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Qualitative Behavior of Some Max-type Difference Equations

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Abstract. In this paper we investigate the boundedness and the periodicity character of solutions of the difference equation

$$x_{n+1} = \max\left\{\frac{1}{x_n}, \frac{A_n^1}{x_{n-1}}, \frac{A_n^2}{x_{n-2}}, ..., \frac{A_n^k}{x_{n-k}}\right\}, \qquad n = 0, 1, ...$$

where $\{A_n^i\}_{n=0}^{\infty}$ are sequences of positive numbers and $A_n^i \in (0, 1]$ for all $n = 0, 1, ...$ and $i = 1, 2, ..., k$.
The solutions of the particular form

$$x_{n+1} = \max\left\{\frac{1}{x_{n-p}}, \frac{1}{x_{n-q}}\right\}, \ n = 0, 1, ...$$

where p and q are nonnegative integer numbers will also be discussed.

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1. Introduction

In [5] Elabbasy *et al.* investigated the boundedness and the periodic nature of solutions of the max type difference equation

$$x_{n+1} = \max\left\{\frac{1}{x_n}, \frac{A_n}{x_{n-1}}\right\},$$

where $\{A_n\}_{n=0}^{\infty}$ is a periodic sequence of period two and $A_n > 1$.

Also, Elabbasy *et al.* [6] studied the semicycles, the boundedness and the periodicity of solutions of the Max-equation

$$x_{n+1} = \max\left\{\frac{1}{x_n}, \frac{A_n}{x_{n-1}}\right\},$$

where $\{A_n\}_{n=0}^{\infty}$ is a periodic sequence of period three and $A_n \in (0,1]$ for all $n = 0, 1, ...$ such that the elements of one of the three subsequences $\{A_{3i}\}_{i=0}^{\infty}$, $\{A_{3i+1}\}_{i=0}^{\infty}$ or $\{A_{3i+2}\}_{i=0}^{\infty}$ equal one.

In [8] Feuer *et al.* investigated the asymptotic behavior, the oscillatory character and periodic nature of solutions of the equation

$$x_{n+1} = \frac{\max\{x_n, A\}}{x_n x_{n-1}},$$

where A is a real constant and $x_{-1},\ x_0$ are nonzero constants.

Also, see [1–10] for some difference equations with the property that every solution is eventually periodic.

The aim of this paper is to study the boundedness and the periodicity character of solutions of the general max-type difference equation

$$x_{n+1} = \max\left\{\frac{1}{x_n}, \frac{A_n^1}{x_{n-1}}, \frac{A_n^2}{x_{n-2}}, ..., \frac{A_n^k}{x_{n-k}}\right\}, \quad n = 0,1,... \tag{1}$$

where $\{A_n^i\}_{n=0}^{\infty}$ are sequences of positive numbers and $A_n^i \in (0,1]$ for all $n = 0, 1, ...$ and $i = 1, 2, ..., k$.

Also the periodicity of solutions of the missing term difference equation

$$x_{n+1} = \max\left\{\frac{1}{x_{n-p}}, \frac{1}{x_{n-q}}\right\}, \quad n = 0,1,... \tag{2}$$

where p and q are nonnegative integer numbers will be investigated.

Max-type equations are important both for purely theoretical reasons, as well as for applied reasons. The first purely theoretical equation, for which it was possible to rigorously show that it has a strange attractor, was "Lozi's map [10]" which is a max-type equation. There are various applied models which use max-type equations.

2. Some Basic Properties and Definitions

In this section we mention some basic properties and definitions for Equation (1)

Equilibrium point

Clearly Equation (1) has a unique equilibrium point $\overline{x} = 1$.

Definitions

(1) Semicycles

(a) *A positive semicycle* of a solution $\{x_n\}_{n=-k}^{\infty}$ of Equation (1) consists of a "string" of terms $\{x_l, x_{l+1},..., x_m\}$, all of which are greater than or equal to the equilibrium $\overline{x}$ with $l \geq -1$ and $m \leq \infty$, such that

$$\text{either} \quad l = -1, \quad \text{or} \quad l > -1 \quad \text{and} \quad x_{l-1} < \overline{x}$$

and

$$\text{either} \quad m = \infty, \quad \text{or} \quad m < \infty \quad \text{and} \quad x_{m+1} < \overline{x}.$$

(b) *A negative semicycle* of a solution $\{x_n\}_{n=-k}^{\infty}$ of Equation (1) consists of a "string" of terms $\{x_l, x_{l+1},..., x_m\}$, all of which are less than the equilibrium $\overline{x}$, with $l \geq -1$ and $m \leq \infty$, such that

$$\text{either} \quad l = -1, \quad \text{or} \quad l > -1 \quad \text{and} \quad x_{l-1} \geq \overline{x}$$

and

$$\text{either} \quad m = \infty, \quad \text{or} \quad m < \infty \quad \text{and} \quad x_{m+1} \geq \overline{x}.$$

The sequence $\{x_n\}_{n=-k}^{\infty}$ is said to be *oscillatory* around an equilibrium point $\overline{x}$ if $\{x_n - \overline{x}\}_{n=-k}^{\infty}$ is oscillatory around zero.

(2) Permanence

The difference equation

$$x_{n+1} = G(x_n, x_{n-1}, ..., x_{n-k}), \quad n = 0, 1, ...$$

is said to be permanent if there exist numbers m and M with $0 < m \leq M < \infty$ such that for any initial conditions $x_{-k}, x_{-k+1}, ..., x_{-1}, x_0 \in (0, \infty)$, there exists a positive integer N which depends on the initial conditions such that

$$m \leq x_n \leq M \quad \text{for all} \quad n \geq N.$$

(3) Periodicity

A sequence $\{x_n\}_{n=-k}^{\infty}$ is said to be periodic with period p if $x_{n+p} = x_n$ for all $n \geq -k$. A sequence $\{x_n\}_{n=-k}^{\infty}$ is said to be periodic with prime period p if p is the smallest positive integer having this property.

3. Boundedness of Solutions

In this section we investigate the boundedness of positive solutions of Equation (1). Our result is the following theorem which is a minor modification of Lemma 2.1 in [7].

Theorem 3.1 *Every positive solution of Equation* (1) *is bounded.*

Proof. First we claim that $\{x_n\}_{n=-k}^{\infty}$ is bounded from above by a positive number $M > 0$ if and only if $\{x_n\}_{n=-k}^{\infty}$ is bounded from below by a positive number $m > 0$.

Indeed, suppose $\{x_n\}_{n=-k}^{\infty}$ is bounded from above by a positive number $M > 0$. We shall show that $\{x_n\}_{n=-k}^{\infty}$ is bounded from below by a positive number $m > 0$.

It follows from Equation (1) that

$$\begin{aligned} x_{n+1} &= \max\left\{\frac{1}{x_n}, \frac{A_n^1}{x_{n-1}} \frac{A_n^2}{x_{n-2}}, ..., \frac{A_n^k}{x_{n-k}}\right\} \\ &\geq \max\left\{\frac{1}{M}, \frac{A_n^1}{M}, \frac{A_n^2}{M}, ..., \frac{A_n^k}{M}\right\} = \frac{1}{M}. \end{aligned}$$

Then for every $n \geq 0$, we see that

$$x_{n+1} \geq \frac{1}{M}.$$

Conversely suppose that $\{x_n\}_{n=-k}^{\infty}$ is bounded from below by a positive number $m > 0$. We shall show that $\{x_n\}_{n=-k}^{\infty}$ is bounded from above by a positive number $M > 0$.

It follows from Equation (1) that

$$\begin{aligned} x_{n+1} &= \max\left\{\frac{1}{x_n}, \frac{A_n^1}{x_{n-1}}, \frac{A_n^2}{x_{n-2}}, ..., \frac{A_n^k}{x_{n-k}}\right\} \\ &\leq \max\left\{\frac{1}{m}, \frac{A_n^1}{m}, \frac{A_n^2}{m}, ..., \frac{A_n^k}{m}\right\} = \frac{1}{m}. \end{aligned}$$

Then for every $n \geq 0$, we see that

$$x_{n+1} \geq \frac{1}{m},$$

and so the proof of the claim is complete. ∎

We are now ready to prove the theorem. To get a contradiction, suppose that $\{x_n\}_{n=-k}^{\infty}$ is not bounded from above. Then there exists $N > 0$ such that

$$\max\{x_n : -k \leq n < N\} < x_N.$$

It follows that there exist integers N_1 and N_2 with $N \le N_2 < N_1$ such that

$$x_{N_1} < \min\{x_n : -k \le n < N_1\},$$

and

$$x_{N_2} = \max\{x_n : -k \le n < N_1\}.$$

Thus

$$x_{N_1} = \max\left\{\frac{1}{x_{N_1-1}}, \frac{A^1_{N_1-1}}{x_{N_1-2}}, ..., \frac{A^k_{N_1-1}}{x_{N_1-k-1}}\right\}$$
$$\ge \max\left\{\frac{1}{x_{N_2}}, \frac{A^1_{N_1-1}}{x_{N_2}} \frac{A^k_{N_1-1}}{x_{N_2}}\right\} = \frac{1}{x_{N_2}}.$$

Then

$$x_{N_1} x_{N_2} \ge 1.$$

We also have

$$x_{N_2} = \max\left\{\frac{1}{x_{N_2-1}}, \frac{A^1_{N_2-1}}{x_{N_2-2}}, ..., \frac{A^k_{N_2-1}}{x_{N_2-k-1}}\right\}$$
$$< \max\left\{\frac{1}{x_{N_1}}, \frac{A^1_{N_2-1}}{x_{N_1}}, ..., \frac{A^k_{N_2-1}}{x_{N_1}}\right\} = \frac{1}{x_{N_1}}.$$

From which it follows that

$$x_{N_1} x_{N_2} < 1.$$

This is a contradiction, and so the proof of the theorem is complete. ■

4. Periodicity of Solutions of Equation (1)

In this section we investigate the existence of periodic solutions of Equation (1). Two separate cases of Equation (1) will be discussed.

4.1. $A^i_n = 1$, $i = 1, 2, ..., k$, $n = 1, 2, ...$

We have the following results.

Lemma 1. *Let $\{x_n\}_{n=-k}^{\infty}$ be a positive solution of Equation (1) which is not eventually constant. Then the following statements are true.*

(1) *With the possible exception of the first negative semi-cycle, every negative semi-cycle of $\{x_n\}_{n=-k}^{\infty}$ has length equal to one.*

(2) *Every positive semi-cycle of $\{x_n\}_{n=-k}^{\infty}$ has length equal to $k+1$.*

Proof. (1) Suppose there exists $N \ge 0$ such that

$$x_{N-1} \ge 1 \quad \text{and} \quad x_N < 1.$$

Then from Equation (1) we see that

$$x_{N+1} = \max\left\{\frac{1}{x_N}, \frac{1}{x_{N-1}}, \frac{1}{x_{N-2}}, ..., \frac{1}{x_{N-k}}\right\} > 1.$$

(2) Suppose there exists $N \geq 0$ such that $x_N < 1$. Then from Equation (1) we see that

$$x_{N+1} = \max\left\{\frac{1}{x_N}, \frac{1}{x_{N-1}}, \frac{1}{x_{N-2}}, ..., \frac{1}{x_{N-k}}\right\} > 1,$$
$$x_{N+2} = \max\left\{\frac{1}{x_{N+1}}, \frac{1}{x_N}, \frac{1}{x_{N-1}}, ..., \frac{1}{x_{N-k+1}}\right\} > 1,$$
$$x_{N+3} = \max\left\{\frac{1}{x_{N+2}}, \frac{1}{x_{N+1}}, \frac{1}{x_N}, ..., \frac{1}{x_{N-k+2}}\right\} > 1,$$
$$\vdots$$
$$x_{N+k+1} = \max\left\{\frac{1}{x_{N+k}}, \frac{1}{x_{N+k-1}}, \frac{1}{x_{N+k-2}}, ..., \frac{1}{x_N}\right\} > 1,$$

and

$$x_{N+k+2} = \max\left\{\frac{1}{x_{N+k+1}}, \frac{1}{x_{N+k}}, \frac{1}{x_{N+k-1}}, ..., \frac{1}{x_{N+1}}\right\} < 1.$$

This completes the proof. ∎

Theorem 4.1. *Every positive solution of Equation* (1) *is eventually a periodic solution of period* $k+2$.

Proof. It follows from Lemma 1 that every positive semi-cycle of every solution $\{x_n\}_{n=-k}^{\infty}$ of Equation (1) eventually of length $k+1$ and every negative semi-cycle of length one. Thus $\{x_n\}_{n=-k}^{\infty}$ is a periodic solution of period $k+2$. The proof is complete. ∎

4.2. $A_n^i = 1,\ 1 \leq i \leq r$ and $A_n^j < 1,\ r+1 \leq j \leq k$.

In this section we consider Equation (1) where $A_n^i = 1,\ 1 \leq i \leq r$ and $A_n^j < 1,\ r+1 \leq j \leq k$. That is

$$x_{n+1} = \max\left\{\frac{1}{x_n}, \frac{1}{x_{n-1}}, ..., \frac{1}{x_{n-r}}, \frac{A_n^{r+1}}{x_{n-r-1}}, ..., \frac{A_n^k}{x_{n-k}}\right\}, \quad n = 0, 1, ... \tag{3}$$

Here we state and prove the following results.

Lemma 2. *Let* $\{x_n\}_{n=-k}^{\infty}$ *be a positive solution of Equation* (3) *which is not eventually constant. Then the following statements are true.*

(1) *With the possible exception of the first negative semi-cycle, every negative semi-cycle of* $\{x_n\}_{n=-k}^{\infty}$ *has length equal to one.*
(2) *Every positive semi-cycle of* $\{x_n\}_{n=-k}^{\infty}$ *has length at least equals* $r+1$ *and at most equal to* $k+1$.

Proof. (1) Suppose there exists $N \geq 0$ such that

$$x_{N-1} \geq 1 \quad \text{and} \quad x_N < 1.$$

Then it follows from Equation (3) that

$$x_{N+1} = \max\left\{\frac{1}{x_N}, \frac{1}{x_{N-1}}, ..., \frac{1}{x_{N-r}}, \frac{A_N^{r+1}}{x_{N-r-1}}, ..., \frac{A_N^k}{x_{N-k}}\right\} > 1.$$

(2) Suppose there exists $N \geq 0$ such that $x_N < 1$. Then from Equation (3), we see that

$$x_{N+1} = \max\left\{\frac{1}{x_N}, \frac{1}{x_{N-1}}, ..., \frac{1}{x_{N-r}}, \frac{A_N^{r+1}}{x_{N-r-1}}, ..., \frac{A_N^k}{x_{N-k}}\right\} > 1,$$

$$x_{N+2} = \max\left\{\frac{1}{x_{N+1}}, \frac{1}{x_N}, ..., \frac{1}{x_{N-r+1}}, \frac{A_{N+1}^{r+1}}{x_{N-r}}, ..., \frac{A_{N+1}^k}{x_{N-k+1}}\right\} > 1,$$

$$\vdots$$

$$x_{N+r+1} = \max\left\{\frac{1}{x_{N+r}}, \frac{1}{x_{N+r-1}}, ..., \frac{1}{x_N}, \frac{A_{N+r}^{r+1}}{x_{N-1}}, ..., \frac{A_{N+r}^k}{x_{N-k+r}}\right\} > 1,$$

and

$$x_{N+r+2} = \max\left\{\frac{1}{x_{N+r+1}}, \frac{1}{x_{N+r}}, ..., \frac{1}{x_{N+1}}, \frac{A_{N+r+1}^{r+1}}{x_N}, ..., \frac{A_{N+r+1}^k}{x_{N-k+r+1}}\right\}.$$

It is clear that

$$\frac{1}{x_{N+i}} < 1 \quad \text{for all} \quad 1 \leq i \leq r+1.$$

However, one of the following inequalities holds

$$\frac{A_{N+r+1}^{r+j}}{x_{N-i}} \geq 1 \quad \text{for some} \quad 1 \leq j \leq k, \ 0 \leq i \leq k-r-1.$$

Therefore

$$\text{either} \quad x_{N+r+2} > 1 \quad \text{or} \quad x_{N+r+2} < 1.$$

Thus every positive semi-cycle has length at least $r+1$.
Similarly, we see that

$$\text{either} \quad x_{N+r+i} \geq 1 \quad \text{or} \quad x_{N+r+i} < 1 \quad \text{for} \quad i = 3, 4, ..., k+1.$$

Now assume that

$$x_{N+i} \geq 1 \quad \text{for all} \quad i = r+2, r+3, ..., k+1.$$

It follows from Equation (3) that

$$x_{N+k+2} = \max\left\{\frac{1}{x_{N+k+1}}, \frac{1}{x_{N+k}}, ..., \frac{1}{x_{N+k-r+1}}, \frac{A^{r+1}_{N+k+1}}{x_{N+k-r}}, ..., \frac{A^{k}_{N+k+1}}{x_{N+1}}\right\} < 1.$$

Thus every positive semi-cycle has length at most $k+1$.
The proof is complete. ■

Theorem 4.2. *Equation* (3) *possesses a periodic solution of period* $r+2$.

Proof. Let $\{x_n\}_{n=-k}^{\infty}$ be a positive solution of Equation (3). Assume there exists an integer $N \geq 0$ such that

$$x_{N-k},\ x_{N-k+1}, ...,\ x_{N-1} \geq 1,$$

and

$$\max\{\sqrt{A^i_n}\} \leq x_N < 1 \quad \text{for all} \quad i = r+1, r+2, ..., k.$$

Then from Equation (3) we obtain

$$x_{N+1} = \max\left\{\frac{1}{x_N}, \frac{1}{x_{N-1}}, ..., \frac{1}{x_{N-r}}, \frac{A^{r+1}_{N}}{x_{N-r-1}}, ..., \frac{A^{k}_{N}}{x_{N-k}}\right\} = \frac{1}{x_N},$$

$$x_{N+2} = \max\left\{\frac{1}{x_{N+1}}, \frac{1}{x_N}, ..., \frac{1}{x_{N-r+1}}, \frac{A^{r+1}_{N+1}}{x_{N-r}}, ..., \frac{A^{k}_{N+1}}{x_{N-k+1}}\right\} = \frac{1}{x_N},$$

$$\vdots$$

$$x_{N+r+1} = \max\left\{\frac{1}{x_{N+r}}, \frac{1}{x_{N+r-1}}, ..., \frac{1}{x_N}, \frac{A^{r+1}_{N+r}}{x_{N-1}}, ..., \frac{A^{k}_{N+r}}{x_{N-k+r}}\right\} = \frac{1}{x_N},$$

$$x_{N+r+2} = \max\left\{\frac{1}{x_{N+r+1}}, \frac{1}{x_{N+r}}, ..., \frac{1}{x_{N+1}}, \frac{A^{r+1}_{N+r+1}}{x_N}, ..., \frac{A^{k}_{N+r+1}}{x_{N-k+r+1}}\right\} = x_N,$$

and

$$x_{N+r+3} = \max\left\{\frac{1}{x_{N+r+2}}, \frac{1}{x_{N+r+1}}, ..., \frac{1}{x_{N+2}}, \frac{A_{N+r+2}^{r+1}}{x_{N+1}}, ..., \frac{A_{N+r+2}^{k}}{x_{N-k+r+2}}\right\} = \frac{1}{x_N}.$$

Thus

$$x_{N+r+3} = x_{N+1}.$$

Then it follows by induction that $\{x_n\}_{n=-k}^{\infty}$ is periodic solution of period $r+2$. This completes the proof. ■

Conjecture 1. *Every positive solution of Equation* (3) *is periodic with period* $r+2$.

Previous results [1-3], show that this conjecture is true. However, we do not require that $\{A_n\}_{n=0}^{\infty}$ is a periodic sequence. Although, we are not able to prove this conjecture in the general case, we prove it for the following particular case.

Consider the difference equation

$$x_{n+1} = \max\left\{\frac{1}{x_n}, \frac{1}{x_{n-1}}, \frac{A_n}{x_{n-2}}\right\}, \quad n = 0, 1, ... \tag{4}$$

where $\{A_n\}_{n=0}^{\infty}$ is a periodic sequence of period two as $\{..., \alpha, \beta, \alpha, \beta, ...\}$, and $\alpha, \beta \in (0,1)$. Suppose that $\alpha > \beta$.

Lemma 3. *Every solution of Equation* (4) *which is bounded below by* $\sqrt{\alpha}$ *belongs to the interval* $\left[\sqrt{\alpha}, \frac{1}{\sqrt{\alpha}}\right]$.

Proof. Let $\{x_n\}_{n=-2}^{\infty}$ be a positive solution of Equation (4) and let there exist $N \geq 0$ such that

$$x_{n-2} \geq \sqrt{\alpha} \quad \text{for all} \quad n \geq N.$$

It follows from Equation (4) that

$$x_{N+1} = \max\left\{\frac{1}{x_N}, \frac{1}{x_{N-1}}, \frac{A_N}{x_{N-2}}\right\} \leq \max\left\{\frac{1}{\sqrt{\alpha}}, \frac{1}{\sqrt{\alpha}}, \frac{A_N}{\sqrt{\alpha}}\right\} = \frac{1}{\sqrt{\alpha}}.$$

Similarly, we see that

$$x_{N+2} = \max\left\{\frac{1}{x_{N+1}}, \frac{1}{x_N}, \frac{A_{N+1}}{x_{N-1}}\right\} \leq \max\left\{\frac{1}{\sqrt{\alpha}}, \frac{1}{\sqrt{\alpha}}, \frac{A_{N+1}}{\sqrt{\alpha}}\right\} = \frac{1}{\sqrt{\alpha}}.$$

Similarly to the above, the proof follows. ■

Lemma 4. *Every positive semi-cycle of any solution of Equation* (4) *which is bounded below by* $\sqrt{\alpha}$ *has exactly length two.*

Proof. Suppose there exists $N \geq 0$ such that

$$x_{N-1} < 1 \quad \text{and} \quad x_N \geq 1.$$

Then from Equation (4) we see that

$$x_{N+1} = \max\left\{\frac{1}{x_N}, \frac{1}{x_{N-1}}, \frac{A_N}{x_{N-2}}\right\} > 1$$

and

$$x_{N+2} = \max\left\{\frac{1}{x_{N+1}}, \frac{1}{x_N}, \frac{A_{N+1}}{x_{N-1}}\right\} < 1,$$

where $x_{N-1} > \sqrt{\alpha} \Rightarrow \dfrac{A_{N+1}}{x_{N-1}} < \dfrac{A_{N+1}}{\sqrt{\alpha}} \leq \dfrac{\alpha}{\sqrt{\alpha}} = \sqrt{\alpha} < 1.$
This completes the proof. ■

Remark 1. By Lemma 2, every negative semi-cycle has length one.

Theorem 4.3. *Every positive solution of Equation* (4) *which is bounded below by* $\sqrt{\alpha}$ *is eventually periodic with period three.*

Proof. Since the positive semi-cycle has length exactly two and the negative semi-cycle is of length exactly one, we consider only the following cases for an integer $N \geq 0$

(a) $x_{N-2},\ x_{N-1} \geq 1$ and $x_N < 1$;
(b) $x_{N-2},\ x_N \geq 1$ and $x_{N-1} < 1$;
(c) $x_{N-1},\ x_N \geq 1$ and $x_{N-2} < 1$.

We will consider only the case (a) (the other cases are similar and the proof will be omitted). Assume (a) holds, then it is easy to see from Equation (4) that the solution is of the form

$$\left\{..., x_N, \frac{1}{x_N}, \frac{1}{x_N}, x_N, \frac{1}{x_N}, \frac{1}{x_N}, ...\right\}.$$

Therefore $\{x_n\}_{n=-2}^{\infty}$ is a periodic solution with period three. ■

Lemma 5. *Assume* $\{x_n\}_{n=-2}^{\infty}$ *is a positive solution of Equation* (4) *and suppose there exists* $m \geq 2$ *such that*

$$x_{m-2} < \sqrt{\alpha}.$$

Then either $\{x_n\}_{n=-2}^{\infty}$ *is an eventually periodic solution with period three or*

$$\liminf_{n\to\infty} x_n \geq \sqrt{\alpha}.$$

Proof. It follows from Equation (4) that

$$x_{m+1} = \max\left\{\frac{1}{x_m}, \frac{1}{x_{m-1}}, \frac{A_m}{x_{m-2}}\right\} = \frac{\alpha}{x_{m-2}},$$

where $x_{m-1}, x_m > \dfrac{1}{\sqrt{\alpha}}$. So $\alpha x_{m-1} > \sqrt{\alpha} > x_{m-2}$, and similarly $\alpha x_m > x_{m-2}$.

$$x_{m+2} = \max\left\{\frac{1}{x_{m+1}}, \frac{1}{x_m}, \frac{A_{m+1}}{x_{m-1}}\right\} = \max\left\{\frac{x_{m-2}}{\alpha}, \frac{1}{x_m}, \frac{\beta}{x_{m-1}}\right\} = \frac{x_{m-2}}{\alpha},$$

and

$$x_{m+3} = \max\left\{\frac{1}{x_{m+2}}, \frac{1}{x_{m+1}}, \frac{A_{m+2}}{x_m}\right\} = \max\left\{\frac{\alpha}{x_{m-2}}, \frac{x_{m-2}}{\alpha}, \frac{\alpha}{x_m}\right\}.$$

We consider the following two cases:

(A1) $x_{m+3} = \dfrac{x_{m-2}}{\alpha}$. In this case, by some simple computations, we see that the solution is of the form

$$\left\{..., \frac{\alpha}{x_{m-2}}, \frac{x_{m-2}}{\alpha}, \frac{x_{m-2}}{\alpha}, \frac{\alpha}{x_{m-2}}, \frac{x_{m-2}}{\alpha}, \frac{x_{m-2}}{\alpha}, ...\right\}.$$

Therefore $\{x_n\}_{n=-2}^{\infty}$ is a periodic solution with period three.

(A2) $x_{m+3} = \dfrac{\alpha}{x_{m-2}}$. In this case we see that

$$x_{m+4} = \max\left\{\frac{1}{x_{m+3}}, \frac{1}{x_{m+2}}, \frac{A_{m+3}}{x_{m+1}}\right\} = \max\left\{\frac{x_{m-2}}{\alpha}, \frac{\alpha}{x_{m-2}}, \frac{\beta x_{m-2}}{\alpha}\right\} = \frac{\alpha}{x_{m-2}},$$

and

$$x_{m+5} = \max\left\{\frac{1}{x_{m+4}}, \frac{1}{x_{m+3}}, \frac{A_{m+4}}{x_{m+2}}\right\} = \max\left\{\frac{x_{m-2}}{\alpha}, \frac{x_{m-2}}{\alpha}, \frac{\alpha^2}{x_{m-2}}\right\}.$$

We consider the following two cases:

(B1) $x_{m+5} = \frac{x_{m-2}}{\alpha}$. In this case we see that the solution is of the form

$$\left\{..., \frac{x_{m-2}}{\alpha}, \frac{\alpha}{x_{m-2}}, \frac{\alpha}{x_{m-2}}, \frac{x_{m-2}}{\alpha}, \frac{\alpha}{x_{m-2}}, \frac{\alpha}{x_{m-2}}, ...\right\}.$$

Therefore $\{x_n\}_{n=-2}^{\infty}$ is a periodic solution with period three.

(B2) $x_{m+5} = \dfrac{\alpha^2}{x_{m-2}}$. Then

$$x_{m+6} = \max\left\{\frac{1}{x_{m+5}}, \frac{1}{x_{m+4}}, \frac{A_{m+5}}{x_{m+3}}\right\} = \max\left\{\frac{x_{m-2}}{\alpha^2}, \frac{x_{m-2}}{\alpha}, \frac{\beta x_{m-2}}{\alpha}\right\} = \frac{x_{m-2}}{\alpha^2},$$

and in this case we see that

$$x_{m-2} < x_{m+2} = \frac{x_{m-2}}{\alpha} < x_{m+6} = \frac{x_{m-2}}{\alpha^2}.$$

Thus

$$\liminf_{n\to\infty} x_n \geq \sqrt{\alpha}.$$

■

Theorem 4.3 and Lemma 5 lead to the following main result of this section.

Theorem 4.4. *Every solution of Equation* (4) *is periodic with period three.*

5. Periodicity of Equation (2)

It is well known that every solution of the difference equation

$$x_{n+1} = \frac{1}{x_{n-j}}, \quad n = 0, 1, ...$$

is periodic with period $2j+2$, where j is a nonnegative integer number.

One may think of a relation between the period of the periodic solution of Equation (2) and the period of the solution of equations

$$x_{n+1} = \frac{1}{x_{n-p}}, \quad \text{and} \quad x_{n+1} = \frac{1}{x_{n-q}}.$$

Amazingly, there is a relation. This relation will be given by the following theorem.

Theorem 5.1. *The following statements are true.*

(1) *If* $q \neq 3p+2$, *then every positive solution of Equation* (2) *is periodic with period* $q+p+2$.

(2) *If* $q = 3p+2$, *then every positive solution of Equation* (2) *is periodic with period* $2p+2$.

Proof. (1) It suffices to show that every positive semi-cycle of any solution of Equation (2) has length $q+1$ and that every negative semi-cycle has length $p+1$.

Assume that there exists an integer $N \geq 0$ such that

$$x_{N-q}, x_{N-q+1}, ..., x_{N-1} \geq 1 \quad \text{and} \quad x_N < 1.$$

It follows from Equation (2) that

$$\begin{aligned}
x_{N+1} &= \max\left\{\frac{1}{x_{N-p}}, \frac{1}{x_{N-q}}\right\} < 1,\\
x_{N+2} &= \max\left\{\frac{1}{x_{N-p+1}}, \frac{1}{x_{N-q+1}}\right\} < 1,\\
&\ \ .\\
&\ \ .\\
&\ \ .\\
x_{N+p-1} &= \max\left\{\frac{1}{x_{N-2}}, \frac{1}{x_{N-q+p-2}}\right\} < 1,\\
x_{N+p} &= \max\left\{\frac{1}{x_{N-1}}, \frac{1}{x_{N-q+p-1}}\right\} < 1,
\end{aligned}$$

and

$$x_{N+p+1} = \max\left\{\frac{1}{x_{N}}, \frac{1}{x_{N-q+p}}\right\} > 1.$$

Therefore the negative semi-cycle has length exactly $p+1$.

Again, we see from Equation (2) that

$$\begin{aligned}
x_{N+p+2} &= \max\left\{\frac{1}{x_{N+1}}, \frac{1}{x_{N-q+p+1}}\right\} > 1,\\
x_{N+p+3} &= \max\left\{\frac{1}{x_{N+2}}, \frac{1}{x_{N-q+p+2}}\right\} > 1,\\
&\ \ .\\
&\ \ .\\
&\ \ .\\
x_{N+p+q} &= \max\left\{\frac{1}{x_{N+q-1}}, \frac{1}{x_{N+p-1}}\right\} > 1,\\
x_{N+p+q+1} &= \max\left\{\frac{1}{x_{N+q}}, \frac{1}{x_{N+p}}\right\} > 1,
\end{aligned}$$

and

$$x_{N+p+q+2} = \max\left\{\frac{1}{x_{N+q+1}}, \frac{1}{x_{N+p+1}}\right\} < 1.$$

This gives that the positive semi-cycle has exactly length $q+1$. The proof is complete.

(2) As in Case (1), assume that there exists an integer $N \geq 0$ such that

$$x_{N-3p-2}, x_{N-3p-1}, ..., x_{N-1} \geq 1 \quad \text{and} \quad x_N < 1.$$

It follows from Equation (2) that

$$x_{N+1} = \max\left\{\frac{1}{x_{N-p}}, \frac{1}{x_{N-3p-2}}\right\} < 1,$$
$$x_{N+2} = \max\left\{\frac{1}{x_{N-p+1}}, \frac{1}{x_{N-3p-1}}\right\} < 1,$$
$$\vdots$$
$$x_{N+p-1} = \max\left\{\frac{1}{x_{N-2}}, \frac{1}{x_{N-2p-4}}\right\} < 1,$$
$$x_{N+p} = \max\left\{\frac{1}{x_{N-1}}, \frac{1}{x_{N-2p-3}}\right\} < 1,$$

and

$$x_{N+p+1} = \max\left\{\frac{1}{x_{N}}, \frac{1}{x_{N-2p-2}}\right\} > 1.$$

Therefore the negative semi-cycle has exactly length $p+1$.

Again, we see from Equation (2) that

$$x_{N+p+2} = \max\left\{\frac{1}{x_{N+1}}, \frac{1}{x_{N-2p-1}}\right\} > 1,$$
$$x_{N+p+3} = \max\left\{\frac{1}{x_{N+2}}, \frac{1}{x_{N-2p}}\right\} > 1,$$
$$\vdots$$
$$x_{N+2p} = \max\left\{\frac{1}{x_{N+p-1}}, \frac{1}{x_{N-p-3}}\right\} > 1,$$
$$x_{N+2p+1} = \max\left\{\frac{1}{x_{N+p}}, \frac{1}{x_{N-p-2}}\right\} > 1,$$

and

$$x_{N+2p+2} = \max\left\{\frac{1}{x_{N+p+1}}, \frac{1}{x_{N-p-1}}\right\} < 1.$$

This means that the positive semi-cycle has exactly length $p+1$. The proof is complete. ∎

Remark 2. Note that the period of the difference equation

$$x_{n+1} = \max\left\{\frac{1}{x_n}, \frac{1}{x_{n-1}}, \frac{1}{x_{n-2}}, ..., \frac{1}{x_{n-k}}\right\},$$

is the average of the periods of the equations

$$x_{n+1} = \frac{1}{x_{n-i}}, \quad i = 0, 1, ..., k.$$

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