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Hausdorff First Countable ω-bounded Space is Strongly ω-bounded

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Abstract. In this paper we obtain an answer to Problem 7, Ch 8., §1 (p. 131) in the book Open Problems in Topology by Jan van Mill and George M. Reed. Namely, we show that if a Hausdorff first countable topological space is ω-bounded, then it is strongly ω-bounded.

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1. Introduction

In the Preliminairies, paragraph 2., we state the problem in the Abstract and the respective definitions. In Paragraph 3., the Result, we prove that if $(X, \mathcal{T})$ is a Hausdorff first countable topological space that is ω-bounded, then also $(X, \mathcal{T})$ is strongly ω-bounded.

2. Preliminaries

Recall that a topological space $(X, \mathcal{T})$ is said to be first countable if each point has a countable base of neighborhoods. $(X, \mathcal{T})$ is separated or a Hausdorff space if each two different points have disjoint neighborhoods.

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Definition 1. *Following* [1], *we say that a subset W of a topological space $(X, \mathcal{T})$ is σ-compact if W is a countable union of compact subsets of X.*

Definition 2. *Following* [2], *a topological space $(X, \mathcal{T})$ is said to be ω-bounded if the closure of each countable subset of X is compact. We say that $(X, \mathcal{T})$ is strongly ω-bounded if each σ-compact subset of X has compact closure.*

Problem 7 in [2] (Ch. 8, §1, p. 131) is the question whether a (separated) first countable space that is ω-bounded is necessarily strongly ω-bounded or not.

Recall ([1]) that a net $uo\alpha : M \to X$ in X, $\alpha : M \to I$, where $(M, \prec), (I, \leq)$ are directed sets, is a subnet of the net $u = (x_i)$ if and only if the map α has the property that, for each given $i \in I$, there is some $m(i) \in M$ such that the implication

$$\forall m \in M, m \succ m(i) \Rightarrow \alpha(m) \geq i$$

holds.

Lemma 1. *Let $(X, \mathcal{T})$ be a first countable topological space. If (x_n) is a sequence in X such that (x_n) has no convergent subsequence and $S = \{x_n : n \in \mathbf{N}\}$ is the set of all terms, then the closure $\overline{S}$ is not compact.*

Proof. We have to prove that there is a net (x_i) in $\overline{S}$ such that (x_i) has no convergent subnet. Take (x_i) to be the sequence (x_n) and let $p \in X$. We show that, $(x_{\alpha(m)})$ being a subnet of (x_n), the hypothesis $x_{\alpha(m)} \to p$ leads to a contradiction. We may consider a countable base of neighborhoods $\{V_k : k = 1, 2, ...\}$ of p such that $V_k \supset V_{k+1}$ for each k. Assuming that $x_{\alpha(m)} \to p$, then for each given $k = 1, 2, ...$, there is some $m(k)$ in M, where $(M, \prec)$ is the directed set for $(x_{\alpha(m)})$, such that the implication $\forall m \in M, m \succ m(k) \Rightarrow x_{\alpha(m)} \in V_k$ is true; $(x_{\alpha(m)})$ being a subnet, we may consider, following the natural number $\alpha(m(1))$, some $m(2) \succ m(1)$ such that $m \succ m(2) \Rightarrow \alpha(m) \gneqq \alpha(m(1))$ and we have obtained $x_{\alpha(m(1))} \in V_1$, $x_{\alpha(m(2))} \in V_2$, $\alpha(m(2)) \gneqq \alpha(m(1))$. Using the countable Axiom of Choice concerning the class constituted by the nonempty sets $A_1 = \{m(1) \in M : x_{\alpha(m(1))} \in V_1\}$, $A_2 = \{m(2) \in M : m(2) \succ m(1), \alpha(m(2)) \gneqq \alpha(m(1))\}$,..., $A_{k+1} = \{m(k+1) \in M : m(k+1) \succ m(k) \succ ... \succ m(1), \alpha(m(k+1)) \gneqq \alpha(m(k)) \gneqq ... \gneqq \alpha(m(1)\}$ we see by induction that a subsequence $(x_{\alpha(m(k))})$ of (x_n) exists such that $x_{\alpha(m(k))} \to p$. We get a contradiction and the lemma follows. ∎

3. The Rusult

Theorem 1. *If $(X, \mathcal{T})$ is a Hausdorff first countable ω-bounded topological space, then $(X, \mathcal{T})$ is strongly ω-bounded.*

Proof. We have to prove that, the existence of a countable class $\{C_n : n \in \mathbf{N}\}$ of compact subsets of X such that $C = \bigcup_{n=1}^{\infty} C_n$ and $\overline{C}$ is not compact, where we may suppose that $C_n \subsetneq C_{n+1}$, implies that there is a countable set $\{x_n : n \in$

$\mathbf{N}\} \subset X$ such that the closure $\overline{\{x_n : n \in \mathbf{N}\}}$ is not compact. Let $\{O_\gamma : \gamma \in \Gamma\}$ be an open cover of $\overline{C}$ having no finite subcover. By hypothesis, it follows that the open cover $\{O_\gamma : \gamma \in \Gamma, O\}$ of X, where $O = X \backslash \overline{C}$, is such that neither any finite intersection $O^c \cap (\bigcap\{F_\gamma : \gamma \in J\}) = \phi$ nor $\bigcap\{F_\gamma : \gamma \in J\} = \phi$, where we denote $O^c = X \backslash O = \overline{C}$, $F_\gamma = X \backslash O_\gamma = O^c_\gamma$, since X is compact otherwise. We have that $C_n \subset \bigcup\{O_\gamma : \gamma \in I_n\}$ where $I_n \subset \Gamma$, I_n is finite and we may suppose that $I_n \subsetneqq I_{n+1}$ for each n. Hence

(1) $\overline{C} \subset \bigcup\{O_\gamma : \gamma \in I_n, n \in \mathbf{N}\}$

(2) $\bigcap\{F_\gamma : \gamma \in I_n\} \neq \phi$ for each n.

(3) $\overline{C} \cap (\bigcap\{F_\gamma : \gamma \in I_n, n \in \mathbf{N}\} = \phi$.

Also for each n, there is a smallest $k(n) \in \mathbf{N}$, $k(n) \gneqq n$ such that $C_{k(n)} \nsubseteq \bigcup\{O_\gamma : \gamma \in I_n\}$ because no finite union of the open sets O_γ contains C and $C_n \subsetneqq C_{n+1}$ for each n. Hence we may consider $c_{k[1]} = c_{k(1)} \in C_{k(1)} \backslash \bigcup\{O_\gamma : \gamma \in I_1\}$, next $c_{k[2]} = c_{k(k[1])} \in C_{k[2]} \backslash \bigcup\{O_\gamma : \gamma \in I_{k[1]}\}$, $k[2] \gnsim k[1]$ and so on, thus obtaining a sequence $(c_{k[n]})$ such that each $c_{k[n+1]} \in C_{k[n+1]} \backslash \bigcup\{O_\gamma : \gamma \in I_{k[n]}\}$. We claim that $(c_{k[n]})$ has no convergent subsequence. In fact, we have that $c_{k[n+1]} \in \bigcap\{F_\gamma : \gamma \in I_{k[n]}\}$ for each n, where $k[n] \to \infty$, $c_{k[n]} \in C$. Supposing that some subsequence $c_{k[n(j)]} \to p$, then V being any neighborhood of the point p, we have that V contains a set $\{c_{k[n(j)]} : j \geq j(V)\} \neq \phi$ with a suitable $j(V) \in \mathbf{N}$. Since each F_γ is closed, it follows that $p \in \overline{C} \cap (\bigcap\{F_\gamma : \gamma \in I_{k[n(j)]}, j \geq j(V)\}) = \overline{C} \cap (\bigcap\{F_\gamma : \gamma \in I_n, n \in \mathbf{N}\})$ which contradicts (3). According to Lemma 1, we found a countable subset $S = \{c_{k[n(j)]} : j = 1, 2, ...\}$ of X such that the closure $\overline{S}$ is not compact, thus the theorem is proved. ■

References

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