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A Note on Maximal Nonhamiltonian Burkard–Hammer Graphs

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Dedicated to Professor Do Long Van on the occasion of his 65^{th} birthday

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Abstract. A graph $G = (V, E)$ is called a split graph if there exists a partition $V = I \cup K$ such that the subgraphs $G[I]$ and $G[K]$ of G induced by I and K are empty and complete graphs, respectively. In 1980, Burkard and Hammer gave a necessary condition for a split graph G with $|I| < |K|$ to be hamiltonian. We will call a split graph G with $|I| < |K|$ satisfying this condition a Burkard–Hammer graph. Further, a split graph G is called a maximal nonhamiltonian split graph if G is nonhamiltonian but $G+uv$ is hamiltonian for every $uv \notin E$ where $u \in I$ and $v \in K$. In an earlier work, the author and Iamjaroen have asked whether every maximal nonhamiltonian Burkard–Hammer graph G with the minimum degree $\delta(G) \geq |I| - k$ where $k \geq 3$ possesses a vertex adjacent to all vertices of G and whether every maximal nonhamiltonian Burkard–Hammer graph G with $\delta(G) = |I| - k$ where $k > 3$ and $|I| > k+2$ possesses a vertex with exactly $k-1$ neighbors in I. The first question and the second one have been proved earlier to have a positive answer for $k = 3$ and $k = 4$, respectively. In this paper, we give a negative answer both to the first question for all $k \geq 4$ and to the second question for all $k \geq 5$.

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1. Introduction

All graphs considered in this paper are finite undirected graphs without loops or multiple edges. If G is a graph, then $V(G)$ and $E(G)$ (or V and E in short) will denote its vertex-set and its edge-set, respectively. For a subset $W \subseteq V(G)$, the set of all neighbors of W is denoted by $N_G(W)$ or $N(W)$ in short. For a vertex $v \in V(G)$, the degree of v, denoted by $\deg(v)$, is the number $|N(v)|$. The minimum degree of G, denoted by $\delta(G)$, is the number $\min\{\deg(v) \mid v \in V(G)\}$. By $N_{G,W}(v)$ or $N_W(v)$ in short we denote the set $W \cap N_G(v)$. The subgraph of G induced by W is denoted by $G[W]$. Unless otherwise indicated, our graph-theoretic terminology will follow [1].

A graph $G = (V, E)$ is called a *split graph* if there exists a partition $V = I \cup K$ such that the subgraphs $G[I]$ and $G[K]$ of G induced by I and K are empty and complete graphs, respectively. We will denote such a graph by $S(I(G) \cup K(G), E(G))$ or $S(I \cup K, E)$ in short. Further, a split graph $G = S(I \cup K, E)$ is called a *complete split graph* if every $u \in I$ is adjacent to every $v \in K$. The notion of split graphs was introduced in 1977 by Földes and Hammer [4]. These graphs are interesting because they are related to many problems in combinatorics (see [3, 5, 10]) and in computer science (see [6, 7]).

In 1980, Burkard and Hammer gave a necessary condition for a split graph $G = S(I \cup K, E)$ with $|I| < |K|$ to be hamiltonian [2] (see Sec. 2 for more detail). We will call this condition the *Burkard–Hammer condition*. Also, we will call a split graph $G = S(I \cup K, E)$ with $|I| < |K|$, which satisfies the Burkard–Hammer condition, a *Burkard–Hammer graph*.

Thus, by [2] any hamiltonian split graph $G = S(I \cup K, E)$ with $|I| < |K|$ is a Burkard–Hammer graph. In general, the converse is not true. The first nonhamiltonian Burkard–Hammer graph has been indicated in [2]. Further infinite families of nonhamiltonian Burkard–Hammer graphs have been constructed recently in [13].

A split graph $G = S(I \cup K, E)$ is called a *maximal nonhamiltonian split graph* if G is nonhamiltonian but the graph $G + uv$ is hamiltonian for every $uv \notin E$ where $u \in I$ and $v \in K$. It is known from a result in [12] that any nonhamiltonian Burkard–Hammer graph is contained in a maximal nonhamiltonian Burkard–Hammer graph. So knowledge about maximal nonhamiltonian Burkard–Hammer graphs provides us certain information about nonhamiltonian Burkard–Hammer graphs.

It has been shown in [12] (see Theorem 2 in Sec. 2) that there are no nonhamiltonian Burkard–Hammer graphs $G = S(I \cup K, E)$ with $\delta(G) \geq |I| - 2$ and no nonhamiltonian Burkard–Hammer graphs $G = S(I \cup K, E)$ with $\delta(G) = |I| - 3$ and $|I| > 5$. Therefore, without loss of generality we may assume that all considered in this paper maximal nonhamiltonian Burkard–Hammer graphs $G = S(I \cup K, E)$ have $\delta(G) = |I| - k$ where $|I| \geq k \geq 3$ and all considered maximal nonhamiltonian Burkard–Hammer graphs $G = S(I \cup K, E)$ with $\delta(G) = |I| - k$ and $|I| > k + 2$ have $k > 3$.

It has been proved recently in [14] that a maximal nonhamiltonian Burkard–Hammer graph $G = S(I \cup K, E)$ with $\delta(G) = |I| - k$ where $|I| \geq k \geq 3$ must have $|I| \geq k + 2$ and no vertices with exactly $k + 1, \ldots, |I| - 1$ neighbors in I. Moreover, if $G = S(I \cup K, E)$ has $\delta(G) = |I| - k$ where $k > 3$ and $|I| > k + 2$, then G also has no vertices with exactly k neighbors in I. However, it is shown in [14] that for every integer $k > 3$ and every integer $m > k + 2$ there exists a maximal nonhamiltonian Burkard–Hammer graph $G = S(I \cup K, E)$ with $|I| = m$ and $\delta(G) = |I| - k$ which possesses a vertex with exactly $k-1$ neighbors in I. Ngo Dac Tan and Iamjaroen have asked in [14] whether all maximal nonhamiltonian Burkard–Hammer graphs $G = S(I \cup K, E)$ with $\delta(G) = |I| - k$ where $k \geq 3$ possess a vertex adjacent to all vertices of G and whether all maximal nonhamiltonian Burkard–Hammer graphs $G = S(I \cup K, E)$ with $\delta(G) = |I| - k$ where $k > 3$ and $|I| > k+2$ possess a vertex with exactly $k-1$ neighbors in I. The first question has been proved in [12] to have a positive answer for $k = 3$. Recently, Ngo Dac Tan and Iamjaroen have proved in [14] that the second question also has a positive answer for $k = 4$. In this paper, however, we will give a negative answer both to the first question for all $k \geq 4$ and to the second question for all $k \geq 5$.

We would like to note that there is an interesting discussion about the Burkard–Hammer condition in [9]. Concerning the hamiltonian problem for split graphs, the readers can see also [8] and [11].

2. Preliminaries

Let $G = S(I \cup K, E)$ be a split graph and $I' \subseteq I, K' \subseteq K$. Denote by $B_G(I' \cup K', E')$ the graph $G[I' \cup K'] - E(G[K'])$. It is clear that $G' = B_G(I' \cup K', E')$ is a bipartite graph with the bipartition subsets I' and K'. So we will call $B_G(I' \cup K', E')$ the *bipartite subgraph of G induced by I' and K'*. For a component $G'_j = B_G(I'_j \cup K'_j, E'_j)$ of $G' = B_G(I' \cup K', E')$ we define

$$k_G(G'_j) = k_G(I'_j, K'_j) = \begin{cases} |I'_j| - |K'_j| & \text{if } |I'_j| > |K'_j|, \\ 0 & \text{otherwise.} \end{cases}$$

If $G' = B_G(I' \cup K', E')$ has r components $G'_1 = B_G(I'_1 \cup K'_1, E'_1), \ldots, G'_r = B_G(I'_r \cup K'_r, E'_r)$ then we define

$$k_G(G') = k_G(I', K') = \sum_{j=1}^{r} k_G(G'_j).$$

A component $G'_j = B_G(I'_j \cup K'_j, E'_j)$ of $G' = B_G(I' \cup K', E')$ is called a *T-component* (resp., *H-component, L-component*) if $|I'_j| > |K'_j|$ (resp., $|I'_j| = |K'_j|, |I'_j| < |K'_j|$). Let $h_G(G') = h_G(I', K')$ denote the number of H-components of G'.

In 1980, Burkard and Hammer proved the following necessary but not sufficient condition for hamiltonian split graphs [2].

Theorem 1. [2] *Let $G = S(I \cup K, E)$ be a split graph with $|I| < |K|$. If G is hamiltonian, then*

$$k_G(I', K') + \max\left\{1, \frac{h_G(I', K')}{2}\right\} \leq |N_G(I')| - |K'|$$

holds for all $\emptyset \neq I' \subseteq I, K' \subseteq N_G(I')$ with $(k_G(I', K'), h_G(I', K')) \neq (0, 0)$.

We will shortly call the condition in Theorem 1 the *Burkard–Hammer condition*. We also call a split graph $G = S(I \cup K, E)$ with $|I| < |K|$, which satisfies the Burkard–Hammer condition, a *Burkard–Hammer graph*. Thus, by Theorem 1 any hamiltonian split graph $G = S(I \cup K, E)$ with $|I| < |K|$ is a Burkard–Hammer graph. For split graphs $G = S(I \cup K, E)$ with $|I| < |K|$ and $\delta(G) \geq |I| - 2$ the converse is true [12]. But it is not true in general. The first example of a nonhamiltonian Burkard–Hammer graph has been indicated in [2]. Recently, Tan and Hung [12] have classified nonhamiltonian Burkard–Hammer graphs $G = S(I \cup K, E)$ with $\delta(G) = |I| - 3$. Namely, they have proved the following result.

Theorem 2. [12] *Let $G = S(I \cup K, E)$ be a split graph with $|I| < |K|$ and the minimum degree $\delta(G) \geq |I| - 3$. Then*

(i) *If $|I| \neq 5$ then G has a Hamilton cycle if and only if G satisfies the Burkard–Hammer condition;*

(ii) *If $|I| = 5$ and G satisfies the Burkard–Hammer condition, then G has no Hamilton cycles if and only if G is isomorphic to one of the graphs $H^{1,n}$, $H^{2,n}$, $H^{3,n}$ or $H^{4,n}$ listed in Table 1.*

Table 1. The graphs $H^{1,n}$, $H^{2,n}$, $H^{3,n}$ and $H^{4,n}$

The graph G	The vertex-set $V(G) = I^* \cup K^*$	The edge-set $E(G) = E_1^* \cup \cdots \cup E_5^* \cup E_{K^*}^*$
$H^{1,n}$ ($n > 5$)	$I^* = \{u_1^*, u_2^*, u_3^*, u_4^*, u_5^*\}$, $K^* = \{v_1^*, v_2^*, ..., v_n^*\}$.	$E_1^* = \{u_1^*v_1^*, u_1^*v_2^*\}$, $E_2^* = \{u_2^*v_2^*, u_2^*v_4^*\}$, $E_3^* = \{u_3^*v_2^*, u_3^*v_3^*, u_3v_6^*\}$, $E_4^* = \{u_4^*v_1^*, u_4^*v_4^*, u_4v_6^*\}$, $E_5^* = \{u_5^*v_5^*, u_5^*v_6^*\}$, $E_{K^*}^* = \{v_i^*v_j^* \mid i \neq j;\ i, j = 1, ..., n\}$,
$H^{2,n}$	$V(H^{2,n}) = V(H^{1,n})$	$E(H^{2,n}) = E(H^{1,n}) \cup \{u_4^*v_2^*\}$
$H^{3,n}$	$V(H^{3,n}) = V(H^{1,n})$	$E(H^{3,n}) = E(H^{1,n}) \cup \{u_5^*v_2^*\}$
$H^{4,n}$	$V(H^{4,n}) = V(H^{1,n})$	$E(H^{4,n}) = E(H^{1,n}) \cup \{u_4^*v_2^*, u_5^*v_2^*\}$

Theorem 2 shows that there are only four nonhamiltonian Burkard–Hammer graphs $G = S(I \cup K, E)$ with $K = N(I)$ and $\delta(G) = |I| - 3$, namely, the graphs $H^{1,6}, H^{2,6}, H^{3,6}$ and $H^{4,6}$. In contrast with this result, the number of nonhamiltonian Burkard–Hammer graphs $G = S(I \cup K, E)$ with $K = N(I)$ and $\delta(G) = |I| - 4$ is infinite. This is a recent result of Tan and Iamjaroen [13]. We remind now one of the constructions in this work, which is needed for the next sections.

Let $G_1 = S(I_1 \cup K_1, E_1)$ and $G_2 = S(I_2 \cup K_2, E_2)$ be split graphs with

$$V(G_1) \cap V(G_2) = \emptyset$$

and v be a vertex of K_1. We say that a graph G is an *expansion of* G_1 *by* G_2 *at* v if G is the graph obtained from $(G_1 - v) \cup G_2$ by adding the set of edges

$$E_0 = \{x_i v_j \mid x_i \in V(G_1) \setminus \{v\}, v_j \in K_2 \text{ and } x_i v \in E_1\}.$$

It is clear that such a graph G is a split graph $S(I \cup K, E)$ with $I = I_1 \cup I_2$, $K = (K_1 \setminus \{v\}) \cup K_2$ and is uniquely determined by G_1, G_2 and $v \in K_1$. Because of this, we will denote this graph G by $G_1[G_2, v]$. Further, a graph G is called an *expansion of* G_1 *by* G_2 if it is an expansion of G_1 by G_2 at some vertex $v \in K_1$.

The following results which have been proved in [12 - 14] are needed later.

Lemma 1. [12] *Let $G = S(I \cup K, E)$ be a Burkard–Hammer graph. Then for any $uv \notin E$ where $u \in I$ and $v \in K$, the graph $G+uv$ is also a Burkard–Hammer graph.*

Theorem 3. [13] *Let $G_1 = S(I_1 \cup K_1, E_1)$ be a Burkard–Hammer graph and $G_2 = S(I_2 \cup K_2, E_2)$ be a complete split graph with $|I_2| < |K_2|$. Then any expansion of G_1 by G_2 is a Burkard–Hammer graph.*

Theorem 4. [13] *Let $G_1 = S(I_1 \cup K_1, E_1)$ be an arbitrary split graph and $G_2 = S(I_2 \cup K_2, E_2)$ be a split graph with $|K_2| = |I_2| + 1$. Then an expansion of G_1 by G_2 is a hamiltonian graph if and only if both G_1 and G_2 are hamiltonian graphs.*

Let $G = S(I \cup K, E)$ be a split graph. Set
$B_i(G) = \{v \in K \mid |N_I(v)| = i\}$.

If the graph G is clear from the context then we also write B_i instead of $B_i(G)$.

Theorem 5. [14] *Let $G_1 = S(I_1 \cup K_1, E_1)$ be a complete split graph with $|I_1| < |K_1|$ and $G_2 = S(I_2 \cup K_2, E_2)$ be a maximal nonhamiltonian Burkard–Hammer graph with $\delta(G_2) = |I_2| - k_2$ such that every vertex $u \in I_2$ has $N_{G_2}(u) \neq K_2$. Then any expansion $G = S(I \cup K, E) = G_1[G_2, v_1]$ where $v_1 \in K_1$ is a maximal nonhamiltonian Burkard–Hammer graph with $\delta(G) = \delta(G_2) = |I| - (k_2 + |I_1|)$. Moreover, for any $x \in K_1 \setminus \{v_1\}$, $|N_{G,I}(x)| = |I_1|$ and for any $y \in K_2$, $|N_{G,I}(y)| = |N_{G_2,I_2}(y)| + |I_1|$.*

3. Formulations of the Main Results and Discussions

By Theorem 2 in the previous section there are no nonhamiltonian Burkard–Hammer graphs $G = S(I \cup K, E)$ with $\delta(G) \geq |I| - 2$ and no nonhamiltonian Burkard–Hammer graphs $G = S(I \cup K, E)$ with $\delta(G) = |I| - 3$ and $|I| > 5$. Therefore, in further discussions without loss of generality we may assume that all considered maximal nonhamiltonian Burkard–Hammer graphs $G = S(I \cup K, E)$ with $\delta(G) = |I| - k$ have $|I| \geq k \geq 3$ and all considered maximal nonhamiltonian Burkard–Hammer graphs $G = S(I \cup K, E)$ with $\delta(G) = |I| - k$ and $|I| > k + 2$ have $k > 3$. We start our discussions with the following result proved in [14].

Theorem 6. [14] *Let $G = S(I \cup K, E)$ be a maximal nonhamiltonian Burkard–Hammer graph with the minimum degree $\delta(G) = |I| - k$ where $|I| \geq k \geq 3$. Then $|I| \geq k + 2$ and $B_{k+1} = \cdots = B_{|I|-1} = \emptyset$. Furthermore, if $k > 3$ and $|I| > k + 2$ then B_k is also empty.*

Two questions raised from Theorem 6 are whether a maximal nonhamiltonian Burkard–Hammer graph $G = S(I \cup K, E)$ with $\delta(G) = |I| - k$ where $k \geq 3$ must have $B_{|I|} = \emptyset$ and whether a maximal nonhamiltonian Burkard–Hammer graph $G = S(I \cup K, E)$ with $\delta(G) = |I| - k$ where $k > 3$ and $|I| > k + 2$ also must have $B_{k-1} = \emptyset$. The following results proved in [14] show that both these questions have negative answers.

Theorem 7. [14]

(a) *For every integer $k \geq 3$ there exists a maximal nonhamiltonian Burkard–Hammer graph $G = S(I \cup K, E)$ with $|I| = k + 2$ and $\delta(G) = |I| - k$, which has $B_k \neq \emptyset$ and $B_{|I|} \neq \emptyset$.*

(b) *For every integer $k > 3$ and every integer $m > k + 2$ there exists a maximal nonhamiltonian Burkard–Hammer graph $G = S(I \cup K, E)$ with $|I| = m$ and $\delta(G) = |I| - k$, which has $B_{k-1}(G) \neq \emptyset$ and $B_{|I|} \neq \emptyset$.*

Two natural questions raised from the results in Theorem 7 are whether every maximal nonhamiltonian Burkard–Hammer graph $G = S(I \cup K, E)$ with $\delta(G) = |I| - k$ where $k \geq 3$ has $B_{|I|} \neq \emptyset$ and whether every maximal nonhamiltonian Burkard–Hammer graph $G = S(I \cup K, E)$ with $\delta(G) = |I| - k$ where $k > 3$ and $|I| > k + 2$ has $B_{k-1} \neq \emptyset$. These questions have been posed in [14]. Theorem 2 shows that the first question has a positive answer for $k = 3$ and Theorem 8 below proved in [14] shows that the second question has a positive answer for $k = 4$. These make the questions more attractive for investigation.

Theorem 8. [14] *Let $G = S(I \cup K, E)$ be a maximal nonhamiltonian Burkard–Hammer graph with $|I| \geq 7$ and the minimum degree $\delta(G) = |I| - 4$. Then $B_4 = B_5 = \cdots = B_{|I|-1} = \emptyset$ but $B_3 \neq \emptyset$.*

In this paper, we get complete answers to the above two questions. Namely, we will prove the following results.

Theorem 9.

(a) *For every integer $k \geq 4$ there exists a maximal nonhamiltonian Burkard–Hammer graph $G = S(I \cup K, E)$ with $\delta(G) = |I| - k$, which has $B_{|I|} = \emptyset$.*

(a) *For every integer $k \geq 5$ and every integer $m > k + 2$ there exists a maximal nonhamiltonian Burkard–Hammer graph $G = S(I \cup K, E)$ with $|I| = m$ and $\delta(G) = |I| - k$, which has $B_{k-1} = \emptyset$ but $B_{k-2} \neq \emptyset, B_{k-3} \neq \emptyset$ and $B_{k-4} \neq \emptyset$.*

Thus, by Theorem 9 both the first question for all $k \geq 4$ and the second question for all $k \geq 5$ have negative answers, although the former question has a positive answer for $k = 3$ and the latter one has a positive answer for $k = 4$.

4. Proof of Theorem 9

First of all we prove the following lemmas.

Lemma 2. *Let* $L = S(I(L) \cup K(L), E(L))$ *be the split graph with*

$$\begin{aligned} I(L) &= \{u_1^*, u_2^*, \ldots, u_6^*\}, \\ K(L) &= \{v_1^*, v_2^*, \ldots, v_7^*\}, \\ E(L) &= E_1^* \cup E_2^* \cup \cdots \cup E_6^* \cup E_K^*, \end{aligned}$$

where

$$\begin{aligned} E_1^* &= \{u_1^*v_1^*, u_1^*v_2^*, u_1^*v_3^*\}, \\ E_2^* &= \{u_2^*v_2^*, u_2^*v_4^*\}, \\ E_3^* &= \{u_3^*v_3^*, u_3^*v_4^*, u_3^*v_6^*\}, \\ E_4^* &= \{u_4^*v_1^*, u_4^*v_4^*, u_4^*v_7^*\}, \\ E_5^* &= \{u_5^*v_2^*, u_5^*v_5^*, u_5^*v_7^*\}, \\ E_6^* &= \{u_6^*v_3^*, u_6^*v_7^*\}, \\ E_K^* &= \{v_i^*v_j^* \mid i \neq j; i, j \in \{1, \ldots, 7\}\} \end{aligned}$$

(see Fig. 1). *Then* L *is a maximal nonhamiltonian Burkard–Hammer graph with* $B_{|I(L)|} = \emptyset$.

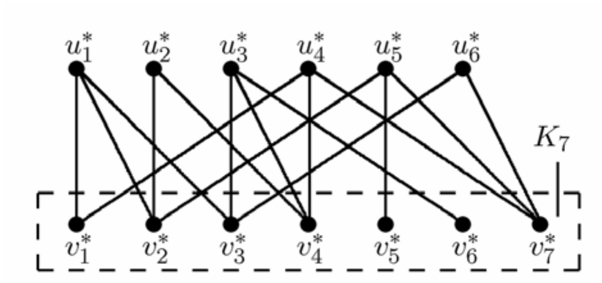

Fig. 1. The graph L

Table 2. The Hamilton cycle for $L - u_i^*$

Graph $L - u_i^*$	Hamilton cycle $C_{u_i^*}$ for $L - u_i^*$
$L - u_1^*$	$C_{u_1^*} = u_2^*v_2^*u_5^*v_5^*v_3^*u_6^*v_7^*u_4^*v_1^*v_6^*u_3^*v_4^*u_2^*$
$L - u_2^*$	$C_{u_2^*} = u_1^*v_1^*u_4^*v_4^*u_3^*v_6^*v_2^*v_5^*u_5^*v_7^*u_6^*v_3^*u_1^*$
$L - u_3^*$	$C_{u_3^*} = u_1^*v_2^*u_2^*v_4^*u_4^*v_1^*v_6^*v_5^*u_5^*v_7^*u_6^*v_3^*u_1^*$
$L - u_4^*$	$C_{u_4^*} = u_1^*v_1^*v_3^*u_6^*v_7^*u_5^*v_5^*v_6^*u_3^*v_4^*u_2^*v_2^*u_1^*$
$L - u_5^*$	$C_{u_5^*} = u_1^*v_1^*u_4^*v_7^*u_6^*v_3^*v_5^*v_6^*u_3^*v_4^*u_2^*v_2^*u_1^*$
$L - u_6^*$	$C_{u_6^*} = u_1^*v_1^*u_4^*v_7^*u_5^*v_5^*v_3^*v_6^*u_3^*v_4^*u_2^*v_2^*u_1^*$

Proof. For any vertex $u_i^* \in I(L)$, the graph $L - u_i^*$ has a Hamilton cycle $C_{u_i^*}$ which is shown in Table 2. Therefore, by Theorem 1 the Burkard–Hammer condition holds for any $\emptyset \neq I' \subseteq I(L)$ and $K' \subseteq N_L(I')$ with $|I'| \leq 5$ and $(k(I', K'), h(I', K')) \neq (0, 0)$. For $I' = I(L)$ and $K' \subseteq N_L(I(L))$, by direct computations we can verify that the Burkard–Hammer condition also holds. (It is tedious to do this, but we don't know other ways to verify the last assertion.) Thus, L satisfies the Burkard–Hammer condition.

Now suppose that L has a Hamilton cycle C. Since $\deg(u_2^*) = \deg(u_6^*) = 2$, C must contain the paths $v_2^*u_2^*v_4^*$ and $v_3^*u_6^*v_7^*$. We consider separately the following possibilities for C:

(i) $v_2^*u_1^*v_3^*$ is in C.

In this case C must contain the path $v_4^*u_2^*v_2^*u_1^*v_3^*u_6^*v_7^*$. So both $v_2^*u_5^*$ and $v_3^*u_3^*$ cannot be in C. Therefore, $v_5^*u_5^*v_7^*$ and $v_4^*u_3^*v_6^*$ must be in C because $\deg(u_3^*) = \deg(u_5^*) = 3$. It follows that both $u_4^*v_4^*$ and $u_4^*v_7^*$ cannot be in C. Hence, u_4^* is not in C because $\deg(u_4^*) = 3$, contradicting our assumption that C is a Hamilton cycle of L. Thus, this case cannot occur.

(ii) $v_1^*u_1^*v_2^*$ is in C.

In this case, C must contain the path $v_1^*u_1^*v_2^*u_2^*v_4^*$. Therefore, $v_2^*u_5^*$ cannot be in C. Since $\deg(u_5^*) = 3$, $v_5^*u_5^*v_7^*$ must be in C. It follows that $v_7^*u_4^*$ cannot be in C because $v_7^*u_5^*$ and $v_7^*u_6^*$ are already in C. So, $v_1^*u_4^*v_4^*$ must be in C because $\deg(u_4^*) = 3$. Thus, $v_1^*u_1^*v_2^*u_2^*v_4^*u_4^*v_1^*$ is a proper subcycle of C, which is impossible. This means that this case also cannot occur.

(iii) $v_1^*u_1^*v_3^*$ is in C.

By arguments similar to those of Case (ii), we can get a contradiction for this case. Hence, this case also cannot occur.

Thus, the assumption that L has a Hamilton cycle is false. So L must be nonhamiltonian.

Now we prove that L is a maximal nonhamiltonian split graph. Since L is nonhamiltonian as we have proved above, it remains to prove that $L + u_i^*v_j^*$ is hamiltonian for any $u_i^*v_j^* \notin E(L)$ where $u_i^* \in I(L)$ and $v_j^* \in K(L)$. This is done in Table 3.

Finally, the fact that $B_{|I(L)|} = \emptyset$ is trivial. The proof of Lemma 2 is complete.

■

Lemma 3. *Let $H^{4,6}$ be a graph defined in Table 1 and $X = S(I(X) \cup K(X), E(X))$ be the complete split graph with $I(X) = \{u_{x,1}\}$ and $K(X) = \{v_{x,1}, v_{x,2}\}$. Then the graph*

$$T = S(I(T) \cup K(T), E(T)) = H^{4,6}[X, v_1^*] + u_{x,1}v_2^*$$

(see Fig. 2) *is a maximal nonhamiltonian Burkard–Hammer graph with $B_4(T) = \emptyset$ but $B_3(T) \neq \emptyset, B_2(T) \neq \emptyset$ and $B_1(T) \neq \emptyset$.*

Proof. The following assertions (a) and (b) are true for T.

Table 3. The Hamilton cycle for $L + u_i^* v_j^*$

Graph $L + u_i^* v_j^*$	Hamilton cycle $C_{u_i^* v_j^*}$ for $L + u_i^* v_j^*$
$L + u_1^* v_4^*$	$C_{u_1^* v_4^*} = u_1^* v_1^* u_4^* v_7^* u_6^* v_3^* u_3^* v_6^* v_5^* u_5^* v_2^* u_2^* v_4^* u_1^*$
$L + u_1^* v_5^*$	$C_{u_1^* v_5^*} = u_1^* v_1^* u_4^* v_4^* u_2^* v_2^* u_5^* v_7^* u_6^* v_3^* u_3^* v_6^* v_5^* u_1^*$
$L + u_1^* v_6^*$	$C_{u_1^* v_6^*} = u_1^* v_1^* u_4^* v_4^* u_2^* v_2^* u_5^* v_5^* v_7^* u_6^* v_3^* u_3^* v_6^* u_1^*$
$L + u_1^* v_7^*$	$C_{u_1^* v_7^*} = u_1^* v_1^* u_4^* v_4^* u_2^* v_2^* u_5^* v_5^* v_6^* u_3^* v_3^* u_6^* v_7^* u_1^*$
$L + u_2^* v_1^*$	$C_{u_2^* v_1^*} = u_1^* v_1^* u_2^* v_4^* u_4^* v_7^* u_6^* v_3^* u_3^* v_6^* v_5^* u_5^* v_2^* u_1^*$
$L + u_2^* v_3^*$	$C_{u_2^* v_3^*} = u_1^* v_1^* u_4^* v_4^* u_3^* v_6^* v_5^* u_5^* v_7^* u_6^* v_3^* u_2^* v_2^* u_1^*$
$L + u_2^* v_5^*$	$C_{u_2^* v_5^*} = u_1^* v_1^* u_4^* v_4^* u_2^* v_5^* v_6^* u_3^* v_3^* u_6^* v_7^* u_5^* v_2^* u_1^*$
$L + u_2^* v_6^*$	$C_{u_2^* v_6^*} = u_1^* v_1^* u_4^* v_4^* u_3^* v_3^* u_6^* v_7^* u_5^* v_5^* v_6^* u_2^* v_2^* u_1^*$
$L + u_2^* v_7^*$	$C_{u_2^* v_7^*} = u_1^* v_1^* u_4^* v_4^* u_2^* v_7^* u_6^* v_3^* u_3^* v_6^* v_5^* u_5^* v_2^* u_1^*$
$L + u_3^* v_1^*$	$C_{u_3^* v_1^*} = u_1^* v_2^* u_2^* v_4^* u_4^* v_1^* u_3^* v_6^* v_5^* u_5^* v_7^* u_6^* v_3^* u_1^*$
$L + u_3^* v_2^*$	$C_{u_3^* v_2^*} = u_1^* v_1^* u_4^* v_4^* u_2^* v_2^* u_3^* v_6^* v_5^* u_5^* v_7^* u_6^* v_3^* u_1^*$
$L + u_3^* v_5^*$	$C_{u_3^* v_5^*} = u_1^* v_1^* u_4^* v_4^* u_2^* v_2^* v_6^* u_3^* v_5^* u_5^* v_7^* u_6^* v_3^* u_1^*$
$L + u_3^* v_7^*$	$C_{u_3^* v_7^*} = u_1^* v_1^* u_4^* v_4^* u_2^* v_2^* u_5^* v_5^* v_6^* u_3^* v_7^* u_6^* v_3^* u_1^*$
$L + u_4^* v_2^*$	$C_{u_4^* v_2^*} = u_1^* v_1^* u_4^* v_2^* u_2^* v_4^* u_3^* v_6^* v_5^* u_5^* v_7^* u_6^* v_3^* u_1^*$
$L + u_4^* v_3^*$	$C_{u_4^* v_3^*} = u_1^* v_1^* u_4^* v_3^* u_6^* v_7^* u_5^* v_5^* v_6^* u_3^* v_4^* u_2^* v_2^* u_1^*$
$L + u_4^* v_5^*$	$C_{u_4^* v_5^*} = u_1^* v_1^* u_4^* v_5^* u_5^* v_2^* u_2^* v_4^* u_3^* v_6^* v_7^* u_6^* v_3^* u_1^*$
$L + u_4^* v_6^*$	$C_{u_4^* v_6^*} = u_1^* v_1^* u_4^* v_6^* v_5^* u_5^* v_7^* u_6^* v_3^* u_3^* v_4^* u_2^* v_2^* u_1^*$
$L + u_5^* v_1^*$	$C_{u_5^* v_1^*} = u_1^* v_1^* u_5^* v_5^* v_6^* u_3^* v_3^* u_6^* v_7^* u_4^* v_4^* u_2^* v_2^* u_1^*$
$L + u_5^* v_3^*$	$C_{u_5^* v_3^*} = u_1^* v_1^* u_4^* v_7^* u_6^* v_3^* u_5^* v_5^* v_6^* u_3^* v_4^* u_2^* v_2^* u_1^*$
$L + u_5^* v_4^*$	$C_{u_5^* v_4^*} = u_1^* v_1^* u_4^* v_7^* u_6^* v_3^* u_3^* v_6^* v_5^* u_5^* v_4^* u_2^* v_2^* u_1^*$
$L + u_5^* v_6^*$	$C_{u_5^* v_6^*} = u_1^* v_1^* u_4^* v_7^* u_6^* v_3^* v_5^* u_5^* v_6^* u_3^* v_4^* u_2^* v_2^* u_1^*$
$L + u_6^* v_1^*$	$C_{u_6^* v_1^*} = u_1^* v_1^* u_6^* v_3^* u_3^* v_6^* v_5^* u_5^* v_7^* u_4^* v_4^* u_2^* v_2^* u_1^*$
$L + u_6^* v_2^*$	$C_{u_6^* v_2^*} = u_1^* v_1^* u_4^* v_7^* u_5^* v_5^* v_6^* u_3^* v_4^* u_2^* v_2^* u_6^* v_3^* u_1^*$
$L + u_6^* v_4^*$	$C_{u_6^* v_4^*} = u_1^* v_1^* u_4^* v_7^* u_5^* v_5^* v_6^* u_3^* v_3^* u_6^* v_4^* u_2^* v_2^* u_1^*$
$L + u_6^* v_5^*$	$C_{u_6^* v_5^*} = u_1^* v_1^* u_4^* v_4^* u_2^* v_2^* u_5^* v_7^* u_6^* v_5^* v_6^* u_3^* v_3^* u_1^*$
$L + u_6^* v_6^*$	$C_{u_6^* v_6^*} = u_1^* v_1^* u_4^* v_7^* u_6^* v_6^* v_5^* u_5^* v_2^* u_2^* v_4^* u_3^* v_3^* u_1^*$

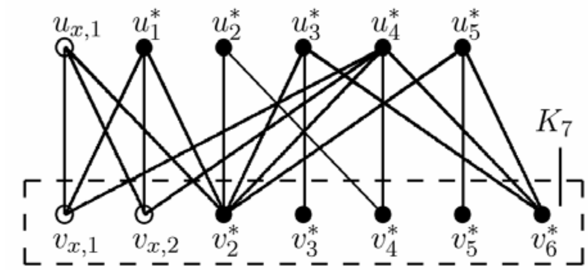

Fig. 2. The graph T

(a) *T is a Burkard–Hammer graph.*

In fact, since $H^{4,6}$ is a Burkard–Hammer graph, by Theorem 3 the graph $H^{4,6}[X, v_1^*]$ is a Burkard–Hammer graph. Therefore, by Lemma 1 the graph T is a Burkard–Hammer graph.

(b) *T is a maximal nonhamiltonian split graph.*

Since $H^{4,6}$ is nonhamiltonian, by Theorem 4 the graph $H^{4,6}[X, v_1^*]$ is nonhamiltonian. Therefore, if T has a Hamilton cycle C then C must contain the edge $u_{x,1}v_2^*$. So C must contain the path $u_{x,1}v_2^*u_2^*v_4^*$ because $N_T(u_2^*) = \{v_2^*, v_4^*\}$. It follows that the edges $u_1^*v_2^*, u_3^*v_2^*, u_5^*v_2^*$ are not in C. Hence, C must contain the paths $v_{x,1}u_1^*v_{x,2}$ and $v_3^*u_3^*v_6^*u_5^*v_5^*$ because u_1^*, u_3^* and u_5^* have degree 3 in T. From these facts we see that both $u_4^*v_2^*$ and $u_4^*v_6^*$ cannot be in C. Now if $u_{x,1}v_{x,1}$ is in C then $u_4^*v_{x,1}$ also cannot be in C because the edges $u_{x,1}v_{x,1}$ and $u_1^*v_{x,1}$ are already in C. Therefore $C_1 = u_{x,1}v_2^*u_2^*v_4^*u_4^*v_{x,2}u_1^*v_{x,1}u_{x,1}$ is a proper subcycle of C, a contradiction. Similarly, if $u_{x,1}v_{x,2}$ is in C then $u_4^*v_{x,2}$ cannot be in C and therefore $C_2 = u_{x,1}v_2^*u_2^*v_4^*u_4^*v_{x,1}u_1^*v_{x,2}u_{x,1}$ is a proper subcycle of C, a contradiction again. Thus, T must be nonhamiltonian.

To prove Assertion (b) it remains to prove that $T + uv$ is hamiltonian for every $uv \notin E(T)$ where $u \in I(T)$ and $v \in K(T)$.

First suppose that $u \in I^*$ and $v \in K^* \setminus \{v_1^*\}$. Then uv also is not an edge of $H^{4,6}$. Since $H^{4,6}$ is a maximal nonhamiltonian split graph by Theorem 2, the graph $H^{4,6} + uv$ is hamiltonian. Therefore, $(H^{4,6} + uv)[X, v_1^*]$ is hamiltonian by Theorem 4 because the graph X trivially has a Hamilton cycle. It is clear that in this case $T + uv = (H^{4,6} + uv)[X, v_1^*] + u_{x,1}v_2^*$. Hence, $T + uv$ is hamiltonian if $u \in I^*$ and $v \in K^* \setminus \{v_1^*\}$.

Next suppose that $u \in I^*$ and $v \in \{v_{x,1}, v_{x,2}\}$. Then u is not adjacent to v_1^* in $H^{4,6}$. Since $H^{4,6}$ is a maximal nonhamiltonian split graph, $H^{4,6} + uv_1^*$ has a Hamilton cycle C containing the edge uv_1^*. Now it is not difficult to see that if $v = v_{x,1}$ (resp., $v = v_{x,2}$) then we can get a Hamilton cycle for $T + uv$ by replacing the vertex v_1^* in C with the path $v_{x,1}u_{x,1}v_{x,2}$ (resp., $v_{x,2}u_{x,1}v_{x,1}$).

Finally suppose that $u = u_{x,1}$ and v is one of the vertices v_3^*, v_4^*, v_5^* or v_6^*. Then

$$C_3 = u_{x,1}v_3^*u_3^*v_6^*u_5^*v_5^*v_2^*u_2^*v_4^*u_4^*v_{x,2}u_1^*v_{x,1}u_{x,1},$$

$$C_4 = u_{x,1}v_4^*u_2^*v_2^*u_3^*v_3^*v_5^*u_5^*v_6^*u_4^*v_{x,2}u_1^*v_{x,1}u_{x,1},$$

$$C_5 = u_{x,1}v_5^*u_5^*v_6^*u_3^*v_3^*v_2^*u_2^*v_4^*u_4^*v_{x,2}u_1^*v_{x,1}u_{x,1}$$

and

$$C_6 = u_{x,1}v_6^*u_5^*v_5^*v_3^*u_3^*v_2^*u_2^*v_4^*u_4^*v_{x,2}u_1^*v_{x,1}u_{x,1}$$

are Hamilton cycles of $T+u_{x,1}v_3^*, T+u_{x,1}v_4^*, T+u_{x,1}v_5^*$ and $T+u_{x,1}v_6^*$, respectively.

Thus, T is a maximal nonhamiltonian split graph.

By Assertions (a) and (b) the graph $T = S(I(T)\cup K(T), E(T)) = H^{4,6}[X, v_1^*] +u_{x,1}v_2^*$ is a maximal nonhamiltonian Burkard–Hammer graph. Furthermore, it is clear that $B_4(T) = \emptyset$ but $B_3(T) \neq \emptyset, B_2(T) \neq \emptyset$ and $B_1(T) \neq \emptyset$.

The proof of Lemma 12 is complete. ■

Lemma 4. *Let $T = S(I(T) \cup K(T), E(T))$ be the maximal nonhamiltonian Burkard–Hammer graph constructed in Lemma* 3 *and $Y_t = S(I(Y_t)\cup K(Y_t), E(Y_t))$ be a complete split graph with $I(Y_t) = \{u_{y,1}, u_{y,2}, \ldots, u_{y,t}\}$ and $K(Y_t) = \{v_{y,1}, v_{y,2}, \ldots, v_{y,t}, v_{y,t+1}\}$ where $t \geq 1$ is an integer. Then the graph $H_t = S(I(H_t) \cup K(H_t), E(H_t)) = T[Y_t, v_2^*]$ is a maximal nonhamiltonian Burkard–Hammer graph with $|I(H_t)| = 6 + t, \delta(H_t) = t + 1 = |I(H_t)| - 5$. Moreover, $B_4(H_t) = \emptyset$ but $B_3(H_t) \neq \emptyset, B_2(H_t) \neq \emptyset$ and $B_1(H_t) \neq \emptyset$.*

Proof. By Lemma 3, graph T is a nonhamiltonian Burkard–Hammer graph. Therefore, by Theorems 3 and 4, the graph H_t is a nonhamiltonian Burkard–Hammer graph. We prove now that $H_t + uv$ is hamiltonian for every $uv \notin E(H_t)$ where $u \in I(H_t)$ and $v \in K(H_t)$. There are two separate cases to consider.

Case 1: $u \in I(T), v \in K(T) \setminus \{v_2^*\}$.

In this case, $uv \notin E(T)$ and $H_t + uv = (T + uv)[Y_t, v_2^*]$. Since T is a maximal nonhamiltonian Burkard–Hammer graph by Lemma 3, the graph $T+uv$ is hamiltonian. The graph $Y_t = S(I(Y_t) \cup K(Y_t), E(Y_t))$ is also hamiltonian because it is a complete split graph with $|K(Y_t)| = |I(Y_t)| + 1$. By Theorem 4, the graph $(T + uv)[Y_t, v_2^*]$ has a Hamilton cycle. Hence, the graph $H_t + uv$ is hamiltonian.

Case 2: $u \in I(Y_t), v \in K(T) \setminus \{v_2^*\}$.

Since $v \in K(T) \setminus \{v_2^*\}$, we have $|N_{I(T)}(v)| \leq 3$. Therefore, there exists a vertex $w \in I(T)$ such that $wv \notin E(T)$. By Case 1, the graph $H_t + wv$ has a Hamilton cycle C which must contain the edge wv because H_t is nonhamiltonian. Let $\overrightarrow{C}$ be the cycle C with an orientation. By $\overleftarrow{C}$ we denote the cycle C with the reverse orientation. If $x, y \in V(C)$, then $x\overrightarrow{C}y$ denotes the consecutive vertices of C from x to y in the direction specified by $\overrightarrow{C}$. The same vertices in the reverse order are given by $y\overleftarrow{C}x$. If $x \in V(C)$ then x^+ denotes the successor of x on $\overrightarrow{C}$, and x^- denotes its predecessor. Without loss of generality, we may assume that $w^+ = v$ in $\overrightarrow{C}$. By the definitions of T and $T[Y_t, v_2^*]$, vertex w is adjacent to both u^+ and u^-. Therefore, $C' = v\overrightarrow{C}u^-w\overrightarrow{C}uv$ is a Hamilton cycle in $H_t + uv$.

Thus, $H_t + uv$ is hamiltonian for every $uv \notin E(H_t)$ where $u \in I(H_t)$ and $v \in K(H_t)$. Therefore, H_t is a maximal nonhamiltonian split graph. Further, we have

$$|I(H_t)| = |I(T)| + |I(Y_t)| = 6 + t,$$

$$\delta(H_t) = |K(Y_t)| = t + 1 = |I(H_t)| - 5.$$

It is also clear that $B_4(H_t) = \emptyset$ but $B_3(H_t) \neq \emptyset, B_2(H_t) \neq \emptyset$ and $B_1(H_t) \neq \emptyset$. The proof of Lemma 4 is complete. ■

Proof of Theorem 9.

(a) Let $k = 4$. Then the graph $L = S(I(L) \cup K(L), E(L))$ of Lemma 2 is a maximal nonhamiltonian Burkard–Hammer graph with $\delta(L) = 2 = |I(L)| - 4$ and $B_{|I(L)|} = \emptyset$. Thus, Assertion (a) is true for $k = 4$.

Now suppose that $k > 4$. Let $G_1 = S(I_1 \cup K_1, E_1)$ be a complete split graph with $|K_1| > |I_1| = k-4$ and v be a vertex of K_1. Since the graph L of Lemma 2 is a maximal nonhamiltonian Burkard–Hammer graph which has $N_L(u) \neq K(L)$ for every $u \in I(L)$, by Theorem 6 the graph $G = S(I \cup K, E) = G_1[L, v]$ is a maximal nonhamiltonian Burkard–Hammer graph with $\delta(G) = \delta(L) = |I| - (4 + |I_1|) = |I| - k$. Moreover, by Theorem 5 and Lemma 2, $B_{|I|} = \emptyset$. Thus, Assertion (a) is also true for $k > 4$.

(b) Let $k = 5$ and m be an integer with $m > 7$. Further, let $H_t = T[Y_t, v_2^*]$ be a graph constructed from T and Y_t with $|I(Y_t)| = t = m - 6$ as in Lemma 4. Then by this lemma, the graph H_t is a maximal nonhamiltonian Burkard–Hammer graph with $|I(H_t)| = |I(T)| + |I(Y_t)| = 6 + (m-6) = m$ and $\delta(H_t) = |I(H_t)| - 5$. Also by Lemma 4, $B_4(H_t) = \emptyset$ but $B_3(H_t) \neq \emptyset, B_2(H_t) \neq \emptyset$ and $B_1(H_t) \neq \emptyset$. Thus, Assertion (b) is true for $k = 5$ and any integer $m > 7$.

Now suppose that k and m are integers with $k \geq 6$ and $m > k + 2$. Let $G_1 = S(I_1 \cup K_1, E_1)$ be a complete split graph with $|K_1| > |I_1| = k - 5$ and v be a vertex of K_1. Further, let $G_2 = S(I_2 \cup K_2, E_2)$ be the graph $H_l = T[Y_l, v_2^*]$ defined in Lemma 4 where $l = m - k - 1$. Then by Lemma 4, the graph G_2 is a maximal nonhamiltonian Burkard–Hammer graph with $|I_2| = |I(H_l)| = m-k+5, \delta(G_2) = \delta(H_l) = |I(G_2)|-5$ and $B_4(G_2) = \emptyset$ but $B_3(G_2) \neq \emptyset, B_2(G_2) \neq \emptyset, B_1(G_2) \neq \emptyset$. Moreover, it is clear that for every vertex $u \in I_2$, $N_{G_2}(u) \neq K_2$. Therefore, by Theorem 6 the graph $G = S(I \cup K, E) = G_1[G_2, v]$ is a maximal nonhamiltonian Burkard–Hammer graph. Further, we have $|I| = |I_1| + |I_2| = (k - 5) + (m - k + 5) = m$ and by Theorem 5 and Lemma 4

$$\begin{aligned}
\delta(G) = \delta(G_2) &= |I| - (5 + |I_1|) = |I| - k, \\
B_{k-1}(G) &= B_{4+|I_1|}(G) = \emptyset, \\
B_{k-2}(G) &= B_{3+|I_1|}(G) \neq \emptyset, \\
B_{k-2}(G) &= B_{2+|I_1|}(G) \neq \emptyset \text{ and} \\
B_{k-4}(G) &= B_{1+|I_1|}(G) \neq \emptyset.
\end{aligned}$$

Thus, Assertion (b) is also true for any $k \geq 6$ and $m > k + 2$.

The proof of Theorem 10 is complete. ■

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