{1 + \lambda z} \right),
\]
On the Fekete-Szegő Problem for Certain Subclasses of Analytic Functions

\[ P^\lambda(0) = 0 = (F^\lambda)'(0) - 1 \]

and
\[ (1 - \alpha) \left( \frac{G^\lambda(z)}{z} \right)^\beta + \alpha [G^\lambda]'(z) \left( \frac{z}{G^\lambda(z)} \right)^{\beta - 1} = \phi \left( \frac{z(z + \lambda)}{1 + \lambda z} \right), \]

\[ G^\lambda(0) = 0 = (G^\lambda)'(0). \]

Clearly the functions \( K^\phi_{2,\alpha}, F^\lambda_{2,\alpha}, G^\lambda_{2,\alpha} \in M_{\alpha}(\phi) \). Also we write \( K^\phi_{2,\alpha} := K^\phi_{2,2}. \)

If \( \mu < \sigma_1 \) or \( \mu > \sigma_2 \), then the equality holds if and only if \( f \) is \( K^\phi_{2,\alpha} \) or one of its rotations. When \( \sigma_1 < \mu < \sigma_2 \), the equality holds if and only if \( f \) is \( F^\lambda_{2,\alpha} \) or one of its rotations. If \( \mu = \sigma_1 \) then the equality holds if and only if \( f \) is \( F^\lambda_{2,\alpha} \) or one of its rotations.

Remark 2.2. If \( \sigma_1 \leq \mu \leq \sigma_2 \), then, in view of Lemma 1.2, Theorem 2.1 can be improved. Let \( \sigma_3 \) be given by
\[ \sigma_3 := \frac{2(\beta + \alpha)^2 B_2 + \left( \beta + 2\alpha \right)(\beta - 1)B_1^2}{2(\beta + 2\alpha)B_1^2}. \]

If \( \sigma_1 \leq \mu \leq \sigma_3 \), then
\[ |a_3 - \mu a_2^2| + \frac{(\beta + \alpha)^2}{(\beta + 2\alpha)B_1^2} \left[ B_1 - B_2 + \frac{(\beta - 1 + 2\mu)(2\alpha + \beta)}{2(\beta + \alpha)^2}B_1^2 \right] |a_2|^2 \leq \frac{B_1}{(\beta + 2\alpha)}. \]

If \( \sigma_3 \leq \mu \leq \sigma_2 \), then
\[ |a_3 - \mu a_2^2| + \frac{(\beta + \alpha)^2}{(\beta + 2\alpha)B_1^2} \left[ B_1 + B_2 + \frac{(\beta - 1 + 2\mu)(2\alpha + \beta)}{2(\beta + \alpha)^2}B_1^2 \right] |a_2|^2 \leq \frac{B_1}{(\beta + 2\alpha)}. \]

3. Applications to Functions Defined by Fractional Derivatives

In order to introduce the class \( M^\lambda_{\alpha,\beta}(\phi) \), we need the following.

Definition 3.1. (see [4, 5]; see also [10, 11]). Let \( f(z) \) be analytic in a simply connected region of the \( z \)-plane containing the origin. The fractional derivative of \( f \) of order \( \lambda \) is defined by
\[ D^\lambda f(z) := \frac{1}{\Gamma(1 - \lambda)} \frac{d}{dz} \int_0^z f(\zeta)(z - \zeta)^{\lambda - 1} d\zeta \quad (0 \leq \lambda < 1) \]
where the multiplicity of \((z - \zeta)^{\lambda}\) is removed by requiring that \(\log(z - \zeta)\) is real for \(z - \zeta > 0\).

Using the above Definition 3.1 and its known extensions involving fractional derivatives and fractional integrals, Owa and Srivastava [4] introduced the operator \(\Omega^{\lambda} : \mathcal{A} \to \mathcal{A}\) defined by

\[
(\Omega^{\lambda} f)(z) = \Gamma(2 - \lambda)z^{\lambda}D_{z}^{\lambda}f(z), \quad (\lambda \neq 2, 3, 4, \ldots).
\]

The class \(M_{\alpha,\beta}^{\lambda} (\phi)\) consists of functions \(f \in \mathcal{A}\) for which \(\Omega^{\lambda} f \in M_{\alpha,\beta}^{\lambda} (\phi)\). Note that \(M_{1,0}^{1} (\phi) \equiv S^*(\phi)\) and \(M_{\alpha,\beta}^{\lambda} (\phi)\) is the special case of the class \(M_{\alpha,\beta}^{\lambda} (\phi)\) when

\[
g(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)} z^{n}.
\]

Let \(g(z) = z + \sum_{n=2}^{\infty} g_{n} z^{n}\) \((g_{n} > 0)\). Since \(f(z) = z + \sum_{n=2}^{\infty} g_{n} z^{n} \in M_{\alpha,\beta}^{\lambda} (\phi)\) if and only if \((f * g)(z) = z + \sum_{n=2}^{\infty} g_{n} a_{n} z^{n} \in M_{\alpha,\beta}^{\lambda} (\phi)\), we obtain the coefficient estimate for functions in the class \(M_{\alpha,\beta}^{\lambda} (\phi)\), from the corresponding estimate for functions in the class \(M_{\alpha,\beta}^{\lambda} (\phi)\). Applying Theorem 2.1 for the function \((f * g)(z) = z + g_{2}a_{2} z^{2} + g_{3}a_{3} z^{3} + \cdots\), we get the following Theorem 3.2 after an obvious change of the parameter \(\mu\).

**Theorem 3.2.** Let the function \(\phi(z)\) be given by \(\phi(z) = 1 + B_{1} z + B_{2} z^{2} + B_{3} z^{3} + \cdots\). If \(f(z)\) given by (1.1) belongs to \(M_{\alpha,\beta}^{\lambda} (\phi)\), then

\[
|a_{3} - \mu a_{2}^{2}| \leq \begin{cases} 
\frac{1}{g_{3}} \left[ \frac{B_{2}}{\beta + 2\alpha} - \frac{\mu g_{3}}{g_{2}^{2}(\beta + \alpha)^{2}} B_{1}^{2} + \frac{1 - \beta}{2(\beta + \alpha)^{2}} B_{1}^{2} \right], & \text{if } \mu \leq \sigma_{1} \\
\frac{1}{g_{3}} \left[ \frac{B_{1}}{\beta + 2\alpha} \right], & \text{if } \sigma_{1} \leq \mu \leq \sigma_{2} \\
\frac{1}{g_{3}} \left[ -\frac{B_{2}}{\beta + 2\alpha} + \frac{\mu g_{3}}{g_{2}^{2}(\beta + \alpha)^{2}} B_{1}^{2} - \frac{1 - \beta}{2(\beta + \alpha)^{2}} B_{1}^{2} \right], & \text{if } \mu \geq \sigma_{2}
\end{cases}
\]

where

\[
\sigma_{1} := \frac{g_{3}^{2} 2(\beta + \alpha)^{2}(B_{2} - B_{1}) + (1 - \beta)(\beta + 2\alpha)B_{1}^{2}}{2(\beta + 2\alpha)B_{1}^{4}}
\]

\[
\sigma_{2} := \frac{g_{3}^{2} 2(\beta + \alpha)^{2}(B_{2} + B_{1}) + (1 - \beta)(\beta + 2\alpha)B_{1}^{2}}{2(\beta + 2\alpha)B_{1}^{4}}.
\]

The result is sharp.

Since

\[
(\Omega^{\lambda} f)(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)} a_{n} z^{n},
\]
we have
\[ g_2 := \frac{\Gamma(3)\Gamma(2 - \lambda)}{\Gamma(3 - \lambda)} = \frac{2}{2 - \lambda} \quad (3.2) \]
and
\[ g_3 := \frac{\Gamma(4)\Gamma(2 - \lambda)}{\Gamma(4 - \lambda)(3 - \lambda)} = \frac{6}{(2 - \lambda)(3 - \lambda)}. \quad (3.3) \]

For \( g_2 \) and \( g_3 \) given by (3.2) and (3.3), Theorem 3.2 reduces to the following:

**Theorem 3.3.** Let the function \( \phi(z) \) be given by
\[ \phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots \]
If \( f(z) \) given by (1.1) belongs to \( M_{\alpha, \beta}^{\lambda}(\phi) \), then
\[
|a_3 - \mu a_2^2| \leq \begin{cases} 
\frac{(2 - \lambda)(3 - \lambda)}{6} \gamma & \text{if } \mu \leq \sigma_1 \\
\frac{(2 - \lambda)(3 - \lambda)}{6} \frac{B_3}{2(1 + 2\alpha)} & \text{if } \sigma_1 \leq \mu \leq \sigma_2 \\
\frac{(2 - \lambda)(3 - \lambda)}{6} \gamma & \text{if } \mu \geq \sigma_2 
\end{cases}
\]
where
\[
\gamma := \frac{B_2}{\beta + 2\alpha} - \frac{3(2 - \lambda)}{2(3 - \lambda)} \frac{\mu}{(\beta + \alpha)^2} B_1^2 + \frac{1 - \beta}{2(\beta + \alpha)^2} B_1^2 \\
\sigma_1 := \frac{2(3 - \lambda)}{3(2 - \lambda)} \frac{2(\beta + \alpha)^2(B_2 - B_1) + (1 - \beta)(\beta + 2\alpha)B_1^2}{2(\beta + 2\alpha)B_1^2} \\
\sigma_2 := \frac{2(3 - \lambda)}{3(2 - \lambda)} \frac{2(\beta + \alpha)^2(B_2 + B_1) + (1 - \beta)(\beta + 2\alpha)B_1^2}{2(\beta + 2\alpha)B_1^2}
\]
The result is sharp.

**Remark 3.4.** When \( \alpha = 1, \beta = 0, B_1 = 8/\pi^2 \) and \( B_2 = 16/(3\pi^2) \), the above Theorem 3.2 reduces to a recent result of Srivastava and Mishra [8, Theorem 8, p. 64] for a class of functions for which \( \Omega_{\lambda} f(z) \) is a parabolic starlike function [2, 6].

**References**