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On Systems of Quasivariational Inclusion Problems of Type I and Related Problems*

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Dedicated to Professor Do Long Van on the occasion of his 65^{th} birthday

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Abstract. The systems of quasivariational inclusion problems are introduced and sufficient conditions on the existence of their solutions are shown. As special cases, we obtain several results on the existence of solutions of quasivariational inclusion problems, general vector ideal (proper, Pareto, weak) quasi-optimization problems, quasivariational inequalities, and vector quasi-equilibrium problems etc.

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1. Introduction

Let Y be a topological vector space with a cone C. For a given subset $A \subset Y$, one can define efficient points of A with respect to C in different senses as: Ideal, Pareto, proper, weak,... (see Definition 2.1 below). The set of these efficient

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points is denoted by $\alpha\text{Min}(A/C)$ with $\alpha = \text{I}; \alpha = \text{P}\ ; \alpha = \ \text{Pr}\ ; \alpha = \text{W}; ...$ for the case of ideal, Pareto, proper, weak efficient points, respectively. Let D be a subset of another topological vector space X. By 2^D we denote the family of all subsets in D. For a given multivalued mapping $F : D \to 2^Y$, we consider the problem of finding $\bar{x} \in D$ such that

$$F(\bar{x}) \cap \alpha\text{Min}(F(D)/C) \neq \emptyset. \qquad (GVOP)_\alpha$$

This is called a general vector α optimization problem corresponding to D, F and C. The set of such points $\bar{x}$ is said to be the solution set of $(GVOP)_\alpha$. The elements of $\alpha\text{Min}(F(D)/C)$ are called α optimal values of $(GVOP)_\alpha$.

Now, let X, Y and Z be Hausdorff locally convex topogical vector spaces, let $D \subset X, K \subset Z$ be nonempty subsets and let $C \subset Y$ be a cone. Given the following multivalued mappings

$$S : D \times K \to 2^D,$$
$$T : D \times K \to 2^K,$$
$$F : D \times K \times D \to 2^Y,$$

we are interested in the problem of finding $(\bar{x}, \bar{y}) \in D \times K$ such that

$$\bar{x} \in S(\bar{x}, \bar{y}),$$
$$\bar{y} \in T(\bar{y}, \bar{x}), \qquad (GVQOP)_\alpha$$

and

$$F(\bar{y}, \bar{x}, \bar{x}) \cap \alpha\text{Min}(F(\bar{x}, \bar{y}, S(\bar{x}, \bar{y})) \neq \emptyset.$$

This is called a general vector α quasi-optimization problem (α is one of the following qualifications: ideal, Pareto, proper, weak, respectively). Such a pair $(\bar{x}, \bar{y})$ is said to be a solution of $(GVQOP)_\alpha$. The above multivalued mappings S, T, and F are said to be a constraint, a potential, and a utility mapping, respectively. These problems play a central role in the vector optimization theory concerning multivalued mappings and have many relations to the following problems

(UIQEP), Upper Ideal Quasi-Equilibrium Problem: Find $(\bar{x}, \bar{y}) \in D \times K$ such that

$$\bar{x} \in S(\bar{x}, \bar{y}),$$
$$\bar{y} \in T(\bar{x}, \bar{y}),$$
$$F(\bar{x}, \bar{y}, x) \subset C, \ \text{ for all } x \in S(\bar{x}, \bar{y}).$$

(LIQEP), Lower ideal quasi-equilibrium problem: Find $(\bar{x}, \bar{y}) \in D \times K$ such that

$$\bar{x} \in S(\bar{x}, \bar{y}),$$
$$\bar{y} \in T(\bar{x}, \bar{y}),$$
$$F(\bar{x}, \bar{y}, x) \cap C \neq \emptyset, \ \text{ for all } x \in S(\bar{x}, \bar{y}).$$

(UPQEP), Upper Pareto quasi-equilibrium problem: Find $(\bar{x}, \bar{y}) \in D \times K$ such that

$$\bar{x} \in S(\bar{x}, \bar{y}),$$
$$\bar{y} \in T(\bar{x}, \bar{y}),$$
$$F(\bar{x}, \bar{y}, x) \not\subseteq -(C \setminus l(C)), \text{ for all } x \in S(\bar{x}, \bar{y}).$$

(LPQEP), Lower Pareto quasi-equilibrium problem: Find $(\bar{x}, \bar{y}) \in D \times K$ such that

$$\bar{x} \in S(\bar{x}, \bar{y}),$$
$$\bar{y} \in T(\bar{x}, \bar{y}),$$
$$F(\bar{x}, \bar{y}, x) \cap -(C \setminus l(C)) = \emptyset, \text{ for all } x \in S(\bar{x}, \bar{y}).$$

(UWQEP), Upper weak quasi-equilibrium problem: Find $(\bar{x}, \bar{y}) \in D \times K$ such that

$$\bar{x} \in S(\bar{x}, \bar{y}),$$
$$\bar{y} \in T(\bar{x}, \bar{y}),$$
$$F(\bar{x}, \bar{y}, x) \not\subseteq \text{-int}(C), \text{ for all } x \in S(\bar{x}, \bar{y}).$$

(UWQEP), Lower weak quasi-equilibrium problem: Find $(\bar{x}, \bar{y}) \in D \times K$ such that

$$\bar{x} \in S(\bar{x}, \bar{y}),$$
$$\bar{y} \in T(\bar{x}, \bar{y}),$$
$$F(\bar{x}, \bar{y}, x) \cap \text{-int}(C) = \emptyset, \text{ for all } x \in S(\bar{x}, \bar{y}).$$

These problems generalize many well-known problems in the optimization theory as quasi-equilibrium problems, quasivariational inequalities, fixed point problems, complementarity problems, saddle point problems, minimax problems as well as different others which have been studied by many authors, for examples, Park [1], Chan and Pang [2], Parida and Sen [3], Guerraggio and Tan [4] etc. for quasi-equilibrium problems and quasivariational inequalities; Blum and Oettli [5], Tan [7], Minh and Tan [8], Ky Fan [9] etc. for equilibrium and variational inequality problems and by some others in the references therein. If we denote by $\alpha_i, i = 1, 2, 3, 4$, the relations between subsets in Y: $A \subseteq B, A \cap B \neq \emptyset, A \not\subseteq B$ and $A \cap B = \emptyset$ as in [6], then the above problems (UIQEP), (LIQEP) can be written as:

Find $(\bar{x}, \bar{y}) \in D \times K$ such that

$$\bar{x} \in S(\bar{x}, \bar{y}),$$
$$\bar{y} \in T(\bar{x}, \bar{y}),$$
$$\alpha_i(F(\bar{x}, \bar{y}, x), C), \text{ for all } x \in S(\bar{x}, \bar{y}), i = 1, 2, \text{ respectively.}$$

The problems (UPQEP), (LPQEP) can be written as:

Find $(\bar{x}, \bar{y}) \in D \times K$ such that

$$\bar{x} \in S(\bar{x}, \bar{y}),$$
$$\bar{y} \in T(\bar{x}, \bar{y}),$$
$$\alpha_i(F(\bar{x}, \bar{y}, x), -(C \setminus l(C))), \text{ for all } x \in S(\bar{x}, \bar{y}), i = 3, 4, \text{ respectively.}$$

Analogously, the problems (UWQEP), (LWQEP) can be written as:

Find $(\bar{x}, \bar{y}) \in D \times K$ such that

$$\begin{gathered} \bar{x} \in S(\bar{x}, \bar{y}), \\ \bar{y} \in T(\bar{x}, \bar{y}), \\ \alpha_i(F(\bar{x}, \bar{y}, x), -\text{int}C), \text{ for all } x \in S(\bar{x}, \bar{y}), i = 3, 4, \text{ respectively.} \end{gathered}$$

The purpose of this paper is to prove some new results on the existence of solutions to systems concerning the following quasivariational inclusions.

(UQVIP), Upper quasivariational inclusion problem of type I: Find $(\bar{x}, \bar{y}) \in D \times K$ such that

$$\begin{gathered} \bar{x} \in S(\bar{x}, \bar{y}), \\ \bar{y} \in T(\bar{x}, \bar{y}), \\ F(\bar{y}, \bar{x}, x) \subset F(\bar{x}, \bar{x}, \bar{x}) + C, \text{ for all } x \in S(\bar{x}, \bar{y}). \end{gathered}$$

(LQVIP), Lower quasivariational inclusion problem of type I: Find $(\bar{x}, \bar{y}) \in D \times K$ such that

$$\begin{gathered} \bar{x} \in S(\bar{x}, \bar{y}), \\ \bar{y} \in T(\bar{x}, \bar{y}), \\ F(\bar{y}, \bar{x}, \bar{x}) \subset F(\bar{y}, \bar{x}, x) - C, \text{ for all } x \in S(\bar{x}, \bar{y}). \end{gathered}$$

In [7] the author gave some existence theorems on the above problems and their systems. But, he presented some rather strong conditions. For example: The polar cone C' of the cone C is supposed to have weakly compact basis in the weak*topology, the multivalued mapping F has nonempty convex closed values. In this paper, we shall give some weaker sufficient conditions to improve his results by considering the existence of solutions of the systems of the above quasivariational inclusion problems: Let X, Z, D, K, S and T be given as above. Assume that Y_i are other Hausdorff locally convex topological vector spaces with convex closed cones $C_i, i = 1, 2$ and $F_1 : K \times D \times D \to 2^{Y_1}, F_2 : D \times K \times K \to 2^{Y_2}$ are multivalued mappings. We consider

System (A). Find $(\bar{x}, \bar{y}) \in D \times K$ such that

$$\begin{gathered} \bar{x} \in S(\bar{x}, \bar{y}), \\ \bar{y} \in T(\bar{x}, \bar{y}), \\ F_1(\bar{y}, \bar{x}, x) \subset F_1(\bar{y}, \bar{x}, \bar{x}) + C_1, \text{ for all } x \in S(\bar{x}, \bar{y}), \\ F_2(\bar{x}, \bar{y}, y) \subset F_2(\bar{x}, \bar{y}, \bar{y}) + C_2, \text{ for all } y \in T(\bar{x}, \bar{y}). \end{gathered}$$

System (B). Find $(\bar{x}, \bar{y}) \in D \times K$ such that

$$\begin{gathered} \bar{x} \in S(\bar{x}, \bar{y}), \\ \bar{y} \in T(\bar{x}, \bar{y}), \\ F_1(\bar{y}, \bar{x}, x) \subset F_1(\bar{y}, \bar{x}, \bar{x}) + C_1, \text{ for all } x \in S(\bar{x}, \bar{y}), \\ F_2(\bar{x}, \bar{y}, \bar{y}) \subset F_2(\bar{x}, \bar{y}, y) - C_2, \text{ for all } y \in T(\bar{x}, \bar{y}). \end{gathered}$$

System (C). Find $(\bar{x}, \bar{y}) \in D \times K$ such that

$$
\begin{aligned}
&\bar{x} \in S(\bar{x}, \bar{y}),\\
&\bar{y} \in T(\bar{x}, \bar{y}),\\
F_1(\bar{y}, \bar{x}, \bar{x}) \subset F_1(\bar{y}, \bar{x}, x) - C_1, &\quad \text{for all } x \in S(\bar{x}, \bar{y}),\\
F_2(\bar{x}, \bar{y}, y) \subset F_2(\bar{x}, \bar{y}, \bar{y}) + C_2, &\quad \text{for all } y \in T(\bar{x}, \bar{y}).
\end{aligned}
$$

System (D). Find $(\bar{x}, \bar{y}) \in D \times K$ such that

$$
\begin{aligned}
&\bar{x} \in S(\bar{x}, \bar{y}),\\
&\bar{y} \in T(\bar{x}, \bar{y}),\\
F_1(\bar{y}, \bar{x}, \bar{x}) \subset F_1(\bar{y}, \bar{x}, x) - C_1, &\quad \text{for all } x \in S(\bar{x}, \bar{y}),\\
F_2(\bar{x}, \bar{y}, \bar{y}) \subset F_2(\bar{x}, \bar{y}, y) - C_2, &\quad \text{for all } y \in T(\bar{x}, \bar{y}).
\end{aligned}
$$

We shall see that a solution of one of the above systems, under some additional conditions, is also a solution of some other systems of quasi-optimization problems, quasi-equilibrium problems, quasivariational problems etc.

2. Preliminaries and Definitions

Throughout this paper, as in the introduction, by $X, Y, Y_i, i = 1, 2,$ and Z we denote real Hausdorff locally convex topological vector spaces. The space of real numbers is denoted by R. Given a subset $D \subset X$, we consider a multivalued mapping $F : D \to 2^Y$. The definition domain and the graph of F are denoted by

$$
\begin{aligned}
\text{dom}F &= \{x \in D / F(x) \neq \emptyset\},\\
\text{Gr}(F) &= \{(x, y) \in D \times Y / y \in F(x)\},
\end{aligned}
$$

respectively. We recall that F is said to be a closed mapping if the graph $Gr(F)$ of F is a closed subset in the product space $X \times Y$ and it is said to be a compact mapping if the closure $\overline{F(D)}$ of its range $F(D)$ is a compact set in Y.

Further, let Y be a Hausdorff locally convex topological vector space with a cone C. We denote $l(C) = C \cap (-C)$. If $l(C) = \{0\}, C$ is said to be pointed. We recall the following definitions (see Definition 2.1, Chapter 2 in [10]).

Definition 2.1. *Let A be a nonempty subset of Y. We say that:*

(i) $x \in A$ *is an ideal efficient (or ideal minimal) point of A with respect to C (w.r.t. C for short) if $y - x \in C$ for every $y \in A$.*
The set of ideal minimal points of A is denoted by $\text{IMin}(A/C)$.

(ii) $x \in A$ *is an efficient (or Pareto–minimal, or nondominated) point of A w.r.t. C if there is no $y \in A$ with $x - y \in C \setminus l(C)$.*
The set of efficient points of A is denoted by $\text{PMin}(A/C)$.

(iii) $x \in A$ *is a (global) proper efficient point of A w.r.t. C if there exists a convex cone $\tilde{C}$ which is not the whole space and contains $C \setminus l(C)$ in its interior so that $x \in \text{PMin}(A/\tilde{C})$.*
The set of proper efficient points of A is denoted by $\text{PrMin}(A/C)$.

(iv) *Supposing that*$\text{int } C$ *nempty, $x \in A$ is a weak efficient point of A w.r.t. C if $x \in \text{PMin}(A/\{0\} \cup \text{ int } C)$.*
The set of weak efficient points of A is denoted by $\text{WMin}(A/C)$.

We write $\alpha\text{Min}(A/C)$ to denote one of $\text{IMin}(A/C), \text{PMin}(A/C), \ldots.$

We have the following inclusions

$$\text{PrMin}(A/C) \subseteq \text{PMin}(A/C) \subseteq \text{WMin}(A/C).$$

Now, we introduce new definitions of C-continuities.

Definition 2.2. *Let $F : D \to 2^Y$ be a multivalued mapping.*

(i) *F is said to be upper (lower) C-continuous in $\bar{x} \in \text{dom} F$ if for any neighborhood V of the origin in Y there is a neighborhood U of $\bar{x}$ such that:*

$$F(x) \subset F(\bar{x}) + V + C \ \ (F(\bar{x}) \subset F(x) + V - C, \ \textit{respectively})$$

holds for all $x \in U \cap \text{dom} F$.

(ii) *If F is upper C-continuous and lower C-continuous in $\bar{x}$ simultaneously, we say that it is C-continuous in $\bar{x}$.*

(iii) *If F is upper, lower, $\ldots, C$-continuous in any point of $\text{dom} F$, we say that it is upper, lower,$\ldots, C$-continuous on D.*

(iv) *In the case $C = \{0\}$, a trivial one in Y, we shall only say that F is upper, lower continuous instead of upper, lower 0-continuous. And, F is continuous if it is upper and lower continuous simultaneously.*

Definition 2.3. *Let D be convex and F be a multivalued mapping from D to 2^Y. We say that:*

(i) *F is upper C-quasiconvex on D if for any $x_1, \ x_2 \in D, \ t \in [0, 1]$, either*

$$F(x_1) \subset F(tx_1 + (1 - t)x_2) + C$$

or,

$$F(x_2) \subset F(tx_1 + (1 - t)x_2) + C,$$

holds.

(ii) *F is lower C-quasiconvex on D if for any $x_1, x_2 \in D, t \in [0, 1]$, either*

$$F(tx_1 + (1 - t)x_2) \subset F(x_1) - C$$

or ,

$$F(tx_1 + (1 - t)x_2) \subset F(x_2) - C,$$

holds.

Now, we give some necessary and sufficient conditions on the upper and the lower C-continuities which we shall need in the next section.

Proposition 2.4. *Let $F : D \to 2^Y$ and $C \subset Y$ be a convex closed cone.*

1) *If F is upper C-continuous at $x_o \in domF$ with $F(x_o) + C$ closed, then for any net $x_\beta \to x_o, y_\beta \in F(x_\beta) + C, y_\beta \to y_o$ imply $y_o \in F(x_o) + C$. Conversely, if F is compact and for any net $x_\beta \to x_o, y_\beta \in F(x_\beta)+C, y_\beta \to y_o$ imply $y_o \in F(x_o) + C$, then F is upper C-continuous at x_o.*

2) *If F is compact and lower C-continuous at $x_o \in domF$, then any net $x_\beta \to x_o, y_o \in F(x_o)+C$, there is a net $\{y_\beta\}, y_\beta \in F(x_\beta)$, which has a convergent subnet $\{y_{\beta_\gamma}\}, y_{\beta_\gamma} - y_o \to c \in C$(i.e $\ \ y_{\beta_\gamma} \to y_o + c \in y_o + C$).*

Conversely, if $F(x_o)$ is compact and for any net $x_\beta \to x_o, y_o \in F(x_o) + C$, there is a net $\{y_\beta\}, y_\beta \in F(x_\beta)$, which has a convergent subnet $\{y_{\beta_\gamma}\}, y_{\beta_\gamma} - y_o \to c \in C$, then F is lower C-continuous at x_o.

Proof.
1) Assume first that F is upper C-continuous at $x_o \in \mathrm{dom}F$ and $x_\beta \to x_o, y_\beta \in F(x_\beta) + C, y_\beta \to y_o$. We suppose on the contrary that $y_o \notin F(x_o) + C$. We can find a convex closed neighborhood V_o of the origin in Y such that

$$(y_o + V_o) \cap (F(x_o) + C) = \emptyset,$$

or,

$$(y_o + V_o/2) \cap (F(x_o) + V_o/2 + C) = \emptyset.$$

Since $y_\beta \to y_o$, one can find $\beta_1 \geq 0$ such that $y_\beta - y_o \in V/2$ for all $\beta \geq \beta_1$. Therefore, $y_\beta \in y_o + V/2$ and F is upper C-continuous at x_o, this implies that one can find a neighborhood U of x_o such that

$$F(x) \subset F(x_o) + V_o/2 + C \text{ for all } x \in U \cap \mathrm{dom}\, F.$$

Since $x_\beta \to x_o$, one can find $\beta_2 \geq 0$ such that $x_\beta \in U$ and

$$y_\beta \in F(x_\beta) + C \subset F(x_o) + V/2 + C \text{ for all } x \in U \cap \text{ dom}\, F.$$

It follows that

$$y_\beta \in (y_o + V/2) \cap (F(x_o) + V/2 + C) = \emptyset \text{ for all } \beta \geq \max\{\beta_1, \beta_2\}$$

and we have a contradiction. Thus, we conclude $y_o \in F(x_o) + C$. Now, assume that F is compact and for any net $x_\beta \to x_o, y_\beta \in F(x_\beta) + C, y_\beta \to y_o$ imply $y_o \in F(x_o) + C$. On the contrary, we assume that F is not upper C-continuous at x_o. It follows that there is a neighborhood V of the origin in Y such that for any neighborhood U_β of x_o one can find $x_\beta \in U_\beta$ such that

$$F(x_\beta) \not\subset F(x_o) + V + C.$$

We can choose $y_\beta \in F(x_\beta)$ with $y_\beta \notin F(x_o) + V + C$. Since $\overline{F(D)}$ is compact, we can assume, without loss of generality, that $y_\beta \to y_o$, and hence $y_o \in F(x_o) + C$. On the other hand, since $y_\beta \to y_o$, there is $\beta_o \geq 0$ such that $y_\beta - y_o \in V$ for all $\beta \geq \beta_o$. Consequently,

$$y_\beta \in y_o + V \subset F(x_o) + V + C, \text{ for all } \beta \geq \beta_o$$

and we have a contradiction.

2) Assume that F is compact and lower C-continuous at $x_o \in \mathrm{dom}\, F$, and $x_\beta \to x_o, y_o \in F(x_o)$. For any neighborhood V of the origin in Y there is a neighborhood U of x_o such that

$$F(x_o) \subset F(x) + V - C, \text{ for all } x \in U \cap \text{ dom}\, F.$$

Since $x_\beta \to x_o$, there is $\beta_o \geq 0$ such that $x_\beta \in U$ and then

$$F(x_o) \subset F(x_\beta) + V - C, \text{ for all } \beta \geq \beta_o.$$

For $y_o \in F(x_o)$, we can write

$$y_o = y_\beta + v_\beta - c_\beta \text{ with } y_\beta \in F(x_\beta) \subset \overline{F(D)}, v_\beta \in V, c_\beta \in C.$$

Since $\overline{F(D)}$ is compact, we can choose $y_{\beta_\gamma} \to y^*, v_{\beta_\gamma} \to 0$. Therefore, $c_{\beta_\gamma} = y_{\beta_\gamma} + v_{\beta_\gamma} - y_o \to y^* - y_o \in C$, or $y_{\beta_\gamma} \to y^* \in y_o + C$. Thus, for any $x_\beta \to x_o, y_o \in F(x_o)$, one can find $y_{\beta_\gamma} \in F(x_{\beta_\gamma})$ with $y_{\beta_\gamma} \to y^* \in y_o + C$.

Now, we assume that $\overline{F(x_o)}$ is compact and for any net $x_\beta \to x_o, y_o \in F(x_o) + C$, there is a net $\{y_\beta\}$, $y_\beta \in F(x_\beta)$ which has a convergent subnet $y_{\beta_\gamma} - y_o \to c \in C$. On the contrary, we suppose that F is not lower C-continuous at x_o. It follows that there is a neighborhood V of the origin in Y such that for any neighborhood U_β of x_o one can find $x_\beta \in U_\beta$ such that

$$F(x_o) \not\subset F(x_\beta) + V - C.$$

We can choose $z_\beta \in F(x_o)$ with $z_\beta \notin (F(x_\beta) + V - C)$. Since $F(x_o)$ is compact, we can assume, without loss of generality, that $z_\beta \to z_o \in F(x_o)$, and hence $z_o \in F(x_o) + C$. We may assume that $x_\beta \to x_o$. Therefore, there is a net $\{y_\beta\}, y_\beta \in F(x_\beta)$ which has a convergent subnet $\{y_{\beta_\gamma}\}, y_{\beta_\gamma} - z_o \to c \in C$. Without loss of generality, we suppose $y_\beta \to y^* \in z_o + C$. It follows that there is $\beta_1 \geq 0$ such that $z_\beta \in z_o + V/2, y_\beta \in y^* + V/2$ and $z_o \in y_\beta + V/2 - C$ for all $\beta \geq \beta_1$. Consequently,

$$z_\beta \in y_\beta + V/2 + V/2 - C \subset F(x_\beta) + V - C, \text{ for all } \beta \geq \beta_1$$

and we have a contradiction. ∎

In the proof of the mains results in Sec. 3, we need the following theorem.

Theorem 2.5. [11] *Let D be a nonempty convex compact subset of X and $F : D \to 2^D$ be a multivalued mapping satisfying the following conditions:*
1) *For all $x \in D, x \notin F(x)$ and $F(x)$ is convex;*
2) *For all $y \in D, F^{-1}(y)$ is open in D.*

Then there exists $\bar{x} \in D$ such that $F(\bar{x}) = \emptyset$.

3. Main Results

Throughout this section, unless otherwise specify, by X, $Y, Y_i, i = 1, 2$ and Z we denote Hausdorff locally convex topogical vector spaces. Let $D \subset X, K \subset Z$ be nonempty subsets, C, C_i, $i = 1, 2$ are convex closed cones in Y, Y_i, respectively. Given multivalued mappings S, T and F as in the introduction, we first prove the following proposition.

Proposition 3.1. *Let $B \subset D$ be a nonempty convex compact subset, $G : B \to 2^Y$ be an upper C-quasiconvex and lower $(-C)$-continuous multivalued mapping with nonempty closed values. Then there exists $\bar{z} \in B$ such that*

$$G(z) \subset G(\bar{z}) + C, \text{ for all } z \in B.$$

Proof. We define the multivalued mapping $N : B \to 2^B$ by

$$N(z) = \{z' \in B \,|\, G(z') \not\subset G(z) + C\}.$$

It is clear that $z \notin N(z)$ for all $z \in B$. If $z_1, z_2 \in N(z)$, then

$$G(z_1) \not\subset G(z) + C,$$
$$G(z_2) \not\subset G(z) + C.$$

Together with the upper C-quasiconvexity of G we conclude

$$G(tz_1 + (1-t)z_2) \not\subset G(z) + C.$$

This implies $tz_1 + (1-t)z_2 \in N(z)$ for all $t \in [0,1]$ and hence $N(z)$ is a convex set for any $z \in B$.

Further, we have

$$N^{-1}(z') = \{z \in B \,|\, G(z') \not\subset G(z) + C\}.$$

Take $z \in N^{-1}(z')$, we deduce $z' \in N(z)$ and so

$$G(z') \not\subset G(z) + C.$$

The upper C-continuity of G implies that for any neighborhood V of the origin in Y there is a neighborhood U_V of z such that

$$G(x) \subset G(z) + V + C, \text{ for some } x \in U_V \cap B.$$

This implies that if for all V

$$G(z') \subset G(x) + C, \text{ for some } x \in U_V \cap B,$$

then

$$G(z') \subset G(x) + C \subset G(z) + V + C$$

and so

$$G(z') \subset G(z) + V + C, \text{ for all } V.$$

Since $G(z)$ and C are closed, the last inclusion shows that $G(z') \subset G(z) + C$ and we have a contradiction. Therefore, there exists V_0 such that

$$G(z') \not\subset G(x) + C, \text{ for all } x \in U_{V_0} \cap B.$$

This gives

$$U_{V_0} \cap B \subset N^{-1}(z')$$

and so $N^{-1}(z')$ is an open set in B. As it has been shown: $z \notin N(z), N(z)$ is convex for any $z \in B$ and $N^{-1}(z')$ is open in B for any $z' \in B$. Consequently, applying Theorem 2.5 in Sec. 2, we conclude that there exists $\bar{z} \in B$ with $N(\bar{z}) = \emptyset$. This implies

$$G(z) \subset G(\bar{z}) + C, \text{ for all } z \in B.$$

Thus, the proof is complete. ∎

Analogously, we can prove the following proposition.

Proposition 3.2. *Let $B \subset D$ be a nonempty convex compact subset, $G : B \to 2^Y$ be a lower C-quasiconvex and upper C-continuous multivalued mapping with nonempty closed values. Then there exists $\bar{z} \in B$ such that*

$$G(\bar{z}) \subset G(z) - C, \text{ for all } z \in B.$$

Corollary 3.3. *Assume that all assumptions of Proposition* 3.1 *are satisfied and for any $z \in B, IMin(G(z)/C) \neq \emptyset$. Then there exists $\bar{z} \in B$ such that*

$$G(\bar{z}) \cap IMin(G(B)/C) \neq \emptyset.$$

(*This means that the general vector ideal optimization problem concerning G, B, C has a solution*).

Proof. Proposition 3.1 implies that there exists $\bar{z} \in B$ such that

$$G(z) \subset G(\bar{z}) + C, \text{ for all } z \in B. \tag{1}$$

Take $v^* \in IMin(G(\bar{z})/C)$, we have $G(\bar{z}) \subset v^* + C$. Then, (1) yields

$$G(z) \subset v^* + C, \text{ for all } z \in B.$$

This shows that $v^* \in IMin(G(B)/C)$ and the proof is complete. ∎

Similarly, we have

Corollary 3.4. *Assume that all assumptions of Proposition* 3.2 *are satisfied. Then there exists $\bar{z} \in B$ such that*

$$G(\bar{z}) \cap PMin(G(B)/C) \neq \emptyset.$$

(*This means that the general vector Pareto optimization problem concerning G, B, C has a solution*).

Corollary 3.5. *If $B \subset D$ is a nonempty convex compact subset having the following property: For any $x_1, x_2 \in B, t \in [0,1]$ either $x_1 - (tx_1 + (1-t)x_2) \in C$ or, $x_1 - (tx_1 + (1-t)x_2) \in C$, then there exist $x^*, x^{**} \in B$ such that*

$$x^{**} \succeq x \succeq x^*, \text{ for all } x \in B,$$

where $x \succeq y$ denotes $x - y \in C$.

Proof. Apply Corollaries 3.3 and 3.4 with $G(z) = -z$ and then $G(z) = z$. ∎

Theorem 3.6. *Let D, K be nonempty convex closed subsets of Hausdorff locally convex topological vector spaces X, Z, respectively. Let $C_i \subset Y_i, i = 1, 2$ be closed convex cones. Then System (A) has a solution provided that the following conditions are satisfied:*

1) *The multivalued mappings $S : D \times K \to 2^D, T : D \times K \to 2^K$ are compact continuous with nonempty convex closed values.*
2) *The multivalued mappings $F_1 : K \times D \times D \to 2^{Y_1}$ and $F_2 : D \times K \times K \to 2^{Y_2}$ are lower $(-C)$ and upper C-continuous with nonempty closed values.*

3) *For any fixed* $(x, y) \in D \times K$, *the multivalued mapping* $F_1(y, x, .) : D \to 2^{Y_1}$ *is upper* C_1*-quasiconvex and the multivalued mapping* $F_2(x, y, .) : K \to 2^{Y_2}$ *is upper* C_2*-quasiconvex.*

Proof. We define the multivalued mapping $M_1 : D \times K \to 2^D$, $M_2 : D \times K \to 2^K$ by

$$M_1(x, y) = \{x' \in S(x, y) \,|\, F_1(y, x, z) \subset F_1(y, x, x') + C_1, \text{ for all } z \in S(x, y)\},$$
$$M_2(x, y) = \{y' \in T(x, y) \,|\, F_2(x, y, v) \subset F_2(x, y, y') + C_2, \text{ for all } v \in T(x, y)\}.$$

For any fixed $(y, x) \in D \times K$ we apply Proposition 3.1 with $B = S(x, y)$ and $G(z) = F_1(y, x, z)$ to show that there exists $\bar{z} \in B$ with

$$F_1(y, x, z) \subset F_1(y, x, \bar{z}) + C_1, \quad \text{for all } z \in S(x, y).$$

This implies $\bar{z} \in M_1(x, y)$ and therefore $M_1(x, y)$ is nonempty. Now, we prove that $M_1(x, y)$ is convex. Indeed, for any $x_1, x_2 \in M_1(x, y)$ and $t \in [0, 1]$, we have from the convexity of $S(x, y)$, $tx_1 + (1 - t)x_2 \in S(x, y)$ and

$$F_1(y, x, z) \subset F_1(y, x, x_1) + C_1, \text{ for all } z \in S(x, y);$$
$$F_1(y, x, z) \subset F_1(y, x, x_2) + C_1, \text{ for all } z \in S(x, y).$$

Since $F_1(y, x, .)$ is upper C_1-quasiconvex, we then conclude

$$F_1(y, x, z) \subset F_1(y, x, tx_1 + (1 - t)x_2) + C_1, \text{ for all } t \in [0, 1], z \in S(x, y).$$

This shows that $tx_1 + (1 - t)x_2 \in M_1(x, y)$ and $M_1(x, y)$ is a convex set.

Further, we claim that M_1 is a closed multivalued mapping. Indeed, assume that $x_\beta \to x, y_\beta \to y, {x'}_\beta \in M_1(x_\beta, y_\beta), {x'}_\beta \to x'$. We have to show $x' \in M_1(x, y)$. Since ${x'}_\beta \in S(x_\beta, y_\beta)$, the upper continuity of S with closed values implies $z' \in S(x, y)$. For $z_\beta \in M_1(x_\beta, y_\beta)$, one can see

$$F_1(y_\beta, x_\beta, z) \subset F_1(y_\beta, x_\beta, {x'}_\beta) + C_1, \text{ for all } z \in S(x_\beta, y_\beta). \tag{2}$$

The lower continuity of S and $x_\beta \to x$, $y_\beta \to y$ imply that for any $z \in S(x, y)$ there exist $z_\beta \in S(x_\beta, y_\beta), z_\beta \to z$ and (2) gives

$$F_1(y_\beta, x_\beta, z_\beta) \subset F_1(y_\beta, x_\beta, {x'}_\beta) + C_1, \text{ for all } z_\beta \in S(x_\beta, y_\beta). \tag{3}$$

Since $(y_\beta, x_\beta, z_\beta) \to (y, x, z)$ and F_1 is lower $(-C)$-continuous at (y, x, z), for any neighborhood V of the origin in Y_1, there is β_1 such that

$$F_1(y, x, z) \subset F_1(y_\beta, x_\beta, z_\beta) + V + C_1, \text{ for all } \beta \geq \beta_1. \tag{4}$$

Since $(y_\beta, x_\beta, {x'}_\beta) \to (y, x, x')$ and F_1 is upper C-continuous at (y, x, x'), there exists β_2 such that

$$F_1(y_\beta, x_\beta, {x'}_\beta) \subset F_1(y, x, x') + V + C_1, \text{ for all } \beta \geq \beta_2. \tag{5}$$

Setting $\beta_0 = \max\{\beta_1, \beta_2\}$, the combination of (3), (4) and (5) yields

$$F_1(y, x, z) \subset F_1(y, x, x') + 2V + C_1, \text{ for all } z \in S(x, y).$$

The closedness of C and the closed values of F_1 show that

$$F_1(y, x, z) \subset F_1(y, x, x') + C_1, \text{ for all } z \in S(x, y).$$

This means that $x' \in M_1(x,y)$ and then M is a closed multivalued mapping. By the same arguments we verify that M_2 is also a closed multivalued mapping with nonempty convex values.

Lastly, we define the multivalued mapping $P : D \times K \to 2^{D\times K}$ by

$$P(x,y) = M_1(x,y) \times M_2(x,y), \ \ (x,y) \in D \times K.$$

We can easily see that $P(x,y) \neq \emptyset, P(x,y)$ is convex for all $(x,y) \in D \times K$ and P is a closed multivalued mapping. Moreover, since $P(D \times K) \subset M_1(D \times K) \times M_2(D \times K) \subset S(D \times K) \times T(D \times K)$, then P is a compact multivalued mapping. Applying the fixed point theorem of Himmelberg type (see for example, in (Ref.1)), we conclude that there exists a point $(\bar{x},\bar{y}) \in D \times K$ with $(\bar{y},\bar{x}) \in M_1(\bar{x},\bar{y}) \times M_2(\bar{x},\bar{y})$. This implies $\bar{x} \in S(\bar{x},\bar{y}), \bar{y} \in T(\bar{x},\bar{y})$ and

$$F_1(\bar{y},\bar{x},x) \subset F_1(\bar{y},\bar{x},\bar{x}) + C_1, \ \text{ for all } x \in S(\bar{x},\bar{y}),$$
$$F_2(\bar{x},\bar{y},y) \subset F_2(\bar{x},\bar{y},\bar{y}) + C_2, \ \text{ for all } x \in T(\bar{x},\bar{y}).$$

Thus, the proof of the theorem is complete. ■

Theorem 3.7. *Let $D,K,S,T,C_i,Y_i, i = 1,2$ be as the same in Theorem 3.6. Then System (B) has a solution provided that the following conditions are satisfied.*

1) *The multivalued mapping $F_1 : K \times D \times D \to 2^{Y_1}$ is lower $(-C_1)$ and upper C_1-continuous with nonempty closed values and the multivalued mapping $F_2 : D \times K \times K \to 2^{Y_2}$ is lower C_2-continuous and upper $(-C_2)$-continuous with nonempty closed values;*
2) *For any fixed $(x,y) \in D \times K$, the multivalued mapping $F_1(y,x,.) : D \to 2^{Y_1}$ is upper C_1-quasiconvex and the multivalued mapping $F_2(x,y,.) : K \to 2^{Y_2}$ is lower C_2-quasiconvex.*

Proof. We define the multivalued mappings $M_1 : D \times K \to 2^D, M_2 : D \times K \to 2^K$ by

$$M_1(x,y) = \{x' \in S(x,y) \mid F_1(y,x,z) \subset F_1(y,x,x') + C_1, \ \text{ for all } z \in S(x,y)\},$$
$$M_2(x,y) = \{y' \in T(x,y) \mid F_2(x,y,y') \subset F_2(x,y,v) - C_2, \ \text{ for all } v \in T(x,y)\}$$

and use the same proof as in Theorem 3.6. ■

Theorem 3.8. *Let $D,K,S,T,C_i,Y_i, i = 1,2$ be the same as in Theorem 3.6. Then System (C) has a solution provided that the following conditions are satisfied.*

1) *The multivalued mappings $F_1 : K \times D \times D \to 2^{Y_1}$ is lower C_1 and upper $(-C_1)$-continuous with nonempty closed values and the multivalued mapping $F_2 : D \times K \times K \to 2^{Y_2}$ is lower $(-C_2)$-continuous and upper C_2-continuous with nonempty closed values;*
2) *For any fixed $(x,y) \in D \times K$, the multivalued mapping $F_1(y,x,.) : D \to 2^{Y_1}$ is lower C_1-quasiconvex and the multivalued mapping $F_2(x,y,.) : K \to 2^{Y_2}$ is upper C_2-quasiconvex.*

Proof. We define the multivalued mappings $M_1 : D\times K \to 2^D, M_2 : D\times K \to 2^K$ by

$$M_1(x,y) = \{x' \in S(x,y) \mid F_1(y,x,x') \subset F_1(y,x,z) - C_1, \text{ for all } z \in S(x,y)\},$$
$$M_2(x,y) = \{y' \in T(x,y) \mid F_2(x,y,v) \subset F_2(x,y,y') + C_2, \text{ for all } v \in T(x,y)\}$$

and use the same proof as in Theorem 3.6. ∎

Theorem 3.9. *Let $D, K, S, T, C_i, Y_i, i = 1, 2$ be the same as in Theorem 3.6. Then System (D) has a solution provided that the following conditions are satisfied.*

1) *The multivalued mapping $F_1 : K \times D \times D \to 2^{Y_1}$ is lower C_1 and upper $(-C_1)$-continuous with nonempty closed values and the multivalued mapping $F_2 : D \times K \times K \to 2^{Y_2}$ is lower C_2-continuous and upper $(-C_2)$-continuous with nonempty closed values;*
2) *For any fixed $(x,y) \in D \times K$, the multivalued mapping $F_1(y,x,.) : D \to 2^{Y_1}$ is lower C_1-quasiconvex and the multivalued mapping $F_2(x,y,.) : K \to 2^{Y_2}$ is lower C_2-quasiconvex.*

Proof. We define the multivalued mappings $M_1 : D \times K \to 2^D$, $M_2 : D \times K \to 2^K$ by

$$M_1(x,y) = \{x' \in S(x,y) \mid F_1(y,x,x') \subset F_1(y,x,z) - C_1, \text{ for all } z \in S(x,y)\},$$
$$M_2(x,y) = \{y' \in T(x,y) \mid F_2(x,y,y') \subset F_2(x,y,v) - C_2, \text{ for all } v \in T(x,y)\}$$

and use the same proof as in Theorem 3.6. ∎

The following corollaries are special cases of Theorems 3.6, 3.7, 3.8, and 3.9. Their proofs follow immediately from the above theorems.

Corollary 3.10. *Let D be a nonempty convex closed subset of Hausdorff locally convex topological vector space X. Let $C \subset Y$ be a closed convex cone. Let $S : D \times K \to 2^D, T : D \times K \to 2^K$ be compact continuous multivalued mappings with nonempty convex closed values. Let $F : K \times D \times D \to 2^Y$ be a lower $(-C)$-continuous and upper C-continuous mapping with nonempty closed values such that for any fixed $(x,y) \in D \times K$, the multivalued mapping $F(y,x,.) : D \to 2^Y$ is upper C-quasiconvex.*

Then there exists $(\bar{x},\bar{y}) \in D \times K$ such that $\bar{x} \in S(\bar{x},\bar{y}), \bar{y} \in T(\bar{x},\bar{y})$ and

$$F(\bar{y},\bar{x},x) \subset F(\bar{y},\bar{x},\bar{x}) + C, \text{ for all } x \in S(\bar{x},\bar{y}).$$

Corollary 3.11. *Let D, K, Y, C be as in Corollary 3.10. Let $F : K \times D \times D \to 2^Y$ be a lower C-continuous and upper $(-C)$-continuous mapping with nonempty closed values such that for any fixed $(x,y) \in D \times K$, the multivalued mapping $F(y,x,.) : D \to 2^Y$ is lower C-quasiconvex.*

Then there exists $(\bar{x},\bar{y}) \in D \times K$ such that $\bar{x} \in S(\bar{x},\bar{y}), \bar{y} \in T(\bar{x},\bar{y})$ and

$$F(\bar{y},\bar{x},\bar{x}) \subset F(\bar{y},\bar{x},x) - C, \text{ for all } x \in S(\bar{x},\bar{y}).$$

Corollary 3.12. *Let D, K, C, S, T and F_i, $i = 1, 2$, be as in Theorem 3.6. In addition, assume that $F_1(y,x,x) \subset C_1, F_2(x,y,y) \subset C_2$ for all $(x,y) \in D \times K$. Then there exists $(\bar{x},\bar{y}) \in D \times K$ such that $\bar{x} \in S(\bar{x},\bar{y}), \bar{y} \in T(\bar{x},\bar{y})$ and*

$$F_1(\bar{y},\bar{x},x) \subset C_1, \text{ for all } x \in S(\bar{x},\bar{y}),$$
$$F_2(\bar{x},\bar{y},y) \subset C_1, \text{ for all } y \in T(\bar{x},\bar{y}).$$

Proof. It is obvious. ■

Corollary 3.13. *Let D, K, S, T and F_i, $i = 1, 2$, be as in Theorem 3.6 and* $\text{IMin}(F_1(y,x,x)/C_1) \neq \emptyset, \text{IMin}(F_2(x,y,y)/C_2) \neq \emptyset$ *for all $(x,y) \in D \times K$. Then $(\bar{x},\bar{y})$ satisfies*

$$\begin{gathered} \bar{x} \in S(\bar{x},\bar{y}), \\ \bar{y} \in T(\bar{x},\bar{y}), \\ F_1(\bar{y},\bar{x},x) \subset F_1(\bar{y},\bar{x},\bar{x}) + C_1, \text{ for all } x \in S(\bar{x},\bar{y}), \\ F_2(\bar{x},\bar{y},y) \subset F_2(\bar{x},\bar{y},\bar{y}) + C_2, \text{ for all } y \in T(\bar{x},\bar{y}) \end{gathered} \tag{7}$$

if and only if

$$\begin{gathered} \bar{x} \in S(\bar{x},\bar{y}), \\ \bar{y} \in T(\bar{x},\bar{y}), \end{gathered}$$

such that

$$\begin{gathered} F_1(\bar{y},\bar{x},\bar{x}) \cap \text{IMin}(F_1(\bar{y},\bar{x},S(\bar{x},\bar{y}))/C_1) \neq \emptyset, \\ F_2(\bar{x},\bar{y},\bar{y}) \cap \text{IMin } (F_2(\bar{x},\bar{y},T(\bar{x},\bar{y}))/C_2) \neq \emptyset. \end{gathered} \tag{8}$$

Proof. First we assume that $(\bar{x},\bar{y})$ satisfies (7). Take $v^* \in \text{IMin}(F_1(\bar{y},\bar{x},\bar{x})/C_1)$. It is clear that $F_1(\bar{y},\bar{x},\bar{y},\bar{x}) \subset v^* + C_1$. Together with (7) we have

$$F_1(\bar{x},\bar{y},x) \subset F_1(\bar{y},\bar{x},\bar{x}) + C_1 \subset v^* + C_1 \quad \text{for all } \ x \in S(\bar{x},\bar{y}).$$

This implies $v^* \in \text{IMin}(F_1(\bar{x},\bar{y},S(\bar{x},\bar{y})/C_1)$ and hence

$$F_1(\bar{y},\bar{x},\bar{x}) \cap \text{IMin}(F_1(\bar{y},\bar{x},S(\bar{x},\bar{y}))/C_1) \neq \emptyset.$$

Analogously, we get

$$F_2(\bar{x},\bar{y},\bar{y}) \cap \text{IMin}(F_2(\bar{x},\bar{y},T(\bar{x},\bar{y}))/C_2).$$

Now, assume that (8) holds. Take $v^* \in F_1(\bar{y},\bar{x},\bar{x}) \cap \text{IMin}(F_1(\bar{y},\bar{x},S(\bar{x},\bar{y}))/C_1)$, we have

$$F_1(\bar{y},\bar{x},x) \subset v^* + C_1 \subset F_1(\bar{y},\bar{x},\bar{x}) + C_1 \quad \text{for all} \quad x \in S(\bar{x},\bar{y}).$$

Similarly, we have

$$F_2(\bar{x},\bar{y},y) \subset F_2(\bar{x},\bar{y},\bar{y}) + C_2 \quad \text{for all } \ y \in T(\bar{x},\bar{y}).$$

This completes the proof of the corollary. ■

Corollary 3.14. *Let D, K, C_i, S, T and F_i, $i = 1, 2$ be as in Theorem 3.6. In addition, assume that there exists a convex cone $\tilde{C}_i$ which is not the whole space and contains $C_i \setminus \{0\}$ in its interior. Then the there exists $(\bar{x},\bar{y}) \in D \times K$ with*

$$\bar{x} \in S(\bar{x}, \bar{y}),$$
$$\bar{y} \in T(\bar{x}, \bar{y}),$$

such that

$$F_1(\bar{y}, \bar{x}, \bar{x}) \cap \text{PrMin}(F_1(\bar{y}, \bar{x}, S(\bar{x}, \bar{y}))/C_1) \neq \emptyset,$$
$$F_2(\bar{x}, \bar{y}, \bar{y}) \cap \text{PrMin}(F_2(\bar{x}, \bar{y}, T(\bar{x}, \bar{y}))/C_2) \neq \emptyset.$$

Proof. Since C_i has the property as above, then any compact set A_i in Y_i has $\text{PrMin}(A_i/C_i) \neq \emptyset$ (by using the cone ${C_i}^* = \{0\} \cup \text{int}\tilde{C_i}$ one can verify $\text{PMin}(A_i/{C_i}^*) \neq \emptyset$, see, for example, Corollary 3.15, Chapter 2 in Ref. 10). We then apply Theorem 3.6 to obtain $(\bar{x}, \bar{y}) \in D \times K$ such that:

$$\bar{x} \in S(\bar{x}, \bar{y}), \bar{y} \in T(\bar{y}, \bar{x})$$

and

$$F_1(\bar{y}, \bar{x}, x) \subset F_1(\bar{y}, \bar{x}, \bar{x}) + C_1, \quad \text{for all } x \in S(\bar{x}, \bar{y}). \tag{9}$$

Since $F_1(\bar{y}, \bar{x}, \bar{x})$ is a compact set, it follows that $\text{PrMin}(F_1(\bar{y}, \bar{x}, \bar{x})/C_1) \neq \emptyset$. Take $\bar{v} \in \text{PrMin}(F_1(\bar{y}, \bar{x}, \bar{x})/C_1)$, we show that $\bar{v} \in \text{PrMin}(F_1(\bar{y}, \bar{x}, S(\bar{x}, \bar{y}))/C_1)$. On the contrary, we suppose that $\bar{v} \notin \text{PrMin}(F_1(\bar{y}, \bar{x}, S(\bar{x}, \bar{y}))/C_1)$. Then, there is $v^* \in F_1(\bar{y}, \bar{x}, S(\bar{x}, \bar{y}))$ such that

$$\bar{v} - v^* \in {C_1}^* \setminus l({C_1}^*). \tag{10}$$

Assume that $v^* \in F_1(\bar{y}, \bar{x}, x^*)$ for some $x^* \in S(\bar{x}, \bar{y})$. We can conclude from (7) that there exists $v^o \in F_1(\bar{y}, \bar{x}, \bar{x})$ such that $v^* - v^o = c \in C_1$. If $c = 0$, then $v^* = v^o$ and then $\bar{v} - v^o \in {C_1}^* \setminus l({C_1}^*)$. If $c \neq 0$, using (10), we conclude

$$\bar{v} - v^o = \bar{v} - v^* + v^* - v^o \in {C_1}^* \setminus l({C_1}^*) + C_1 \setminus \{0\} \subset {C_1}^* \setminus l({C_1}^*).$$

Therefore, we obtain $\bar{v} - v^o \in {C_1}^* \setminus l({C_1}^*)$. Due to $\bar{v} \in \text{PrMin}(F_1(\bar{y}, \bar{x}, \bar{x})/C_1)$ and $v^o \in F_1(\bar{y}, \bar{x}, \bar{x})$, we then have a contradiction. Consequently,

$$F_1(\bar{y}, \bar{x}, \bar{x}) \cap \text{PrMin}(F_1(\bar{y}, \bar{x}, S(\bar{x}, \bar{y}))/C_1) \neq \emptyset$$

By the same arguments, we conclude

$$F_2(\bar{x}, \bar{y}, \bar{y}) \cap \text{PrMin}(F_2(\bar{x}, \bar{y}, T(\bar{x}, \bar{y}))/C_2) \neq \emptyset$$

■

Corollary 3.15. *If* D, K, C, S, T, F_i, $i = 1, 2$, *are as in Theorem* 3.6, *then the there exists* $(\bar{x}, \bar{y}) \in D \times K$ *with*

$$\bar{x} \in S(\bar{x}, \bar{y}),$$
$$\bar{y} \in T(\bar{x}, \bar{y}),$$

such that

$$F_1(\bar{y}, \bar{x}, \bar{x}) \cap \text{PMin}(F_1(\bar{y}, \bar{x}, S(\bar{x}, \bar{y}))/C_1) \neq \emptyset,$$
$$F_2(\bar{x}, \bar{y}, \bar{y}) \cap \text{PMin}(F_2(\bar{x}, \bar{y}, T(\bar{x}, \bar{y}))/C_2) \neq \emptyset.$$

Proof. By Theorem 3.6, there is $(\bar{x}, \bar{y}) \in D \times K$ such that:

$$\bar{x} \in S(\bar{x}, \bar{y}), \bar{y} \in T(\bar{x}, \bar{y})$$

and

$$F_1(\bar{y},\bar{x},x) \subset F_1(\bar{y},\bar{x},\bar{x}) + C_1, \text{ for all } x \in S(\bar{x},\bar{y}). \tag{11}$$

We claim that

$$F_1(\bar{y},\bar{x},\bar{x}) \cap \mathrm{PMin}(F_1(\bar{x},\bar{y},S(\bar{x},\bar{y}))/C_1) \neq \emptyset.$$

The compactness of $F_1(\bar{y},\bar{x},\bar{x})$ shows that $\mathrm{PMin}(F_1(\bar{y},\bar{x},\bar{x})/C_1) \neq \emptyset$. Assume $\bar{v} \in \mathrm{PMin}(F_1(\bar{y},\bar{x},\bar{x})/C_1)$ and $\bar{v} \notin \mathrm{PMin}(F_1(\bar{x},\bar{y},S(\bar{x},\bar{y}))/C_1)$. It follows that there is $v \in F_1(\bar{x},\bar{y},S(\bar{x},\bar{y}))$, say $v \in F_1(\bar{x},\bar{y},x)$ with some $x \in S(\bar{x},\bar{y})$, such that

$$\bar{v} - v \in C_1 \setminus l(C_1). \tag{12}$$

(11) implies that $v \in F_1(\bar{y},\bar{x},\bar{x}) + C_1$ and so

$$v = v^* + c, \text{ with some } v^* \in F_1(\bar{y},\bar{x},\bar{x}), c \in C_1,$$

or

$$v - v^* \in C_1. \tag{13}$$

A combination of (12) and (13) gives

$$\bar{v} - v^* = \bar{v} - v + v - v^* \in C_1 \setminus l(C_1) + C_1 \subset C_1 \setminus l(C_1).$$

This contradicts $\bar{v} \in \mathrm{PMin}(F_1(\bar{y},\bar{x},\bar{x})/C_1)$. Therefore, we obtain

$$F_1(\bar{y},\bar{x},\bar{x}) \cap \mathrm{PMin}(F_1(\bar{x},\bar{y},S(\bar{x},\bar{y}))/C_1) \neq \emptyset.$$

By the same arguments we verify

$$F_2(\bar{y},\bar{x},\bar{x}) \cap \mathrm{PMin}(F_2(\bar{x},\bar{y},S(\bar{x},\bar{y}))/C_2) \neq \emptyset.$$

This completes the proof of the corollary. ∎

Similarly, we can also obtain several results for systems of the other quasi-equilibrium and quasi-optimization problems.

Corollary 3.16. *Let D, K, C, S, T and F_i, $i = 1, 2$, be as in Theorem* 3.9 *with $F_1(y,x,x) \subset C_1, F_2(x,y,y) \subset C_2$, for any $(x,y) \in D \times K$. If $(\bar{x},\bar{y})$ is a solution of the System (D), then it is also a solution of the following system of Pareto quasi-equilibrium problems: Find $(\bar{x},\bar{y}) \in D \times K$ such that*

$$\bar{x} \in S(\bar{x}),$$
$$\bar{y} \in T(\bar{x},\bar{y}),$$
$$F_1(\bar{y},\bar{x},x) \not\subset -(C_1 \setminus l(C_1)), \text{ for all } x \in S(\bar{x},\bar{y}),$$
$$F_2(\bar{x},\bar{y},y) \not\subset -(C_2 \setminus l(C_2)), \text{ for all } x \in T(\bar{x},\bar{y}).$$

Proof. Indeed, on the contrary we assume that there is $x^* \in S(\bar{x},\bar{y})$ such that $F_1(\bar{y},\bar{x},x^*) \subset -(C_1 \setminus l(C_1))$. Since $F_1(\bar{y},\bar{x},x^*) \cap C_1 \neq \emptyset$, we can take $v^* \in F_1(\bar{y},\bar{x},x^*) \cap C_1$. This yields $v^* \in C_1 \cap (-(C_1 \setminus l(C)) \subset -l(C_1), v^* \in F(_1\bar{y},\bar{x},x^*) \subset -(C_1 \setminus l(C_1))$. It is impossible, because $v^* \in -l(C_1)$. Analogously, if there is $y^* \in T(\bar{x},\bar{y})$ such that $F_2(\bar{x},\bar{y},y^*) \subset -(C_2 \setminus l(C_2))$, we then also have a contradiction. This completes the proof of the corollary. ∎

To conclude the paper, we give a corollary of Theorem 3.6 on saddle point problems of vector functions. We have

Corollary 3.17. *Let D, K, C, S, T, be as in Theorem* 3.6. *Let $F : D \times K \to Y$ be a $(-C)$- and C-continuous singlevalued mapping such that for any fixed $y \in K$, the mapping $F(.,y) : D \to Y$ is C-quasiconvex and for any fixed $x \in D$, the mapping $F(x,.) : K \to Y$ is $(-C)$-quasiconvex. Then there exists $(\bar{x}, \bar{y}) \in D \times K$ with*

$$\bar{x} \in S(\bar{x}, \bar{y}),$$
$$\bar{y} \in T(\bar{x}, \bar{y}),$$

such that

$$F(\bar{y}, x) \in F(\bar{y}, \bar{x}) + C, \text{ for all } x \in S(\bar{x}, \bar{y}),$$
$$F(\bar{x}, \bar{y}) \in F(y, \bar{x}) + C, \text{ for all } y \in T(\bar{x}, \bar{y}).$$

Proof. The proof of this corollary follows immediately from Theorem 3.6 with $F_1 : K \times D \times D \to Y, F_2 : D \times K \times K \to Y$ defined by

$$F_1(y, x, x') = F(x', y) - F(x, y), (y, x, x') \in K \times D \times D,$$
$$F_2(x, y, y') = F(x, y) - F(x, y'), (x, y, y') \in D \times K \times K.$$

Applying this theorem, we obtain the proof of the corollary. ∎

Remark. For $u, v \in Y$, we define $u \succeq v$ if $u - v \in C$, then in the conclusion of Corollary 3.17 we can write $\bar{x} \in S(\bar{x}), \bar{y} \in T(\bar{x}, \bar{y})$, and

$$F(\bar{y}, x) \succeq F(\bar{y}, \bar{x}) \succeq F(y, \bar{x}), \text{ for all } x \in S(\bar{x}, \bar{y}) \text{ and } y \in T(\bar{x}, \bar{y}).$$

Such a point $(\bar{x}, \bar{y})$ is said to be a saddle point of F with respect to S, T and C.

References

1. S. Park, Fixed points and quasi-equilibrium problems, nonlinear operator theory, *Math. and Computer Modelling* **32** (2000) 1297–1304.
2. D. Chan and J. S. Pang, The generalized quasi-variational inequality problem, *Math.Oper. Res.* **7** (1982) 211–222.
3. J. Parida and A. Sen, A variational-like inequality for multifunctions with applications, *J. Math. Anal. Appl.* **124** (1986) 73–81.
4. A. Gurraggio, and N. X. Tan, On general vector quasi-optimization problems, *Math. Methods of Operation Research* **55** (2002) 347–358.
5. E. Blum and W. Oettli, From optimization and variational inequalities to equilibrium problems, *The Math. Student* **64** (1993) 1–23.
6. P. H. Sach, On a class of generalized vector equilibrium problems with set-valued maps, Hanoi Inst. Math., *Preprint* 05-07, 2005.
7. N. X. Tan, On the existence of solutions of quasi-variational inclusion problems, *J. Opt. Theory and Appl.* **123** (2004) 619–638.

8. N. B. Minh, and N. X. Tan, Some sufficient conditions for the existence of equilibrium points concerning multivalued mappings, *Vietnam J. Math.* **28** (2000) 295–310.
9. K. Fan, A minimax inequality and application, in *Inequalities* III, O. Shisha (Ed.), Academic Press, New-York, 1972, p. 33.
10. D. T. Luc, *Theory of Vector Optimization,* Lecture Notes in Economics and Mathematical Systems, Vol. 319, Springer-Verlag, Berlin, 1989.
11. N. C. Yannelis and N. D. Prabhaker, Existence of maximal elements and equilibria in linear topological spaces, *J. Math. Economics* **12** (1983) 233–245.