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On Some p-subgroups of Automorphism Group of a Finite p-group

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Abstract. Let G be a group and let $\mathrm{Aut}_{Z(G)}^{G'}(G)$ denote the group of all automorphisms of G fixing both G/G' and $Z(G)$ elementwise. In this paper, using the notion of Frattinian groups, we give some necessary and sufficient conditions on a finite non-abelian p-group G for the groups $\mathrm{Aut}_{Z(G)}^{G'}(G)$ and $\mathrm{Inn}(G)$ coincide.

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1. Introduction

Let G be a group and let N be a normal subgroup of G. Let σ be an automorphism of G. If $N^\sigma = N$ (or $Ng^\sigma = Ng$ for all g in G), we shall say σ normalizes N (centralizes G/N respectively). Now let M and N be normal subgroups of a group G. We let $\mathrm{Aut}^N(G)$ denote the group of all automorphisms of G normalizing N and centralizing G/N, and $\mathrm{Aut}_M(G)$ the group of all automorphisms of G centralizing M. Moreover, $\mathrm{Aut}_M^N(G)$= $\mathrm{Aut}^N(G)\bigcap\mathrm{Aut}_M(G)$. Various authors have studied the groups $\mathrm{Aut}^N(G)$ and $\mathrm{Aut}_M^N(G)$ for some particular characteristic subgroups M and N of a finite p-group G. It is well known that if G is a finite p-group, then so is the group $\mathrm{Aut}^\Phi(G)$, where Φ denotes the Frattini subgroup of G, the intersection of all the maximal subgroups of G. Liebeck in [6] gave an upper bound for the nilpotency class of $\mathrm{Aut}^\Phi(G)$. In [1], Adney and Yen proved that if G is a finite p-group having no nontrivial abelian direct factor, then there is a one-to-one correspondence between $\mathrm{Aut}^Z(G)$ and the group $\mathrm{Hom}(G/G', Z)$ of all homomorphisms of G into $Z = Z(G)$, where G' denotes the derived subgroup of G. For some special values of M and N the group of all inner automorphisms $\mathrm{Inn}(G)$ of G is contained in $\mathrm{Aut}_M^N(G)$.

Several papers have been devoted to study the group $\mathrm{Aut}_M^N(G)/\mathrm{Inn}(G)$ when G is a finite nonabelian p-group. Müller in [7] proved, using techniques from cohomology, that if G is a finite nonabelian p-group, then $\mathrm{Aut}_Z^\Phi(G) = \mathrm{Inn}(G)$ if and only if $\Phi \le Z$ and Φ is cyclic. This turns out that $\mathrm{Aut}^\Phi(G)/\mathrm{Inn}(G)$ is nontrivial if and only if G is neither elementary abelian nor extraspecial. Cheng [3] proved, among others, the following result. Let G be a finite p-group such that $G' = \langle a \rangle$ is cyclic. Assume that either $p > 2$, or $p = 2$ and $[a, G] \le \langle a^4 \rangle$. Then $\mathrm{Aut}_Z^{G'}(G) = \mathrm{Inn}(G)$. Curran and McCaughan in [5] proved that if G is a finite p-group, then $\mathrm{Aut}^Z(G) = \mathrm{Inn}(G)$ if and only if $G' = Z(G)$ and $Z(G)$ is cyclic. Finally Curran [4] showed that for any nonabelian group G, $\mathrm{Aut}_Z^Z(G) \cong \mathrm{Hom}(G/G'Z, Z)$ obtaining some results concerning the group $\mathrm{Aut}_Z^Z(G)$, where G is a finite nonabelian p-group. In particular, he showed that $\mathrm{Aut}^Z(G) = Z(\mathrm{Inn}(G))$ if and only if $\mathrm{Hom}(G/G', Z) \cong Z(G/Z)$.

In this paper we study closely the groups $\mathrm{Aut}_Z^{G'}(G)$ and $\mathrm{Aut}^{G'}(G)$ for a finite nonabelian p-group G. We also give an alternative short proof for the main result of Müller mentioned earlier using an elegant theorem of Schmid [8].

In Sec. 2 we give some preliminary results that are needed for the main results of the paper. In Sec. 3 we prove the main results of the paper. Finally in Sec. 4 we give a new short proof for the Müller's result which was mentioned earlier. This proof, based on an elegant result of Schmid [8], simplifies greatly the Müller's proof. We use standard notation in group theory: we use the notation $\mathrm{Hom}(G, A)$ to denote the group of homomorphisms of G into an abelian group A, $\Omega_i(G)$ the subgroup of G generated by its elements of order dividing p^i. Recall that a group G is called a central product of its subgroups A and B if A and B commute elementwise and together generate G. In this situation, we write $G = A * B$.

2. Some Basic Results

In this section we give some known results which will be used in the rest of the paper.

Let G be a finite p-group. Following Schmid, we call G Frattinian provided $Z(G) \neq Z(M)$ for all maximal subgroups M of G. In [8], Schmid proved the following structural theorem for the Frattinian groups.

Theorem 2.1 [8]. *Suppose G is a nonabelian Frattinian p-group. Then one of the following holds:*

(i) *G is the central product of nonabelian p-groups of order $p^2|Z(G)|$, amalgamating their centres.*

(ii) *$G = E * F$ is the central product of Frattinian subgroups E and F with $C_F(Z(\Phi(F))) = \Phi(F)$, $E = C_G(F)$ and $\Phi(E) \le Z(G)$.*

It is worth noting that in case (i) of the above theorem the factors of the central product are minimal nonabelian p-groups. Accordingly, in this case we have

$Z(G) = \Phi(G)$. The following simple lemmas will be used in the rest of the paper.

Lemma 2.2. *Let G be a group and let M, N be normal subgroups of G with $N \leq M$ and $C_N(M) \leq Z(G)$. Then $Aut_M^N(G) \cong Hom(G/M, C_N(M))$.*

Proof. It is easy to verify that the map $f_\sigma : Mx \mapsto x^{-1}x^\sigma$ defines a homomorphism from G/M into $C_N(M)$ for every σ in $\mathrm{Aut}_M^N(G)$. On the other hand, the map $\sigma_f : x \mapsto xf(x)$ defines an automorphism of G for every f in $\mathrm{Hom}(G/M, C_N(M))$. This automorphism lies in $\mathrm{Aut}_M^N(G)$ and the map $\sigma \mapsto f_\sigma$ is an isomorphism from $\mathrm{Aut}_M^N(G)$ to $\mathrm{Hom}(G/M, C_N(M))$. ∎

Lemma 2.3. *Let $G = E * F$ be a central product of subgroups E and F. Assume that $\psi(G)$ is $\Phi(G)$, G' or $Z(G)$. If $\alpha \in Aut_{Z(E)}^{\psi(E)}(E)$ then the map $\hat{\alpha} : xy \mapsto x^\alpha y$, where $x \in E$ and $y \in F$, defines an automorphism of G lying in $Aut_{Z(G)}^{\psi(G)}(G)$.*

Proof. Straightforward. ∎

Throughout the paper we write Z and Φ for $Z(G)$ and $\Phi(G)$, respectively.

3. The Groups $\mathbf{Aut}_Z^{G'}(G)$ and $\mathbf{Aut}^{G'}(G)$

In this section we study the groups $\mathrm{Aut}_Z^{G'}(G)$ and $\mathrm{Aut}^{G'}(G)$ for a finite nonabelian p-group G.

We begin by an elementary lemma which is a consequence of Lemma 2.2.

Lemma 3.1. *If G is a group of class 2 , then*
(i) $\mathrm{Aut}^{G'}(G) \cong \mathrm{Hom}(G/G', G')$.
(ii) $\mathrm{Aut}_Z^{G'}(G) \cong \mathrm{Hom}(G/Z(G), G')$.

Proposition 3.2. *Let G be a finite p-group of class 2. Then $\mathrm{Aut}_Z^{G'}(G) = \mathrm{Inn}(G)$ if and only if G' is cyclic.*

Proof. Assume that G' is cyclic. Since $\exp(G/Z) = \exp(G')$, $\mathrm{Aut}_Z^{G'}(G) \cong \mathrm{Hom}(G/Z, G') \cong G/Z$, as required. The converse of the result is evident from the fact that $\mathrm{Hom}(G/Z, G') \cong G/Z$. ∎

Theorem 3.3. *Let G be a finite nonabelian p-group of class 2. Then $\mathrm{Aut}^{G'}(G) = \mathrm{Inn}(G)$ if and only if G' is cyclic and $Z(G) = G'G^{p^n}$ where $|G'| = p^n$.*

Proof. Assume that G' is cyclic and $Z(G) = G'G^{p^n}$, where $|G'| = p^n$. By Proposition 3.2, $\mathrm{Aut}_Z^{G'}(G) = \mathrm{Inn}(G)$. Let $\alpha \in \mathrm{Aut}^{G'}(G)$ and $a \in G$. We may write $\alpha(a) = ad$ with $d \in G'$. Now we observe that $\alpha(a^{p^n}) = \alpha(a)^{p^n} = a^{p^n}d^{p^n} = a^{p^n}$, which shows that α fixes any element of $Z(G)$. Consequently $\mathrm{Aut}^{G'}(G) \leq \mathrm{Aut}_Z^{G'}(G)$, and the proof is complete.

Conversely suppose that $\mathrm{Aut}^{G'}(G) = \mathrm{Inn}(G)$. We deduce that G' is cyclic, because $\mathrm{Aut}_Z^{G'}(G) = \mathrm{Inn}(G)$. Since G is of class 2, $G' \le G'G^{p^n} \le Z(G)$. It follows that

$$\mathrm{Inn}(G) \cong \mathrm{Hom}(G/Z(G), G') \rightarrowtail \mathrm{Hom}(G/G'G^{p^n}, G') \rightarrowtail \mathrm{Hom}(G/G', G')$$
$$\cong \mathrm{Aut}^{G'}(G) = \mathrm{Inn}(G).$$

So that $\mathrm{Hom}(G/G'G^{p^n}, G') \cong \mathrm{Hom}(G/Z(G), G')$.
However $\exp(G') = \exp(G/Z(G)) = |G'|$, which gives $|G/Z(G)| = |G/G'G^{p^n}|$, as required. ■

Remark. In [2], Berkovich shows that if G is a finite p-group with $\mathrm{rank}(G/G')=r$ and $|G'|^r \le |G/Z|$, then $\mathrm{Aut}^{G'}(G) = \mathrm{Inn}(G)$.

As an application of Theorem 3.3, we get another proof of the main result of [5].

Corollary 3.4. [5]. *If G is a finite p-group then $\mathrm{Aut}^Z(G) = \mathrm{Inn}(G)$ if and only if $G' = Z(G)$ and $Z(G)$ is cyclic.*

Proof. If $G' = Z(G)$ and $Z(G)$ is cyclic then G' is cyclic and obviously $Z(G) = G'G^{p^n}$, and hence $\mathrm{Aut}^{G'}(G) = \mathrm{Aut}^Z(G) = \mathrm{Inn}(G)$, by Theorem 3.3. Conversely, suppose that $\mathrm{Aut}^Z(G) = \mathrm{Inn}(G)$. So G is of class 2 and we have $\mathrm{Aut}^{G'}(G) \le \mathrm{Aut}^Z(G)$. It follows that $\mathrm{Aut}^{G'}(G) = \mathrm{Inn}(G)$. Therefore G' is cyclic and $Z(G) = G'G^{p^n}$, from which we conclude that G has no nontrivial abelian direct factor. So, by [1], we have

$$|\mathrm{Hom}(G/G', Z(G))| = |\mathrm{Aut}^Z(G)| = |\mathrm{Aut}^{G'}(G)| = |\mathrm{Hom}(G/G', G')|.$$

Using [5, Lemma I],

$$|\mathrm{Aut}^Z(G)| = |\mathrm{Hom}(G/G', Z(G))| \ge |\mathrm{Hom}(G/Z(G), G')||Z(G) : G'|$$
$$= |\mathrm{Aut}_Z^{G'}(G)||Z(G) : G'|.$$

Thus $Z(G) = G'$ as required. ■

Corollary 3.5. *If G is a finite nonabelian p-group, then $\mathrm{Aut}_Z^Z(G) = \mathrm{Inn}(G)$ if and only if G is of class 2 and $Z(G)$ is cyclic.*

Proof. Let $\mathrm{Aut}_Z^Z(G) = \mathrm{Inn}(G)$. Obviously G is of class 2. By Lemma 2.2, $\mathrm{Aut}_Z^Z(G) \cong \mathrm{Hom}(G/Z, Z)$. Now since $\exp(G/Z) = \exp(G') \le \exp(Z)$, we conclude that Z is cyclic. The converse of the result is immediate. ■

Theorem 3.6. *Let G be a finite nonabelian p-group such that $Z(\Phi(G)) \le Z(G)$. Then $\mathrm{Aut}_Z^{G'}(G) = \mathrm{Inn}(G)$ if and only if G is of class 2 and G' is cyclic.*

Proof. Assume that $\mathrm{Aut}_Z^{G'}(G) = \mathrm{Inn}(G)$. We distinguish two cases:

Case I. $Z(G) \not\leq \Phi(G)$.
We may write $G = MZ(G)$ for some maximal subgroup M of G. It is evident that $Z(\Phi(M)) \leq Z(\Phi(G))$, whence $Z(\Phi(M)) \leq Z(G) \bigcap M = Z(M)$. Let $\alpha \in \mathrm{Aut}_{Z(M)}^{M'}(M)$. Then the map $\bar{\alpha} : xz \mapsto x^{\alpha} z$, where $x \in M$ and $z \in Z(G)$, defines an automorphism of G which lies in $\mathrm{Aut}_Z^{G'}(G) = \mathrm{Inn}(G)$. Since $G = MZ(G)$, it implies that $\alpha \in \mathrm{Inn}(M)$. Therefore $\mathrm{Aut}_{Z(M)}^{M'}(M) = \mathrm{Inn}(M)$. Using induction, we conclude that M' is cyclic and M is of class 2. It follows that G' is cyclic and G is of class 2.

Case II. $Z(G) \leq \Phi(G)$.
In this case we show that G is Frattinian. Let M be an arbitrary maximal subgroup of G, and $z \in G \backslash M$. We write $G = M\langle z\rangle$ and choose an element u in $\Omega_1(G' \bigcap Z(G))$. Clearly the map $\alpha : hz^i \mapsto h(zu)^i$, where $h \in M$ and $0 \leq i < p$, defines an automorphism of G which is in $\mathrm{Aut}_Z^{G'}(G) = \mathrm{Inn}(G)$. Assume that α is the inner automorphism of G induced by x. It turns out that $x \in C_G(M) = Z(M)$ which shows that $Z(G) \neq Z(M)$. So G is Frattinian and one of the statements (i),(ii) of Theorem 2.1 holds. If (i) is fulfilled, then $\Phi(G) = Z(G)$ and G is of class 2. So the result follows at once from Proposition 3.2. However, the second statement of Theorem 2.1 cannot occur, because in this case, by $\Phi(G) = \Phi(E)\Phi(F) \leq Z(G)\Phi(F)$, we have $Z(\Phi(F)) \leq Z(\Phi(G)) \leq Z(G)$, which gives the contradiction $F = C_F(Z(\Phi(F))) = \Phi(F)$. The converse follows at once from Proposition 3.2. ∎

Theorem 3.7. *Let G be a finite nonabelian p-group such that $Z(\Phi(G)) \leq Z(G)$. Then $\mathrm{Aut}^{G'}(G) = \mathrm{Inn}(G)$ if and only if $Z(G) = \Phi(G)$ and G' is of order p.*

Proof. We claim that $Z(G) \leq \Phi(G)$. Assume that this is false, then $G = M\langle z\rangle$ for some maximal subgroup M of G and for some z in $Z(G)\backslash M$. We choose an element u in $\Omega_1(G' \bigcap Z(G))$. The map $\alpha : hz^i \mapsto h(zu)^i$, where $h \in M$ and $0 \leq i < p$, is in $\mathrm{Aut}^{G'}(G) = \mathrm{Inn}(G)$ from which we conclude that $u = 1$, a contradiction. So $Z(G) \leq \Phi(G)$. By a similar argument given for the proof of Theorem 3.6, G is Frattinian. Thus one of the statements of Theorem 2.1 holds. However, the second statement of Theorem 2.1 cannot occur by a similar argument given for the proof of Theorem 3.6. If the first statement occurs, then $\Phi(G) = Z(G)$. Hence by Proposition 3.2, G is of class 2 and G' is cyclic. Now since $\exp(G') = \exp(G/Z(G)) = \exp(G/\Phi(G))$, we conclude that $|G'| = p$.

The converse is immediate. ∎

4. The Groups $\mathrm{Aut}_Z^{\Phi}(G)$ and $\mathrm{Aut}^{\Phi}(G)$

In this section we give an alternative proof for the Müller's result on the groups $\mathrm{Aut}_Z^{\Phi}(G)/\mathrm{Inn}(G)$ and $\mathrm{Aut}^{\Phi}(G)/\mathrm{Inn}(G)$ using Theorem 2.1 and the following Proposition due to Schmid [8] which is readily proved by cohomological methods.

Proposition 4.1 [8, Proposition 3]. *Let G be a finite Frattinian p-group. If $\mathrm{Aut}_Z^{\Phi}(G) = \mathrm{Inn}(G)$ then $C_G(Z(\Phi(G))) \neq \Phi(G)$.*

Theorem 4.2 [7, Proposition 3.1]. *Let G be a finite nonabelian p-group. Then* $\mathrm{Aut}_Z^{\Phi}(G) = \mathrm{Inn}(G)$ *if and only if* $\Phi(G) \leq Z(G)$ *and* $\Phi(G)$ *is cyclic.*

Proof. Assume first that $\Phi(G) \leq Z(G)$ and $\Phi(G)$ is cyclic. By Lemma 2.2, $\mathrm{Aut}_Z^{\Phi}(G) \cong \mathrm{Hom}(G/Z, \Phi)$. Now since G is of class 2, $\exp(G/Z) = \exp(G') \leq \exp(\Phi(G))$, whence $\mathrm{Aut}_Z^{\Phi}(G) \cong G/Z$.

Conversely let $\mathrm{Aut}_Z^{\Phi}(G) = \mathrm{Inn}(G)$. Assume either $\Phi(G) \nleq Z(G)$ or $\Phi(G)$ is noncyclic. We consider two cases:

Case I. $Z(G) \nleq \Phi(G)$.
We choose a maximal subgroup M of G such that $Z(G) \nleq M$. So $G = MZ(G)$ and $Z(M) = Z(G) \bigcap M$. Now if $\Phi(M) \leq Z(M)$ then $\Phi(G) \leq Z(G)$ and hence by Lemma 2.2, $\mathrm{Aut}_Z^{\Phi}(G) \cong \mathrm{Hom}(G/Z, \Phi)$. Since $\Phi(G)$ is noncyclic, it follows that $|\mathrm{Aut}_Z^{\Phi}(G)| > |G/Z|$ which is impossible. So we suppose that $\Phi(M) \nleq Z(M)$. In this situation we may use induction to deduce that $\mathrm{Aut}_{Z(M)}^{\Phi(M)}(M) \neq \mathrm{Inn}(M)$. Let $\beta \in \mathrm{Aut}_{Z(M)}^{\Phi(M)}(M) \backslash \mathrm{Inn}(M)$. We write $G = M\langle z \rangle$ where $z \in Z(G)\backslash M$, and extend β to an automorphism $\hat{\beta} \in \mathrm{Aut}_Z^{\Phi}(G)$ by setting $(hz^i)^{\hat{\beta}} = h^{\beta} z^i$, where $h \in M$ and $0 \leq i < p$. We therefore have $\hat{\beta} \in \mathrm{Inn}(G)$. It follows that $\beta \in \mathrm{Inn}(M)$, a contradiction.

Case II. $Z(G) \leq \Phi(G)$.
In this case we claim that G is Frattinian. To see this, let M be an arbitrary maximal subgroup of G. Choose an element z in $G\backslash M$ and let $u \in \Omega_1(Z(G))$. The map $\alpha : hz^i \mapsto h(zu)^i$, where $h \in M$ and $0 \leq i < p$, defines an automorphism of G which is in $\mathrm{Aut}_Z^{\Phi}(G)$. So α is an inner automorphism of G induced by an element t in G. It follows that $t \in C_G(M) = Z(M)$. Now since $t \notin Z(G)$, we see that G is Frattinian. By Theorem 2.1, one of the statements (i),(ii) of the theorem holds. If the statement (i) holds then $Z(G) = \Phi(G)$ and hence $\mathrm{Inn}(G) = \mathrm{Aut}_Z^{\Phi}(G) = \mathrm{Aut}_Z^{Z}(G)$. Consequently $Z(G)$ is cyclic by Corollary 3.5, a contradiction. We therefore suppose that the second statement of Theorem 2.1 is fulfilled. If E is abelian then $E \leq Z(G) \leq \Phi(G)$ and we have $G = F$ whence $C_G(Z(\Phi(G))) = \Phi(G)$, a contradiction to Proposition 4.1. So we may suppose that E is nonabelian. Now let $\alpha \in \mathrm{Aut}_{Z(E)}^{Z(E)}(E)$ and extend α to an automorphism $\hat{\alpha} \in \mathrm{Aut}_Z^{Z}(G)$ according to Lemma 2.3. It follows that $\hat{\alpha}$ is an inner automorphism of G induced by some element in E. Therefore, $\alpha \in \mathrm{Inn}(E)$, and hence $\mathrm{Aut}_{Z(E)}^{Z(E)}(E) = \mathrm{Inn}(E)$. By Corollary 3.5, $Z(E)$ is cyclic. Since $E = C_G(F)$, we deduce that $Z(G) = Z(E)$ is cyclic. Now if $\Phi(F) \leq Z(G)$, then $\Phi(G) = \Phi(E)\Phi(F) \leq Z(G)$ and hence $Z(G) = \Phi(G)$, a contradiction. Thus $\Phi(F) \nleq Z(G)$ from which we deduce that $\Phi(F) \nleq Z(F)$. Again by induction hypothesis $\mathrm{Aut}_{Z(F)}^{\Phi(F)}(F) \neq \mathrm{Inn}(F)$, which is impossible, by a similar argument given in *Case I.* ∎

Corollary 4.3 [7]. *If G is a finite nonabelian p-group then* $\mathrm{Aut}^{\Phi}(G) = \mathrm{Inn}(G)$ *if and only if G is extraspecial.*

Proof. If $\mathrm{Aut}^{\Phi}(G) = \mathrm{Inn}(G)$, then $\Phi(G) \leq Z(G)$ and $\Phi(G)$ is cyclic by Theorem 4.2. So G is of class 2 and hence $Z(G) \leq \Phi(G)$ by Theorem 3.3. It follows that

$Z(G) = \Phi(G)$. Now according to Corollary 3.4, $G' = Z(G)$ and $Z(G)$ is cyclic. Finally $\exp(G') = \exp(G/\Phi) = p$ which completes the proof of the first part.

The converse is straightforward. ∎

References

1. J. E. Adney and T. Yen, Automorphisms of a p-group, *Ill. J. Math.* **9** (1965) 137-143.
2. Y. Berkovich, On abelian subgroups of p-groups, *J. Algebra,* **199** (1998) 262-280.
3. Y. Cheng, On finite p-groups with cyclic commutator subgroup, *Arch. Math.* **39** (1982) 295-298.
4. M. J. Curran, Finite groups with central automorphism group of minimal order, *Math. Proc. Royal Irish Acad.* **104** (2004) 223-229.
5. M. J. Curran and D. J. McCaughan, Central automorphisms that are almost inner, *Comm. Algebra* **29** (2001) 2081-2087.
6. H. Liebeck, The automorphism group of finite p-groups, *J. Algebra* **4** (1966) 426-432.
7. O. Müller, On p-automorphisms of finite p-groups, *Arch. Math.* **32** (1979) 533-538.
8. P. Schmid, Frattinian p-groups, *Geom. Dedicata* **36** (1990) 359-364.