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Hopf-Lax-Oleinik-Type Estimates for Viscosity Solutions to Hamilton-Jacobi Equations with Concave-Convex Data

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Abstract. We consider the Cauchy problem to Hamilton-Jacobi equations with either concave-convex Hamiltonian or concave-convex initial data and investigate their explicit viscosity solutions in connection with Hopf-Lax-Oleinik-type estimates.

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1. Introduction

Since the early 1980s, the concept of *viscosity solutions* introduced by Crandall and Lions [16] has been used in a large portion of research in a nonclassical theory of first-order nonlinear PDEs as well as in other types of PDEs. For convex Hamilton-Jacobi equations, the viscosity solution-characterized by a semiconcave stability condition, was first introduced by Kruzkov [35]. There is an enormous activity which is based on these studies. The primary virtues of this theory are that it allows merely nonsmooth functions to be solutions of nonlinear PDEs, it provides very general existence and uniqueness theorems, and it yields precise formulations of general boundary conditions. Let us mention here the names: Crandall, Lions, Evans, Ishii, Jensen, Barbu, Bardi, Barles, Barron, Cappuzzo-Dolcetta, Dupuis, Lenhart, Osher, Perthame, Soravia, Souganidis,

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Tataru, Tomita, Yamada, and many others, whose contributions make great progress in nonlinear PDEs. The concept of viscosity solutions is motivated by the classical maximum principle which distinguishes it from other definitions of generalized solutions.

In this paper we consider the Cauchy problem for Hamilton-Jacobi equation, namely,

$$u_t + H(u, Du) = 0 \quad \text{in} \quad \{t > 0,\ x \in \mathbb{R}^n\}, \tag{1}$$

$$u(0, x) = \phi(x) \quad \text{on} \quad \{t = 0,\ x \in \mathbb{R}^n\}. \tag{2}$$

Bardi and Evans [7], [21] and Lions [39] showed that the formulas

$$u(t, x) = \min_{y \in \mathbb{R}^n} \big\{\phi(y) + t \cdot H^*\big((x - y)/t\big)\big\}. \tag{1*}$$

and

$$u(t, x) = \max_{p \in \mathbb{R}^n} \{\langle p, x\rangle - \phi^*(p) - tH(p)\} \tag{2*}$$

give the unique Lipschitz viscosity solution of (1)-(2) under the assumptions that H depends only on $p := Du$ and is convex and ϕ is uniformly Lipschitz continuous for (1*) and H is continuous and ϕ is convex and Lipschitz continuous for (2*). Furthemore, Bardi and Faggian [8] proved that the formula (1*) is still valid for unique viscosity solution whenever H is convex and ϕ is uniformly continuous.

Lions and Rochet [41] studied the multi-time Hamilton-Jacobi equations and obtained a Hopf-Lax-Oleinik type formula for these equations.

The Hopf-Lax-Oleinik type formulas for the Hamilton-Jacobi equations (1) were found in the papers by Barron, Jensen, and Liu [13 - 15], where the first and second conjugates for quasiconvex funcions - functions whose level set are convex - were successfully used.

The paper by Alvarez, Barron, and Ishii [4] is concerned with finding Hopf-Lax-Oleinik type formulas of the problem (1)-(2)with $(t, x) \in (0, \infty) \times \mathbb{R}^n$, when the initial function ϕ is only lower semicontinuous (l.s.c.), and possibly infinite. If $H(\gamma, p)$ is convex in p and the initial data ϕ is quasiconvex and l.s.c., the Hopf-Lax-Oleinik type formula gives the l.s.c. solution of the problem (1)-(2). If the assumption of convexity of $p \to H(\gamma, p)$ is dropped, it is proved that $u = (\phi^{\#} + tH)^{\#}$ still is characterized as the minimal l.s.c. supersolution (here, # means the second quasiconvex conjugate, see [12 - 13]).

The paper [77] is a survey of recent results on Hopf-Lax-Oleinik type formulas for viscosity solutions to Hamilton-Jacobi equations obtained mainly by the author and Thanh in cooperation with Gorenflo and published in Van-Thanh-Gorenflo [69], Van-Thanh [70], Van-Thanh [72]...

Let us mention that if H is a concave-convex function given by a D.C representation

$$H(p', p'') := H_1(p') - H_2(p'')$$

and ϕ is uniformly continuous, Bardi and Faggian [8] have found explicit pointwise upper and lower bounds of Hopf-Lax-Oleinik type for the viscosity solutions. If the Hamiltonian $H(\gamma, p), (\gamma, p) \in \mathbb{R} \times \mathbb{R}^n$, is a D.C. function in p, i.e.,

$$H(\gamma, p) = H_1(\gamma, p) - H_2(\gamma, p), \quad (\gamma, p) \in \mathbb{R} \times \mathbb{R}^n,$$

Barron, Jensen and Liu [15] have given their Hopf-Lax-Oleinik type estimates for viscosity solutions to the corresponding Cauchy problem.

We also want to mention that the Hopf-Lax-Oleinik type and explicit formulas have obtained in the recent papers by Adimurthi and Gowda [1 - 3], Barles and Tourin [10], Barles [Bar1], Rockafellar and Wolenski [53], Joseph and Gowda [24 - 25], LeFloch [38], Manfredi and Stroffolini [43], Maslov and Kolokoltsov [44], Ngoan [47], Sachdev [58], Plaskacz and Quincampoix [51], Thai Son [55], Subbotin [61], Melikyan A. [46], Rublev [54], Silin [59], Stromberg [60],...

This paper is a servey of results on Hopf-Lax-Oleinik type and explicit formulas for the viscosity solutions of (1)-(2) with concave-convex data obtained by the authors, Thanh and Tho in [74, 55, 71, 75]. Namely, we propose to examine a class of *concave-convex functions* as a more general framework where the discussion of the global Legendre transformation still makes sense. Hopf-Lax-Oleinik-type formulas for Hamilton-Jacobi equations with concave-convex Hamiltonians (or with concave-convex initial data) can thereby be considered. The method here is a development of that in Chapter 4 [76], which involves the use of Lemmas 4.1-4.2 (and their generalizations). It is essentially different from the methods in [27, 47]. Also, the class of concave-convex functions under our consideration is larger than that in [47] since we do not assume the twice continuous differentiability condition on its functions.

We shall often suppose that $n := n_1 + n_2$ and that the variables $x,\ p \in \mathbb{R}^n$ are separated into two as $x := (x', x''),\ p := (p', p'')$ with $x',\ p' \in \mathbb{R}^{n_1},\ x'',\ p'' \in \mathbb{R}^{n_2}$. Accordingly, the zero-vector in $\mathbb{R}^n$ will be $0 = (0', 0'')$, where $0'$ and $0''$ stand for the zero-vectors in $\mathbb{R}^{n_1}$ and $\mathbb{R}^{n_2}$, respectively.

Definition 1.1 [Rock, p. 349] *A function $H = H(p', p'')$ is called Concave-convex function if it is a concave function of $p' \in \mathbb{R}^{n_1}$ for each $p'' \in \mathbb{R}^{n_2}$ and a convex function of $p'' \in \mathbb{R}^{n_2}$ for each $p' \in \mathbb{R}^{n_1}$.*

In the next section, conjugate concave-convex functions and their smoothness properties are investigated. Sec. 3 is devoted to the study of Hopf-Lax-Oleinik-type estimates for viscosity solutions in the case either of concave-convex Hamiltonians $H = H(p', p")$ or concave-convex initial data $g = g(x', x")$. In Sec. 4 we obtain Hopf-Lax-Oleinik-type estimates for viscosity solutions to the equations with D.C. Hamiltonians containing u, Du.

2. Conjugate Concave-Convex Functions

Let $H = H(p)$ be a differentiable real-valued function on an open nonempty subset A of $\mathbb{R}^n$. The *Legendre conjugate* of the pair (A, H) is defined to be the pair (B, G), where B is the image of A under the gradient mapping $z = \partial H(p)/\partial p$, and $G = G(z)$ is the function on B given by the formula

$$G(z) := \big\langle z, (\partial H/\partial p)^{-1}(z) \big\rangle - H\big((\partial H/\partial p)^{-1}(z)\big).$$

It is not actually necessary to have $z = \partial H(p)/\partial p$ one-to-one on A in order that $G = G(z)$ be well-defined (i.e., single-valued). It suffices if

$$\langle z, p^1\rangle - H(p^1) = \langle z, p^2\rangle - H(p^2)$$

whenever $\partial H(p^1)/\partial p = \partial H(p^2)/\partial p = z$. Then the value $G(z)$ can be obtained unambiguously from the formula by replacing the set $(\partial H/\partial p)^{-1}(z)$ by any of the vectors it contains.

Passing from (A, H) to the Legendre conjugate (B, G), if the latter is well-defined, is called the *Legendre transformation*. The important role played by the Legendre transformation in the classical local theory of nonlinear equations of first-order is well-known. The global Legendre transformation has been studied extensively for convex functions. In the case where $H = H(p)$ and A are convex, we can extend $H = H(p)$ to be a lower semicontinuous convex function on all of $\mathbb{R}^n$ with A as the interior of its effective domain. If this extended $H = H(p)$ is proper, then the Legendre conjugate (B, G) of (A, H) is well-defined. Moreover, B is a subset of $\operatorname{dom} H^*$ (namely the range of $\partial H/\partial p$), and $G = G(z)$ is the restriction of the Fenchel conjugate $H^* = H^*(z)$ to B. (See Theorem A.9; cf. also Lemma 4.3 in [76]).

For a class of C^2-concave-convex functions, Ngoan [47] has studied the global Legendre transformation and used it to give an explicit global Lipschitz solution to the Cauchy problem (1)-(2) with $H = H(p) = H(p', p'')$ in this class. He shows that in his class the (Fenchel-type) *upper* and *lower conjugates* [Rock, p. 389], in symbols $\bar{H}^* = \bar{H}^*(z', z'')$ and $\underline{H}^* = \underline{H}^*(z', z'')$, are the same as the Legendre conjugate $G = G(z', z'')$ of $H = H(p', p'')$.

Motivated by the above facts, we introduce in this section a wider class of concave-convex functions and investigate regularity properties of their *conjugates*. (Applications will be taken up in Secs. 3 and 4.).

All concave-convex functions $H = H(p', p'')$ under our consideration are assumed to be finite and to satisfy the following two "growth conditions."

$$\lim_{|p''| \to +\infty} \frac{H(p', p'')}{|p''|} = +\infty \quad \text{for each } \; p' \in \mathbb{R}^{n_1}. \tag{3}$$

$$\lim_{|p'| \to +\infty} \frac{H(p', p'')}{|p'|} = -\infty \quad \text{for each } \; ; p'' \in \mathbb{R}^{n_2}. \tag{4}$$

Let $H^{*_2} = H^{*_2}(p', z'')$ (resp. $H^{*_1} = H^{*_1}(z', p'')$) be, for each fixed $p' \in \mathbb{R}^{n_1}$ (resp. $p'' \in \mathbb{R}^{n_2}$), the Fenchel conjugate of a given p''-convex (resp. p'-concave) function $H = H(p', p'')$. In other words,

$$H^{*_2}(p', z'') := \sup_{p'' \in \mathbb{R}^{n_2}} \{\langle z'', p''\rangle - H(p', p'')\} \tag{5}$$

$$(\text{resp. } H^{*_1}(z', p'') := \inf_{p' \in \mathbb{R}^{n_1}} \{\langle z', p'\rangle - H(p', p'')\}) \tag{6}$$

for $(p', z'') \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ (resp. $(z', p'') \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$). If $H = H(p', p'')$ is concave-convex, then the definition (5) (resp. (6)) actually implies the convexity (resp. concavity) of $H^{*_2} = H^{*_2}(p', z'')$ (resp. $H^{*_1} = H^{*_1}(z', p'')$) not only in the variable $z'' \in \mathbb{R}^{n_2}$ (resp. $z' \in \mathbb{R}^{n_1}$) but also in the whole variable $(p', z'') \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ (resp. $(z', p'') \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$. Moreover, under the condition (3) (resp. (4)), the finiteness of $H = H(p', p'')$ clearly yields that of $H^{*_2} = H^{*_2}(p', z'')$ (resp. $H^{*_1} = H^{*_1}(z', p'')$) (cf. Remark 4, Chapter 4 in [76]) with

$$\lim_{|z''| \to +\infty} \frac{H^{*_2}(p', z'')}{|z''|} = +\infty \quad (\text{resp.} \lim_{|z'| \to +\infty} \frac{H^{*_1}(z', p'')}{|z'|} = -\infty)$$

locally uniformly in $p' \in \mathbb{R}^{n_1}$ (resp. $p'' \in \mathbb{R}^{n_2}$). To see this, fix any $0 < r_1, r_2 < +\infty$. As a finite concave-convex function, $H = H(p', p'')$ is continuous on $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ [52, Th. 35.1]; hence,

$$C_{r_1, r_2} := \sup_{\substack{|p''| \le r_2 \\ |p'| \le r_1}} |H(p', p'')| < +\infty. \tag{8}$$

So, with $p'' := r_2 z''/|z''|$ (resp. $p' := -r_1 z'/|z'|$), (5) (resp. (6)) together with (8) implies

$$\inf_{|p'| \le r_1} \frac{H^{*_2}(p', z'')}{|z''|} \ge r_2 - \frac{C_{r_1, r_2}}{|z''|} \to r_2 \quad \text{as} \quad |z''| \to +\infty$$

$$(\text{resp.} \sup_{|p''| \le r_2} \frac{H^{*_1}(z', p'')}{|z'|} \le -r_1 + \frac{C_{r_1, r_2}}{|z'|} \to -r_1 \quad \text{as} \quad |z'| \to +\infty).$$

Since r_1, r_2 are arbitrary, (7) must hold locally uniformly in $p' \in \mathbb{R}^{n_1}$ (resp. $p'' \in \mathbb{R}^{n_2}$) as required.

Remark 1. If (4) (resp. (3)) is satisfied, then (5) (resp. (6)) gives

$$\frac{H^{*_2}(p', z'')}{|p'|} \ge -\frac{H(p', 0'')}{|p'|} \to +\infty \quad \text{as} \quad |p'| \to +\infty \tag{9}$$

$$(\text{resp.} \frac{H^{*_1}(z', p'')}{|p''|} \le -\frac{H(0', p'')}{|p''|} \to -\infty \quad \text{as} \quad |p''| \to +\infty) \tag{10}$$

uniformly in $z'' \in \mathbb{R}^{n_2}$ (resp. $z' \in \mathbb{R}^{n_1}$).

Now let $H = H(p', p'')$ be a concave-convex function on $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$. Beside "partial conjugates" $H^{*_2} = H^{*_2}(p', z'')$ and $H^{*_1} = H^{*_1}(z', p'')$, we shall consider the following two "total conjugates" of $H = H(p', p'')$. The first one, which we denote by $\bar{H}^* = \bar{H}^*(z', z'')$, is defined as the Fenchel conjugate of the concave function $\mathbb{R}^{n_1} \ni p' \mapsto -H^{*_2}(p', z'')$; more precisely,

$$\bar{H}^*(z', z'') := \inf_{p' \in \mathbb{R}^{n_1}} \{\langle z', p' \rangle + H^{*_2}(p', z'')\} \tag{11}$$

for each $(z', z'') \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$. The second, $\underline{H}^* = \underline{H}^*(z', z'')$, is defined as the Fenchel conjugate of the convex function $\mathbb{R}^{n_2} \ni p'' \mapsto -H^{*_1}(z', p'')$; i.e.,

$$\underline{H}^*(z', z'') := \sup_{p'' \in \mathbb{R}^{n_2}} \{\langle z'', p'' \rangle + H^{*_1}(z', p'')\} \tag{12}$$

for $(z', z'') \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$. By (5)-(6) and (11)-(12), we have

$$\bar{H}^*(z', z'') = \inf_{p' \in \mathbb{R}^{n_1}} \sup_{p'' \in \mathbb{R}^{n_2}} \{\langle z', p' \rangle + \langle z'', p'' \rangle - H(p', p'')\}, \tag{13}$$

$$\underline{H}^*(z', z'') = \sup_{p'' \in \mathbb{R}^{n_2}} \inf_{p' \in \mathbb{R}^{n_1}} \{\langle z', p' \rangle + \langle z'', p'' \rangle - H(p', p'')\}. \tag{14}$$

Therefore, in accordance with [55, p. 389], $\bar{H}^* = \bar{H}^*(z', z'')$ and $\underline{H}^* = \underline{H}^*(z', z'')$ will be called the *upper* and *lower conjugates*, respectively, of $H = H(p', p'')$. (Of course, (13)-(14) imply $\bar{H}^*(z', z'') \geq \underline{H}^*(z', z'')$.) For any $z' \in \mathbb{R}^{n_1}$, the function

$$\mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \ni (p', z'') \mapsto h(p', z'') := \langle z', p' \rangle + H^{*_2}(p', z'')$$

is convex. Thus (11) shows that $\bar{H}^* = \bar{H}^*(z', z'')$ as a function of z'' is the *image*

$$\mathbb{R}^{n_2} \ni z'' \mapsto (Ah)(z'') := \inf\{h(p', z'') \ : \ A(p', z'') = z''\}$$

of $h = h(p', z'')$ *under* the (linear) projection $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \ni (p', z'') \mapsto A(p', z'') := z''$. It follows that $\bar{H}^* = \bar{H}^*(z', z'')$ is convex in $z'' \in \mathbb{R}^{n_2}$ [Theorem A.4] in [76]. On the other hand, by definition, $\bar{H}^* = \bar{H}^*(z', z'')$ is necessarily concave in $z' \in \mathbb{R}^{n_1}$. This upper conjugate is hence a concave-convex function on $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$. The same conclusion may dually be drawn for the lower conjugate $\underline{H}^* = \underline{H}^*(z', z'')$.

We have previously seen that if the concave-convex function $H = H(p', p'')$ is finite on the whole $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ and satisfies (3)-(4), its partial conjugates $H^{*_2} = H^{*_2}(p', z'')$ and $H^{*_1} = H^{*_1}(z', p'')$ must both be finite with (9)-(10) holding. Therefore, Remarks 8-9 in Chapter 4 [76] show that $\bar{H}^* = \bar{H}^*(z', z'')$ and $\underline{H}^* = \underline{H}^*(z', z'')$ are then also finite, and hence coincide by [53, Corollary 37.1.2]. In this situation, the *conjugate*

$$H^* = H^*(z', z'') := \bar{H}^*(z', z'') = \underline{H}^*(z', z'') \tag{15}$$

of $H = H(p', p'')$ will simultaneously have the properties:

$$\lim_{|z''| \to +\infty} \frac{H^*(z', z'')}{|z''|} = +\infty \quad \text{for each} \quad z' \in \mathbb{R}^{n_1}, \tag{16}$$

$$\lim_{|z'| \to +\infty} \frac{H^*(z', z'')}{|z'|} = -\infty \quad \text{for each} \quad z'' \in \mathbb{R}^{n_2}. \tag{17}$$

For the next discussions, the following technical preparations will be needed.

Lemma 2.1. *Let* $H = H(p', p'')$ *be a finite concave-convex function on* $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ *with the property* (3) (*resp.* (4)) *holding. Then*

$$\lim_{|p''| \to +\infty} \frac{H(p', p'')}{|p''|} = +\infty \ \textit{locally uniformly in} \ p' \in \mathbb{R}^{n_1} \tag{18}$$

$$\textit{(resp.} \ \lim_{|p'| \to +\infty} \frac{H(p', p'')}{|p'|} = -\infty \ \textit{locally uniformly in} \ p'' \in \mathbb{R}^{n_2}\textit{)}. \tag{19}$$

Proof. First, assume (3). According to the above discussions, (5) determines a finite convex function $H^{*_2} = H^{*_2}(p', z'')$. Further, Theorems A.6-A.7 in [76] shows that

$$H(p', p'') = \sup_{z'' \in \mathbb{R}^{n_2}} \{\langle z'', p''\rangle - H^{*_2}(p', z'')\} \tag{20}$$

for any $(p', p'') \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$. Let $0 < r, M < +\infty$ be arbitrarily fixed. As a finite convex function, $H^{*_2} = H^{*_2}(p', z'')$ is continuous (Theorem A.6 in [76]), and hence locally bounded. It follows that

$$C^{*_2}_{r,M} := \sup_{\substack{|z''| \le M \\ |p'| \le r}} |H^{*_2}(p', z'')| < +\infty. \tag{21}$$

So, with $z'' := Mp''/|p''|$, (20)-(21) imply that

$$\inf_{|p'| \le r} \frac{H(p', p'')}{|p''|} \ge M - \frac{C^{*_2}_{r,M}}{|p''|} \to M$$

as $|p''| \to +\infty$. Since $M > 0$ is arbitrary, we have

$$\lim_{|p''| \to +\infty} \inf_{|p'| \le r} \frac{H(p', p'')}{|p''|} = +\infty$$

for any $r \in (0, +\infty)$. Thus (18) holds. Analogously, (19) can be deduced from (4). ■

Definition 2.2. *A finite concave-convex function $H = H(p', p'')$ on $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ is said to be strict if its concavity in $p' \in \mathbb{R}^{n_1}$ and convexity in $p'' \in \mathbb{R}^{n_2}$ are both strict. It will then also be called a strictly concave-convex function on $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$.*

Lemma 2.3. *Let $H = H(p', p'')$ be a strictly concave-convex function on $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ with* (3) (*resp.* (4)) *holding. Then its partial conjugate $H^{*_2} = H^{*_2}(p', z'')$* (*resp.* $H^{*_1} = H^{*_1}(z', p'')$) *defined by* (5)(*resp.* (6)) *is strictly convex* (*resp. concave*) *in $p' \in \mathbb{R}^{n_1}$* (*resp. $p'' \in \mathbb{R}^{n_2}$*) *and everywhere differentiable in $z'' \in \mathbb{R}^{n_2}$* (*resp. $z' \in \mathbb{R}^{n_1}$*). *Beside that, the gradient mapping $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \ni (p', z'') \mapsto \partial H^{*_2}(p', z'')/\partial z''$* (*resp. $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \ni (z', p'') \mapsto \partial H^{*_1}(z', p'')/\partial z'$*) *is continuous and satisfies the identity*

$$\begin{aligned} H^{*_2}(p', z'') &\equiv \big\langle z'', \partial H^{*_2}(p', z'')/\partial z''\big\rangle - H\big(p', \partial H^{*_2}(p', z'')/\partial z''\big) \\ (\textit{resp. } H^{*_1}(z', p'') &\equiv \big\langle z', \partial H^{*_1}(z', p'')/\partial z'\big\rangle - H\big(\partial H^{*_1}(z', p'')/\partial z', p''\big)). \end{aligned} \tag{22}$$

Proof. For any finite concave-convex function $H = H(p', p'')$ satisfying the property (3) (resp. (4)), the partial conjugate $H^{*_2} = H^{*_2}(p', z'')$ (resp. $H^{*_1} = H^{*_1}(z', p'')$) is finite and convex (resp. concave) as has previously been proved.

Now, assume that $H = H(p', p'')$ is a strictly concave-convex function on $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ with (3) holding. Then Lemma 4.3 in [76] shows that $H^{*_2} = H^{*_2}(p', z'')$ must be differentiable in $z'' \in \mathbb{R}^{n_2}$ and satisfy (22). To obtain the continuity of the gradient mapping $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \ni (p', z'') \mapsto \partial H^{*_2}(p', z'')/\partial z''$, let us go to Lemmas 4.1-4.2 [76] and introduce the temporary notations: $n :=$

$n_2,\ E := \mathbb{R}^{n_2},\ m := n_1 + n_2,\ \mathcal{O} := \mathbb{R}^{n_1+n_2},\ \xi := (p', z'')$, and $p := p''$. It follows from (18) that the continuous function

$$\chi = \chi(\xi, p) = \chi(p', z'', p'') := \langle z'', p'' \rangle - H(p', p'') \tag{23}$$

meets Condition (i) of Lemma 4.1[76]. Therefore, by Lemma 4.2 and Remark 4 in Chapter 4 in [76], the nonempty-valued multifunction

$$L = L(\xi) = L(p', z'') := \{p'' \in \mathbb{R}^{n_2} \ : \ \chi(p', z'', p'') = H^{*_2}(p', z'')\}$$

should be upper semicontinuous. However, since $H = H(p', p'')$ is strictly convex in the variable $p'' \in \mathbb{R}^{n_2}$, (23) implies that $L = L(p', z'')$ is single-valued, and hence continuous in $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$. But $L(p', z'') = \{\partial H^{*_2}(p', z'')/\partial z''\}$, which may be handled by the same method as in the proof of Lemma 4.3 in [76] (we use Lemma 4.1[76], ignoring the variable p'). The continuity of $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \ni (p', z'') \mapsto \partial H^{*_2}(p', z'')/\partial z''$ has accordingly been established.

Next, let us claim that the convexity in $p' \in \mathbb{R}^{n_1}$ of $H^{*_2} = H^{*_2}(p', z'')$ is strict. To this end, fix $0 < \lambda < 1,\ z'' \in \mathbb{R}^{n_2}$ and $p', q' \in \mathbb{R}^{n_1}$. Of course, (5) and (23) yield

$$\begin{aligned}
&H^{*_2}(\lambda p' + (1-\lambda)q', z'') \\
&= \max_{p'' \in \mathbb{R}^{n_2}} \chi(\lambda p' + (1-\lambda)q', z'', p'') \\
&\le \max_{p'' \in \mathbb{R}^{n_2}} \{\lambda \chi(p', z'', p'') + (1-\lambda)\chi(q', z'', p'')\} \\
&\le \lambda \max_{p'' \in \mathbb{R}^{n_2}} \chi(p', z'', p'') + (1-\lambda) \max_{p'' \in \mathbb{R}^{n_2}} \chi(q', z'', p'') \\
&= \lambda H^{*_2}(p', z'') + (1-\lambda) H^{*_2}(q', z'').
\end{aligned}$$

If all the equalities simultaneously occur, then there must exist a point

$$p'' \in L(\lambda p' + (1-\lambda)q', z'') \cap L(p', z'') \cap L(q', z'')$$

with

$$\chi(\lambda p' + (1-\lambda)q', z'', p'') = \lambda \chi(p', z'', p'') + (1-\lambda)\chi(q', z'', p'');$$

hence (23) implies

$$H(\lambda p' + (1-\lambda)q', p'') = \lambda H(p', p'') + (1-\lambda) H(q', p'').$$

This would give $p' = q'$, and the convexity in $p' \in \mathbb{R}^{n_1}$ of $H^{*_2} = H^{*_2}(p', z'')$ is thereby strict.

By duality, one easily proves the remainder of the lemma. ■

We are now in a position to extend Lemma 4.3 in [76] to the case of conjugate concave-convex functions.

Proposition 2.4. *Let* $H = H(p', p'')$ *be a strictly concave-convex function on* $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ *with both* (3) *and* (4) *holding. Then its conjugate* $H^* = H^*(z', z'')$ *defined by* (11)-(15) *is also a concave-convex function satisfying* (16)-(17). *Moreover,* $H^* = H^*(z', z'')$ *is everywhere continuously differentiable with*

$$H^*(z',z'') \equiv \Big\langle z', \frac{\partial H^*}{\partial z'}(z',z'')\Big\rangle + \Big\langle z'', \frac{\partial H^*}{\partial z''}(z',z'')\Big\rangle \\ -H\Big(\frac{\partial H^*}{\partial z'}(z',z''), \frac{\partial H^*}{\partial z''}(z',z'')\Big). \tag{24}$$

Proof. For reasons explained just prior to Lemma 2.1, we see that $\bar{H}^*(z',z'') \equiv \underline{H}^*(z',z'')$, hence that (11)-(15) compatibly determine the conjugate $H^* = H^*(z',z'')$, which is a (finite) concave-convex function on $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ with (16)-(17) holding.

We now claim that $H^* = H^*(z',z'') = \underline{H}^*(z',z'')$ is continuously differentiable everywhere. For this, let us again go to Lemmas 4.1- 4.2 in [76] and introduce the temporary notations: $n := n_2$, $E := \mathbb{R}^{n_2}$, $m := n_1 + n_2$, $\mathcal{O} := \mathbb{R}^{n_1+n_2}$, $\xi := (z',z'')$, and $p := p''$. Since (10) has previously been deduced from (3), we can verify that the function

$$\chi = \chi(\xi,p) = \chi(z',z'',p'') := \langle z'',p''\rangle + H^{*_1}(z',p'') \tag{25}$$

meets Condition (i) of Lemma 4.1 in [76], while the other conditions are almost ready. In fact, as a finite concave function, $H^{*_1} = H^{*_1}(z',p'')$ is continuous (cf. Theorem A.6, in [76]) and so is $\chi = \chi(z',z'',p'')$ (cf. (25)); moreover, Condition (ii) follows from (25) and Lemma 2.3. Therefore, by (2) and (15), this Lemma shows that $H^* = H^*(z',z'') = \underline{H}^*(z',z'')$ should be directionally differentiable in $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ with

$$\partial_{(e',e'')} H^*(z',z'') = \max_{p'' \in L(z',z'')} \Big\{ \langle p'',e''\rangle + \Big\langle \frac{\partial H^{*_1}}{\partial z'}(z',p''), e'\Big\rangle \Big\} \tag{26}$$

for $(z',z'') \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$, $(e',e'') \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$, where

$$L = L(\xi) = L(z',z'') := \{p'' \in \mathbb{R}^{n_2} : \chi(z',z'',p'') = \underline{H}^*(z',z'') = H^*(z',z'')\} \tag{27}$$

is an upper semicontinuous multifunction (see Lemma 4.2 and Remark 4 in Chapter 4 [76]). However, because $H^{*_1} = H^{*_1}(z',p'')$ is strictly concave in $p'' \in \mathbb{R}^{n_2}$ (Lemma 2.3), it may be concluded from (25) and (27) that $L = L(z',z'')$ is single-valued, and thus continuous in $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$. Consequently, according to (26) and the continuity of the gradient mapping $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \ni (z',p'') \mapsto \partial H^{*_1}(z',p'')/\partial z'$ (Lemma 2.3), the maximum theorem [6, Theorem 1.4.16] implies that all the first-order partial derivatives of $H^* = H^*(z',z'')$ exist and are continuous in $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ (cf. also [63, Corollary 2.2]). The conjugate $H^* = H^*(z',z'')$ is hence everywhere continuously differentiable. In particular, since $L = L(z',z'')$ is single-valued, it follows from (26) that

$$L(z',z'') \equiv \Big\{\frac{\partial H^*}{\partial z''}(z',z'')\Big\},$$

and therefore that

$$\frac{\partial H^*}{\partial z'}(z',z'') \equiv \frac{\partial H^{*_1}}{\partial z'}\Big(z', \frac{\partial H^*}{\partial z''}(z',z'')\Big).$$

Thus, (25) and (27) combined give

$$H^*(z', z'') \equiv \Big\langle z'', \frac{\partial H^*}{\partial z''}(z', z'') \Big\rangle + H^{*_1}\Big(z', \frac{\partial H^*}{\partial z''}(z', z'') \Big).$$

Finally, we can invoke (22) to deduce that

$$\begin{aligned} H^*(z', z'') &\equiv \Big\langle z'', \frac{\partial H^*}{\partial z''}(z', z'') \Big\rangle + \Big\langle z', \frac{\partial H^{*_1}}{\partial z'}\Big(z', \frac{\partial H^*}{\partial z''}(z', z'') \Big) \Big\rangle \\ &\quad - H\Big(\frac{\partial H^{*_1}}{\partial z'}\Big(z', \frac{\partial H^*}{\partial z''}(z', z'') \Big), \frac{\partial H^*}{\partial z''}(z', z'') \Big) \\ &\equiv \Big\langle z'', \frac{\partial H^*}{\partial z''}(z', z'') \Big\rangle + \Big\langle z', \frac{\partial H^*}{\partial z'}(z', z'') \Big\rangle \\ &\quad - H\Big(\frac{\partial H^*}{\partial z'}(z', z''), \frac{\partial H^*}{\partial z''}(z', z'') \Big). \end{aligned}$$

The identity (24) has thereby been proved. This completes the proof. ∎

3. Hopf-Lax-Oleinik-Type Estimates for Viscosity Solutions

This Section is directly continuation of the Chapter 5 in [76], where we study the concave-convex Hamilton-Jacobi equations. Consider the Cauchy problem for the simplest Hamilton-Jacobi equation, namely,

$$u_t + H(Du) = 0 \quad \text{in} \quad U := \{t > 0,\ x \in \mathbb{R}^n\}, \tag{28}$$

$$u(0, x) = \phi(x) \quad \text{on} \quad \{t = 0,\ x \in \mathbb{R}^n\}. \tag{29}$$

Let us use the notations from Chapter 5 [76]: $\mathrm{Lip}(\bar{U}) := \mathrm{Lip}(U) \cap C(\bar{U})$, where $\mathrm{Lip}(U)$ is the set of all locally Lipschitz continuous functions $u = u(t, x)$ defined on U. A function $u \in \mathrm{Lip}(\bar{U})$ will be called a *global Lipschitz solution* of the Cauchy problem (28)-(29) if it satisfies (28) almost everywhere in U, with $u(0, \cdot) = \phi$. In [76, Chapter 5] we have got the Hopf-Lax-Oleinik- type formulas for global Lipschitz solutions of (28)-(29).

3.1. Estimates for Concave-Convex Hamiltonians

We still consider the Cauchy problem (28)-(29), but throughout this subsection ϕ is uniformly continuous, and $H = H(p', p'')$ is a general finite concave-convex function. Then this Hamiltonian H is continuous by [52, Theorem 35.1]. Therefore, it is known (see [22]) that the problem under consideration has a unique viscosity solution $u = u(t, x)$ in the space $UC_x\big([0, +\infty) \times \mathbb{R}^n\big)$ of the continuous functions which are uniformly continuous in x uniformly in t.

Without (3) (resp. (4)), the partial conjugate H^{*_2} (resp. H^{*_1}) defined in (5) (resp. (6)) is still, of course, convex (resp. concave), but might be infinite somewhere. One can claim only that

$$\begin{gathered} H^{*_2}(p', z'') > -\infty \quad \forall\, (p', z'') \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \\ (\text{resp. } H^{*_1}(z', p'') < +\infty \quad \forall\, (z', p'') \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}). \end{gathered}$$

Now, let

$$
\begin{aligned}
D_1 :=& \{z' \in \mathbb{R}^{n_1} : H^{*_1}(z', p'') > -\infty \quad \forall\, p'' \in \mathbb{R}^{n_2}\} \\
=& \{z' \in \mathbb{R}^{n_1} : \operatorname{dom} H^{*_1}(z', \cdot) = \mathbb{R}^{n_2}\} = \bigcap_{p'' \in \mathbb{R}^{n_2}} \operatorname{dom} H^{*_1}(\cdot, p''), \qquad (30) \\
D_2 :=& \{z'' \in \mathbb{R}^{n_2} \;:\; H^{*_2}(p', z'') < +\infty \quad \forall\, p' \in \mathbb{R}^{n_1}\} \\
=& \{z'' \in \mathbb{R}^{n_2} \;:\; \operatorname{dom} H^{*_2}(\cdot, z'') = \mathbb{R}^{n_1}\} = \bigcap_{p' \in \mathbb{R}^{n_1}} \operatorname{dom} H^{*_2}(p', \cdot), \qquad (31)
\end{aligned}
$$

and, for $(t, x) \in \bar{U}$, set

$$u_-(t,x) := \sup_{z' \in D_1} \min_{z'' \in \mathbb{R}^{n_2}} \{\phi(x - tz) + t \cdot \underline{H}^*(z)\}, \tag{32}$$

$$u_+(t,x) := \inf_{z'' \in D_2} \max_{z' \in \mathbb{R}^{n_1}} \{\phi(x - tz) + t \cdot \bar{H}^*(z)\}, \tag{33}$$

where the lower and upper conjugates, $\underline{H}^*$ and $\bar{H}^*$, of the Hamiltonian H are the concave-convex functions (with possibly infinite values) defined by (11)-(14). Clearly, if $(t, x) \in U$, we also have

$$
\begin{aligned}
u_-(t,x) &= \sup_{y' \in x' - t\cdot D_1} \min_{y'' \in \mathbb{R}^{n_2}} \{\phi(y) + t \cdot \underline{H}^*((x-y)/t)\}, \\
u_+(t,x) &= \inf_{y'' \in x'' - t\cdot D_2} s \max_{y' \in \mathbb{R}^{n_1}} \{\phi(y) + t \cdot \bar{H}^*((x-y)/t)\},
\end{aligned}
$$

cf. (5.30)-(5.31) in [76]. Our estimates for viscosity solutions in the case of general concave-convex Hamiltonians read as follows:

Theorem 3.1. *Let H be a (finite) concave-convex function, and ϕ be uniformly continuous. Then the unique viscosity solution $u \in UC_x\big([0, +\infty) \times \mathbb{R}^n\big)$ of the Cauchy problem* (28)-(29) *satisfies on $\bar{U}$ the inequalities*

$$u_-(t,x) \le u(t,x) \le u_+(t,x),$$

where u_- and u_+ are defined by (32)-(33).

Proof. For each $\underline{z}' \in D_1$, let $F(p, \underline{z}') = F(p', p'', \underline{z}') := \langle \underline{z}', p' \rangle - H^{*_1}(\underline{z}', p'')$. Then $F(\cdot, \underline{z}')$ is a (finite) convex function on $\mathbb{R}^n$ with its (Fenchel) conjugate $F^*(\cdot, \underline{z}')$ given (cf. (12)) by

$$
\begin{aligned}
F^*(z, \underline{z}') &= \sup_{p \in \mathbb{R}^n} \{\langle z, p \rangle - \langle \underline{z}', p' \rangle + H^{*_1}(\underline{z}', p'')\} \\
&= \begin{cases} +\infty & \text{if } \; z = (z', z'') \ne (\underline{z}', z''), \\ \underline{H}^*(\underline{z}', z'') & \text{if } \; z = (\underline{z}', z''). \end{cases} \qquad (34)
\end{aligned}
$$

Next, consider the Cauchy problem

$$
\begin{aligned}
\partial\psi/\partial t + F(\partial\partial\psi/\partial x, \underline{z}') = 0 \quad &\text{in} \quad \{t > 0,\; x \in \mathbb{R}^n\}, \\
\psi(0, x, \underline{z}') = \phi(x) \quad &\text{on} \quad \{t = 0,\; x \in \mathbb{R}^n\}.
\end{aligned}
$$

This is the Cauchy problem for a convex Hamilton-Jacobi equation (with uniformly continuous initial data). In view of (34), its (unique) viscosity solution $\psi(\cdot, \underline{z}')$ can be represented [8, Th. 2.1] as

$$\begin{aligned}\psi(t,x,\underline{z}') &= \min_{z\in\mathbb{R}^n}\{\phi(x-tz)+t\cdot F^*(z,\underline{z}')\}\\ &= \min_{z''\in\mathbb{R}^{n_2}}\{\phi(x'-t\underline{z}',x''-tz'')+t\cdot \underline{H}^*(\underline{z}',z'')\}.\end{aligned} \tag{35}$$

Since $H(p',p'') \le \langle \underline{z}', p'\rangle - H^{*_1}(\underline{z}',p'') = F(p',p'',\underline{z}')$, we may prove that $\psi(\cdot,\underline{z}')$ is a viscosity subsolution of (28)-(29). (In fact, let $\varphi \in C^1(U)$ be a test function such that $\psi(\cdot,\underline{z}') - \varphi$ has a local maximum at some $(t,x)\in U$. Then

$$\partial\varphi/\partial t + H(\partial\varphi/\partial x) \le \partial\varphi/\partial t + F(\partial\varphi/\partial x, \underline{z}') \le 0 \quad \text{at} \quad (t,x),$$

as claimed.)

Now, a standard comparison theorem for unbounded viscosity solutions (see [16]) gives

$$\psi(t,x,z') \le u(t,x) \quad \forall\, z' \in D_1.$$

Hence, by (32) and (35), $u_-(t,x) = \sup_{z'\in D_1}\psi(t,x,z') \le u(t,x)$ for all $(t,x) \in \bar{U}$. Dually, we have $u_+(t,x) \ge u(t,x)$ on $\bar{U}$. ∎

Remark 1. It can be shown that u_- (resp. u_+) is a subsolution (resp. supersolution) of (28)-(29) in the generalized sense of Ishii [22], provided $D_1 \ne \emptyset$ (resp. $D_2 \ne \emptyset$), cf. (30)-(31). Further, let $H(p',p'') \equiv H_1(p') + H_2(p'')$, with H_1 concave, H_2 convex (both finite). As a consequence of Theorem 3.1, we then see that the (unique) viscosity solution u of the Cauchy problem (28)-(29) satisfies on $\bar{U}$ the inequalities

$$\begin{aligned}&\max_{z'\in\mathbb{R}^{n_1}}\min_{z''\in\mathbb{R}^{n_2}}\{\phi(x-tz)+t\cdot(H_1^*(z')+H_2^*(z''))\} \le u(t,x)\\ &\quad\le \min_{z''\in\mathbb{R}^{n_2}}\max_{z'\in\mathbb{R}^{n_1}}\{\phi(x-tz)+t\cdot(H_1^*(z')+H_2^*(z''))\}.\end{aligned}$$

These are essentially Bardi – Faggian's estimates [8, (3.7)] (with only differences in notation). Here, we follow Rockafellar [52, §30] to take

$$H_1^*(z') := \inf_{p'\in\mathbb{R}^{n_1}}\{\langle z',p'\rangle - H_1(p')\},$$

while

$$H_2^*(z'') := \sup_{p''\in\mathbb{R}^{n_2}}\{\langle z'',p''\rangle - H_2(p'')\}.$$

(*Caution:* in general, $H_1^* \ne -(-H_1)^*$. For the convex function $G := -H_1$, one has, not $H_1^*(z') = -G^*(z')$, but $H_1^*(z') = -G^*(-z')$.)

Of course, for $t\cdot(H_1^*(z')+H_2^*(z''))$ not to be vague (and the desired estimates to hold), we adopt the convention that $0\cdot(\pm\infty)=0$, and we may set $H_1^*(z') + H_2^*(z'') = -\infty + \infty$ to be any value in $[-\infty,+\infty]$ if $z' \notin \mathrm{dom} H_1^*$, $z'' \notin \mathrm{dom} H_2^*$. However, "max" and "min" in the above Bardi – Faggian estimates are actually attained on $D_1 \equiv \mathrm{dom} H_1^* := \{z' \in \mathbb{R}^{n_1} \ : \ H_1^*(z') > -\infty\}$ and $D_2 \equiv \mathrm{dom} H_2^* := \{z'' \in \mathbb{R}^{n_2} \ : \ H_2^*(z'') < +\infty\}$, respectively.

Going to Theorem 5.1 in [76], we now have:

Corollary 3.2. *Assume* (G.I)-(G.III) (*see* §5.3), *with* ϕ *uniformly continuous. Then* (5.31) *determines the* (*unique*) *viscosity solution of the Cauchy problem* (28)-(29).

Remark 2. Since ϕ is uniformly continuous, the inequalities in Corollary 5.1 (Chapter 5 in [76]) are satisfied. This implies that the viscosity solution is locally Lipschitz continuous and solves (28) almost everywhere. Notice also that here we have $D_1 \equiv \mathbb{R}^{n_1}$ and $D_2 \equiv \mathbb{R}^{n_2}$ (cf. (30)-(31)).

If ϕ is Lipschitz continuous, then "min" and "max" in (32) and (33) can be computed on particular compact subsets of $\mathbb{R}^{n_2}$ and $\mathbb{R}^{n_1}$, respectively. In fact, we have the following, where Lip(ϕ) stands for the Lipschitz constant of ϕ.

Lemma 3.3. *Let* ϕ *be* (*globally*) *Lipschitz continuous,* H *be* (*finite and*) *concave-convex,* $L \geq 0$ *be such that, for some* $r > \mathrm{Lip}(\phi)$,

$$|H(p', p'') - H(p', \bar{p}'')| \leq L|p'' - \bar{p}''| \quad \forall\, p' \in \mathbb{R}^{n_1};\ p'', \bar{p}'' \in \mathbb{R}^{n_2},\ |p''|, |\bar{p}''| \leq r$$
$$(\text{resp. } |H(p', p'') - H(\bar{p}', p'')| \leq L|p' - \bar{p}'| \quad \forall\, p'' \in \mathbb{R}^{n_2};\ p', \bar{p}' \in \mathbb{R}^{n_1},\ |p'|, |\bar{p}'| \leq r).$$

Then (32)(*resp.* (33)) *becomes*

$$u_-(t, x) = \sup_{z' \in D_1} \min_{|z''| \leq L} \{\phi(x - tz) + t \cdot \underline{H}^*(z)\}$$
$$(\text{resp. } u_+(t, x) = \inf_{z'' \in D_2} \max_{|z'| \leq L} \{\phi(x - tz) + t \cdot \bar{H}^*(z)\}).$$

To prove Lemma 3.3, we need the following preparations. Given any convex Hamiltonian $H = H(q)$, and any uniformly continuous initial data $v_0 = v_0(\alpha)$ ($\alpha, q \in \mathbb{R}^N$), as was already mentioned, the Hopf-Lax formula

$$v(t, \alpha) := \min_{\omega \in \mathbb{R}^N} \{v_0(\alpha - t\omega) + t \cdot H^*(\omega)\} \quad (t \geq 0,\ \alpha \in \mathbb{R}^N) \tag{36}$$

determines the unique viscosity solution $v = v(t, \alpha)$ in the space $UC_\alpha([0, +\infty) \times \mathbb{R}^N)$ of the Cauchy problem

$$v_t + H(\partial v / \partial \alpha) = 0 \quad \text{in} \quad \{t > 0,\ \alpha \in \mathbb{R}^N\},$$
$$v(0, \alpha) = v_0(\alpha) \quad \text{on} \quad \{t = 0,\ \alpha \in \mathbb{R}^N\}.$$

The next technical lemma is somehow related to the so-called "cone of dependence" for viscosity solutions.

Lemma 3.4. *Let* H *be convex,* v_0 *be* (*globally*) *Lipschitz continuous. Assume that*

$$|H(q) - H(\bar{q})| \leq L|q - \bar{q}| \quad \forall\, q, \bar{q} \in \mathbb{R}^N,\ |q|, |\bar{q}| \leq r$$

for some $L \geq 0,\ r > \mathrm{Lip}(v_0)$. *Then* (36) *becomes*

$$v(t,\alpha) = \min_{|\omega|\leq L}\{v_0(\alpha - t\omega) + t \cdot H^*(\omega)\} \quad (t \geq 0,\ \alpha \in \mathbb{R}^N).$$

Proof. We may suppose $t > 0$. Choose $\omega_0 = \omega_0(t,\alpha) \in \mathbb{R}^N$ so that the value at (t,α) of the viscosity solution v, determined by (36), is

$$v_0(\alpha - t\omega_0) + t \cdot H^*(\omega_0).$$

It suffices to prove $|\omega_0| \leq L$. To this end, we first notice that

$$v_0(\alpha - t\omega_0) + t \cdot H^*(\omega_0) \leq v_0(\alpha - t\omega) + t \cdot H^*(\omega)$$

for any $\omega \in \mathbb{R}^N$. Hence,

$$\begin{aligned} H^*(\omega_0) &\leq t^{-1}[v_0(\alpha - t\omega) - v_0(\alpha - t\omega_0)] + H^*(\omega) \\ &\leq R|\omega - \omega_0| + H^*(\omega) \quad \forall\, \omega \in \mathbb{R}^N, \end{aligned} \tag{37}$$

where $R := \mathrm{Lip}(v_0)$. Now, define

$$h(q) := \begin{cases} +\infty & \text{if } |q| > R, \\ \langle \omega_0, q\rangle & \text{if } |q| \leq R. \end{cases}$$

Then h is a lower semicontinuous proper convex function on $\mathbb{R}^N$, with $h^*(\omega) \equiv R|\omega - \omega_0|$. So (37) implies

$$H(q) \geq \langle \omega_0, q\rangle - H^*(\omega_0) \geq \langle \omega_0, q\rangle - (h^* + H^*)(\omega)$$

for all $\omega \in \mathbb{R}^N$. Therefore,

$$H(q) \geq \langle \omega_0, q\rangle + \sup_{\omega\in\mathbb{R}^N}\{-(h^* + H^*)(\omega)\} = \langle \omega_0, q\rangle + (h^* + H^*)^*(0) \tag{38}$$

for any $q \in \mathbb{R}^N$. Next, consider the "infimum convolute" $h \bigotimes H$ given by the formula

$$(h \bigotimes H)(q) := \inf_{\underline{q}+\bar{q}=q}\{h(\underline{q}) + H(\bar{q})\} \equiv \min_{|\underline{q}|\leq R}\{\langle \omega_0, \underline{q}\rangle + H(q - \underline{q})\}.$$

This infimum convolute [52, Theorem 16.4] is a (finite) convex function with $(h \bigotimes H)^* = h^* + H^*$. It follows from (38) that

$$H(q) \geq \langle \omega_0, q\rangle + (h \bigotimes H)^{**}(0) = \langle \omega_0, q\rangle + (h \bigotimes H)(0),$$

i.e. that

$$H(q) \geq \langle \omega_0, q\rangle + \min_{|\bar{q}|\leq R}\{H(\bar{q}) - \langle \omega_0, \bar{q}\rangle\},$$

for all $q \in \mathbb{R}^N$. Thus, there exists a $\bar{q} \in \mathbb{R}^N,\ |\bar{q}| \leq R$, such that

$$H(q) \geq H(\bar{q}) + \langle \omega_0, q - \bar{q}\rangle \quad \forall\, q \in \mathbb{R}^N. \tag{39}$$

Finally, assume, contrary to our claim, that $|\omega_0| > L$. Then, for any fixed ε with $0 < \varepsilon < (r-R)/|\omega_0|$, take $q := \bar{q} + \varepsilon\omega_0$. Of course, $0 < |q-\bar{q}| = \varepsilon|\omega_0| < r - R$. Hence $|q|, |\bar{q}| < r$. But, by (39), this q would satisfy

$$H(q) - H(\bar{q}) \geq |\omega_0| \cdot |q - \bar{q}| > L|q - \bar{q}|,$$

which contradicts the assumption of the lemma. The proof is thereby complete. ■

Proof of Lemma 3.3. For any temporarily fixed $z' \in D_1,\ t \geq 0,\ x' \in \mathbb{R}^{n_1}$, let

$$F = F(p'') := -H^{*_1}(z', p'') \ \text{ and } \ v_0 = v_0(x'') := \phi(x' - tz', x'').$$

Obviously, F is a (finite) convex function on $\mathbb{R}^{n_2}$, with $F^*(z'') \equiv \underline{H}^*(z', z'')$ (in view of (12)). For definiteness, suppose that

$$|H(p', p'') - H(p', \bar{p}'')| \leq L|p'' - \bar{p}''| \ \ \forall p' \in \mathbb{R}^{n_1};\ p'', \bar{p}'' \in \mathbb{R}^{n_2},\ |p''|, |\bar{p}''| \leq r \quad (40)$$

for some $L \geq 0,\ r > \mathrm{Lip}(\phi)$ $(\geq \mathrm{Lip}(v_0))$. Then it can be shown that

$$|F(p'') - F(\bar{p}'')| \leq L|p'' - \bar{p}''| \quad \forall\, p'', \bar{p}'' \in \mathbb{R}^{n_2},\ |p''|, |\bar{p}''| \leq r. \qquad (41)$$

In fact, given arbitrary $\varepsilon \in (0, +\infty)$ and $p'', \bar{p}'' \in \mathbb{R}^{n_2}$, with $|p''|, |\bar{p}''| \leq r$, since $z' \in D_1$, we could find (using (6) and (30)) a $p' \in \mathbb{R}^{n_1}$ such that

$$+\infty > H^{*_1}(z', \bar{p}'') > \langle z', p' \rangle - H(p', \bar{p}'') - \varepsilon.$$

So, (6) together with (40) implies

$$\begin{aligned} H^{*_1}(z', p'') - H^{*_1}(z', \bar{p}'') &\leq H^{*_1}(z', p'') - \langle z', p' \rangle + H(p', \bar{p}'') + \varepsilon \\ &\leq \langle z', p' \rangle - H(p', p'') - \langle z', p' \rangle + H(p', \bar{p}'') + \varepsilon \\ &\leq |H(p', p'') - H(p', \bar{p}'')| + \varepsilon \leq L|p'' - \bar{p}''| + \varepsilon. \end{aligned}$$

Because $\varepsilon \in (0, +\infty)$ is arbitrarily chosen, we get

$$F(\bar{p}'') - F(p'') = H^{*_1}(z', p'') - H^{*_1}(z', \bar{p}'') \leq L|p'' - \bar{p}''|.$$

Similarly,

$$F(p'') - F(\bar{p}'') \leq L|\bar{p}'' - p''| = L|p'' - \bar{p}''|.$$

Thus (41) has been proved. Therefore, we may apply Lemma 3.4 to these F and ϕ instead of H and v_0. (Here, $N := n_2$, while x'' and z'' stand for α and ω, respectively.) It follows that (for an arbitrary $x'' \in \mathbb{R}^{n_2}$)

$$\begin{aligned} &\min_{z'' \in \mathbb{R}^{n_2}} \{\phi(x' - tz', x'' - tz'') + t \cdot \underline{H}^*(z', z'')\} \\ &= \min_{z'' \in \mathbb{R}^{n_2}} \{v_0(x'' - tz'') + t \cdot H^*(z'')\} = \min_{|z''| \leq L} \{v_0(x'' - tz'') + t \cdot H^*(z'')\} \\ &= \min_{|z''| \leq L} \{\phi(x' - tz', x'' - tz'') + t \cdot \underline{H}^*(z', z'')\}. \end{aligned}$$

Hence, (32) gives us

$$u_-(t,x) = \sup_{z' \in D_1} \min_{|z''| \le L} \{\phi(x - tz) + t \cdot \underline{H}^*(z)\}$$

for any $(t,x) \in \bar{U}$. Dually in the other case corresponding to (33). ∎

As an immediate consequence of Lemma 3.3, we have:

Corollary 3.5. *Let ϕ be (globally) Lipschitz continuous, $H(p', p'') \equiv H_1(p') + H_2(p'')$, with H_1 concave, H_2 convex (both finite). Let $L_1, L_2 \ge 0$ be such that, for some $r > \mathrm{Lip}(\phi)$,*

$$|H_1(p') - H_1(\bar{p}')| \le L_1 |p' - \bar{p}'| \quad \forall\, p', \bar{p}' \in \mathbb{R}^{n_1},\ |p'|, |\bar{p}'| \le r,$$
$$|H_2(p'') - H_2(\bar{p}'')| \le L_2 |p'' - \bar{p}''| \quad \forall\, p'', \bar{p}'' \in \mathbb{R}^{n_2},\ |p''|, |\bar{p}''| \le r.$$

Then the unique viscosity solution $u \in UC_x([0,+\infty) \times \mathbb{R}^n)$ of the Cauchy problem (28)-(29) satisfies on $\bar{U}$ the inequalities

$$\max_{|z'| \le L_1} \min_{|z''| \le L_2} \{\phi(x - tz) + t \cdot (H_1^*(z') + H_2^*(z''))\} \le u(t,x)$$
$$\le \min_{|z''| \le L_2} \max_{|z'| \le L_1} \{\phi(x - tz) + t \cdot (H_1^*(z') + H_2^*(z''))\}.$$

Remark 3. The strict inequality in the hypothesis "$r > \mathrm{Lip}(\phi)$" of Lemma 3.3 and Corollary 3.5 (or that in "$r > \mathrm{Lip}(v_0)$" of Lemma 3.4) is essential for the proofs. In this connection, Corollary 3.5 corrects a result by Bardi and Faggian (cf. [8, Lemma 3.3]), where they take $r := \mathrm{Lip}(\phi)$, but this is impossible, as the following example shows.

Example 1. Let $n_1 := 1 =: n_2$, $\phi \equiv 0$, $H_1 \equiv -(p')^2/2$, and $H_2 \equiv p''$. In this case, $\mathrm{Lip}(\phi) = 0$. Then any $L_1, L_2 > 0$ surely satisfy all the hypotheses of [8, Lemma 3.3], but the desired estimates become

$$\max_{|z'| \le L_1} \min_{|z''| \le L_2} \{t \cdot (\delta(z''|1) - (z')^2/2)\} \le u(t,x)$$
$$\le \min_{|z''| \le L_2} \max_{|z'| \le L_1} \{t \cdot (\delta(z''|1) - (z')^2/2)\}$$

that would not be true if $t > 0$ and $L_2 < 1$.

Example 2. Let $n_1 = n_2 := k > 0$, and $f_1(p) := \langle p', p'' \rangle$ for $p = (p', p'') \in \mathbb{R}^k \times \mathbb{R}^k$. Then f_1 is trivially a concave-convex function. (One can also check directly that it is neither convex nor concave.) There could not be any functions $g_1 = g_1(p')$, $g_2 = g_2(p'')$, with $f_1(p) \equiv g_1(p') + g_2(p'')$. To find a (finite) concave-convex function H on $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ with (5.3)-(5.4) in Chapter 5 holding, for which there do not exist any (concave) $H_1 = H_1(p')$ and (convex) $H_2 = H_2(p'')$ such that $H(p) \equiv H_1(p') + H_2(p'')$, we can now take, for example,

$$H(p) := -|p'|^{\sqrt{2}} + \langle p', p'' \rangle + |p''|^{3/2} \quad \text{for} \quad p = (p', p'') \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}.$$

3.2. Estimates for Concave-Convex Initial Data

We now consider the Cauchy problem (28)-(29) with the following hypothesis:
(IV) Hamiltonian $H = H(p)$ is continuous and the initial function $\phi = \phi(x', x'')$ is concave-convex and Lipschitz continuous (without (3)-(4)).
For $(t, x) \in U$, set

$$u_-(t,x) := \sup_{p'' \in V_2} \inf_{p' \in \mathbb{R}^{n_1}} \{< p, x > - \bar{\phi}^*(p) - tH(p)\} \tag{32a}$$

$$u_+(t,x) := \inf_{p' \in V_1} \sup_{p'' \in \mathbb{R}^{n_2}} \{< p, x > - \underline{\phi}^*(p) - tH(p)\}, \tag{33a}$$

where let

$$V_1 := \{p' \in \mathbb{R}^{n_1} : \ \phi^{*1}(p', x'') > -\infty, \ \forall x'' \in \mathbb{R}^{n_2}\},$$
$$V_2 := \{p'' \in \mathbb{R}^{n_2} : \ \phi^{*2}(x', p'') < +\infty, \ \forall x' \in \mathbb{R}^{n_1}\},$$

hence

$$\text{for all } x'' \in \mathbb{R}^{n_2}, \ \phi^{*1}(p', x'') \ \text{ is finite on } V_1$$
$$\text{for all } x' \in \mathbb{R}^{n_1}, \ \phi^{*2}(x', p'') \ \text{ is finite on } V_2.$$

Remark 4. The concave-convex function $\phi = \phi(x', x'')$ is Lipschitz continuous in the sense: $\phi(x', x'')$ is Lipschitz continuous in $x' \in \mathbb{R}^{n_1}$ for each $x'' \in \mathbb{R}^{n_2}$ and in $x'' \in \mathbb{R}^{n_2}$ for each $x' \in \mathbb{R}^{n_1}$.

Our estimates for viscosity solutions in this subsection read as follows:

Theorem 3.6. *Assume (IV). Then the unique viscosity solution $u = u(t, x) \in UC_x\big([0, +\infty) \times \mathbb{R}^n\big)$ of the Cauchy problem (28)-(29) satisfies on $\bar{U}$ the inequalities*

$$u_-(t,x) \le \ u(t,x) \le \ u_+(t,x),$$

where $u_-(t, x)$ and $u_+(t, x)$ are defined by (32a) and (33a) respectively.

Proof. For each $\underline{p}' \in V_1$, let

$$\begin{aligned}\Phi(x; \underline{p}') = \Phi(x', x''; \underline{p}') &:= < x', \underline{p}' > - \ \phi^{*1}(\underline{p}', x'') \\ &= < x', \underline{p}' > - \inf_{x' \in \mathbb{R}^{n_1}} \big\{< x', \underline{p}' > - \ \phi(x', x'')\big\} \\ &\ge \phi(x', x'') \ \text{ for all } (x', x'') \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}.\end{aligned}$$

Since $\phi^{*1}(\underline{p}', .)$ is a concave and Lipschitz continuous function so $\Phi(x; \underline{p}')$ is convex and Lipschitz continuous with its Fenchel conjugate given by

$$\begin{aligned}\Phi^*(p; \underline{p}') = \Phi^*(p', p''; \underline{p}') &= \sup_{x \in \mathbb{R}^n} \big\{< x, p > - \Phi(x, \underline{p}')\big\} \\ &= \sup_{x \in \mathbb{R}^n} \big\{< x', p' > + < x'', p'' > - < x', \underline{p}' > + \phi^{*1}(\underline{p}', x'')\big\} \\ &= \begin{cases} +\infty & \text{if } \ (p', p'') \neq (\underline{p}', p'') \\ \underline{\phi}^*(\underline{p}', p'') & \text{if } \ (p', p'') = (\underline{p}', p''). \end{cases}\end{aligned}$$

Next, consider the Cauchy problem

$$\frac{\partial v}{\partial t} + H\Big(\frac{\partial v}{\partial x}\Big) = 0 \text{ in } U = \{t > 0,\ x \in \mathbb{R}^n\},$$
$$v(0, x) = \Phi(x; \underline{p}') \text{ on } \{t = 0,\ x \in \mathbb{R}^n\}.$$

This is the Cauchy problem with the continuous Hamiltonian $H = H(p)$ and the convex and Lipschitz continuous initial function $\Phi = \Phi(x; \underline{p}')$ for each $\underline{p}' \in V_1$, its unique viscosity solution $v = v(t, x) \in UC_x\big([0, +\infty) \times \mathbb{R}^n\big)$ is given by

$$v(t, x) = \sup_{p \in \mathbb{R}^n} \{< p, x > -\Phi^*(p; \underline{p}') - \ tH(p)\}$$
$$= \sup_{p'' \in \mathbb{R}^{n_2}} \{< \underline{p}', x' > + < p'', x'' > -\underline{\phi}^*(\underline{p}', p'') - \ tH(\underline{p}', p'')\}$$

with the initial condition

$$v(0, x) = \Phi(x; \underline{p}') \geq \phi(x) = u(0, x)$$

for each $\underline{p}' \in V_1$. Hence, for each $\underline{p}' \in V_1,\ v = v(t, x)$ is a (continuous) super-solution of the problem (28)-(29) (according to a standard comparison theorem for unbounded viscosity solutions (see [22])), that means

$$u(t, x) \leq v(t, x) \text{ for each } \underline{p}' \in V_1,$$

and then

$$u(t, x) \leq \inf_{p' \in V_1} \sup_{p'' \in \mathbb{R}^{n_2}} \{< p, x > -\underline{\phi}^*(p) - \ tH(p)\}$$
$$u(t, x) \leq u_+(t, x) \text{ on } \bar{U}.$$

Dually, we also obtain

$$u(t, x) \geq u_-(t, x) \text{ on } \bar{U}.$$

Theorem 3.6 has been proved completely. ∎

Corollary 3.7. *Assume (M.I)-(M.II) in chapter* 5[76]*. Moreover,* $\phi = \phi(x', x'')$ *is Lipschitz continuous on* $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$*. Then the formula* (5.53) *in Chapter* 5[76] *determines the unique viscosity solution* $u(t, x) \in UC_x\big([0, +\infty) \times \mathbb{R}^n\big)$ *of the Cauchy problem* (28)-(29).

Proof. Since $\phi = \phi(x', x'')$ is a concave-convex and Lipschitz continuous function so domϕ^* is a bounded and nonempty set. Independently of $(t, x) \in \bar{U}$, it follows that

$$\varphi(t, x, p', p'') \longrightarrow -\infty \text{ whenever } \mid p'' \mid \text{ is large enough}$$

and

$$\varphi(t, x, p', p'') \longrightarrow +\infty \text{ whenever } \mid p' \mid \text{ is large enough.}$$

Remark 5 in Section 5.3 [76] implies that hypothesis (M.III) in Chapter 5 [76] holds. Then the conclusion is straightforward from Theorem 3.6. ∎

4. D. C. Hamiltonians Containing u and Du

We now study viscosity solutions of the Cauchy problem for Hamilton-Jacobi equations of the form

$$u_t + H(u, D_x u) = 0 \quad \text{in} \quad (0,T)\times\mathbb{R}^n, \tag{42}$$

$$u(0,x) = u_0(x) \quad \text{in} \quad \mathbb{R}^n, \tag{43}$$

where H, u_0 are continuous functions in $\mathbb{R}^{n+1}$ and $\mathbb{R}^n$, respectively.

We aim here to consider Problem (42)-(43) when the Hamiltonian $H(\gamma, p)$ is a function to be the sum of a convex and a concave function.

The following hypotheses are assumed in this section:

A. *The Hamiltonian $H(\gamma,p), (\gamma,p)\in\mathbb{R}\times\mathbb{R}^n$, is a nonconvex-nonconcave function in p, i.e.,*

$$H(\gamma,p) = H_1(\gamma,p) + H_2(\gamma,p), \quad (\gamma,p)\in\mathbb{R}\times\mathbb{R}^n,$$

where H_1, H_2 are continuous on $\mathbb{R}^{n+1}$, and for each fixed $\gamma\in\mathbb{R}$, $H_1(\gamma,.)$ is convex, $H_2(\gamma,.)$ is concave, $H_1(\gamma,.), H_2(\gamma,.)$ are positively homogeneous of degree one in $\mathbb{R}^n$; for each fixed $p\in\mathbb{R}^n$, $H_1(.,p), H_2(.,p)$ are nondecreasing in $\mathbb{R}$;

B. *The initial function u_0 is continuous in $\mathbb{R}^n$.*

The expected solutions of the problem (42)-(43) are now defined:

$$u_-(t,x) := \sup_z \inf_y \big\{[H_1^{\#}(y) \vee u_0(x - t(y+z))] \wedge H_{2\#}(z)\big\}, (t,x)\in(0,T)\times\mathbb{R}^n, \tag{44}$$

and

$$u_+(t,x) := \inf_y \sup)z \big\{H_1^{\#}(y) \vee [u_0(x - t(y+z)) \wedge H_{2\#}(z)]\big\}, (t,x)\in(0,T)\times\mathbb{R}^n, \tag{45}$$

where the operations "$\vee$", "$^{\#}$" are defined as

$$H^{\#}(q) := \inf\{\gamma\in\mathbb{R} : H(\gamma,p) \geq (p,q), \forall p\in\mathbb{R}^n\},$$

the notation $(.,.)$ stands for the ordinary scalar product on $\mathbb{R}^n$, and

$$a\vee b := \max\{a,b\}$$

and the operations "$\wedge$", "$_{\#}$" act as

$$a\wedge b := \min\{a,b\}, \quad \text{and} \quad H_{\#}(q) = \sup\{\gamma\in\mathbb{R} : H(\gamma,p) \leq (p,q), \forall p\in\mathbb{R}^n\}.$$

The following theorem is the main result of this section.

Theorem 4.1.

i) *The function u_- determined by* (44) *is a viscosity subsolution of the equation* (42) *and satisfies* (43), *i.e.,*

$$\lim_{(t,x')\to(0,x)} u_-(t,x') = u_0(x), \quad \forall x \in \mathbb{R}^n. \tag{46}$$

ii) *The function u_+ determined by* (45) *is a viscosity supersolution of the equation* (42) *and satisfies* (43), *i.e.,*

$$\lim_{(t,x')\to(0,x)} u_+(t,x') = u_0(x), \quad \forall x \in \mathbb{R}^n. \tag{47}$$

Relying on the results of Theorem 4.1, one can obtain the upper and lower bounds for the unique viscosity solution of the problem (42)-(43).

Corollary 4.2. *If, in addition, $u_0 \in BUC(\mathbb{R}^n)$, then Problem* (42)-(43) *admits a unique viscosity solution u in $BUC([0,T]\times\mathbb{R}^n)$ such that*

$$u_- \le u \le u_+, \quad \text{in} \quad [0,T]\times\mathbb{R}^n, \tag{48}$$

where u_- and u_+ are defined in (44) *and* (45) *respectively.*

Note that two expressions under the brackets "{.}" in (44) and (45) are, in general, not the same since the operations "$\wedge$" and"$\vee$" are not "commutative". However, for every fixed $(t,x) \in (0,T)\times\mathbb{R}^n$, the supremum in z and the infimum in y may be taken over the convex sets in which these two expressions coincide. The min-max theorems then yield the coincidence of u_+ and u_- in many cases (see [57], for example). In those cases, the unique viscosity solution of Problem (42)-(43) is easily computed.

By means of the above results, we can deduce several admired conclusions: if $H_2 = 0$, then $u_+ = u_- = u$, u is a viscosity solution for the initial data u_0 to be just continuous in $\mathbb{R}^n$ (not necessarily bounded and Lipschitz continuous as in [13]). If $H_1 = 0$, then $u_- = u_+$ and we get a formula for viscosity solutions with a concave Hamiltonian. Actually, for instance if $H_2 = 0$, then a direct calculation gives

$$H_{2\#}(z) = \begin{cases} +\infty & \text{if} \quad z = 0 \\ -\infty & \text{if} \quad z \neq 0. \end{cases}$$

The formulas (44) and (45) then yield

$$u(t,x) = u_-(t,x) = u_+(t,x), \forall (t,x) \in (0,T)\times\mathbb{R}^n.$$

In order to prove Theorem 4.1, we need some properties of the quasiconvex duality [13]. Let a continuous function $H = H(\gamma,p), (\gamma,p) \in \mathbb{R}\times\mathbb{R}^n$, be given.

Using the operations "$(.)^{\#}$", "$(.)_{\#}$", "$\wedge$" and "$\vee$" defined as in Appendix [76], we set

$$H^{\#*}(\gamma,p) := \sup\{(p,q) : q \in \mathbb{R}^n, H^{\#}(q) \le \gamma\}, \quad (\gamma,p) \in \mathbb{R}\times\mathbb{R}^n,$$

and

$$H_{\#*}(\gamma,p) := \inf\{(p,q) : q \in \mathbb{R}^n, H_{\#}(q) \ge \gamma\}, \quad (\gamma,p) \in \mathbb{R}\times\mathbb{R}^n.$$

Some basic features of this dual can be summarized in the following lemma.

Lemma 4.3.

i) *Let H be nondecreasing in γ, convex and positively homogeneous of degree one in p. Then $H^{\#}$ is quasiconvex, lower semicontinuous and*

$$H^{\#}(z) \to +\infty \quad \textit{as} \quad |z| \to \infty, \quad \textit{and} \quad H^{\#*} = H.$$

Moreover, there exists $p^* \in \mathbb{R}^n$ such that

$$H^{\#}(p^*) = -\infty.$$

ii) *Let H be nondecreasing in γ, concave and positively homogeneous of degree one in p. Then $H_{\#}$ is quasiconcave, upper semicontinuous and*

$$H_{\#}(z) \to -\infty \quad \textit{as} \quad |z| \to \infty, \quad \textit{and} \quad H_{\#*} = H.$$

Moreover, there exists $q^* \in \mathbb{R}^n$ such that

$$H_{\#}(q^*) = +\infty.$$

Proof.
i) The first assertion of i) was proved by Barron, Jensen and Liu [13]. Let us verify that there exists a $p^* \in \mathbb{R}^n$ such that $H^{\#}(p^*) = -\infty$. Assume the contrary, that

$$H^{\#}(z) > -\infty, \forall z \in \mathbb{R}^n.$$

Since $H^{\#} \to +\infty$ as $|z| \to \infty$, there exists $N > 0$ so that $H^{\#}(z) > 0$, for all $|z| > N$. Thus, we get

$$-\infty = \inf_{z\in\mathbb{R}^n} H^{\#}(z) = \inf_{|z|\leq N} H^{\#}(z). \tag{49}$$

Since $H^{\#}$ is lower semicontinuous, $H^{\#}(z) > -\infty, \forall z \in \mathbb{R}^n$, the function

$$h(z) := \min\{H^{\#}(z), 0\}, \quad z \in \mathbb{R}^n$$

is clearly finite and lower semicontinuous on $\mathbb{R}^n$. Hence,

$$\inf_{|z|\leq N} H^{\#}(z) \geq \inf_{|z|\leq N} h(z) := M > -\infty,$$

which contradicts (49). This contradiction proved the second part of i).
ii) Using $[-H(-\gamma, -p)]^{\#}(z) = -[H_{\#}(\gamma, p)](z)$ we symmetrically obtain ii) ■

In order to investigate the functions u_-, u_+ we need two auxiliary functions to be determined by

$$v(t,x) := \inf_{y\in\mathbb{R}^n} \big\{H_1^{\#}\big(\frac{x-y}{t}\big) \vee u_0(y)\big\}, \quad (t,x) \in (0,T] \times \mathbb{R}^n, \tag{50}$$

$$w(t,x) := \sup_{y\in\mathbb{R}^n} \big\{H_{2\#}\big(\frac{x-y}{t}\big) \wedge u_0(y)\big\}, \quad (t,x) \in (0,T] \times \mathbb{R}^n. \tag{51}$$

The continuity of v, w can be certified by the following lemma.

Lemma 4.4. *The functions v, w are continuous on $[0,T] \times \mathbb{R}^n$ with*

$$v(0,x) := u_0(x), \qquad w(0,x) := u_0(x), \quad x \in \mathbb{R}^n.$$

Proof. We need only show that v is continuous on $[0,T] \times \mathbb{R}^n$. The argument for w would be similar.

It is convenient to rewrite the function v in (50) as

$$v(t,x) = \inf_{z \in \mathbb{R}^n} \Big\{ H_1^{\#}(z) \vee u_0(x - tz) \Big\}, \quad \forall (t,x) \in (0,T] \times \mathbb{R}^n. \tag{52}$$

By virtue of Lemma 4.3 i), we can take a point $p^* \in \mathbb{R}^n$ such that $H_1^{\#}(p^*) = -\infty$ and keep it fixed. Let $r > 0$ be arbitrarily selected. Then for each $(t,x) \in (0,T] \times B(0;r)$,

$$v(t,x) \leq H_1^{\#}(p^*) \vee u_0(x - tp^*) = u_0(x - tp^*) \leq \max_{|y| \leq r + T|p^*|} u_0(y) := K < +\infty.$$

Since $H_1^{\#}(z) \to +\infty$ as $|z| \to \infty$, there exists a constant $N > 0$ such that

$$H_1^{\#}(z) > K, \quad \forall |z| \geq N.$$

Hence, the infimum in (52) has to be taken over the ball $\overline{B}(0;N)$ for all $(t,x) \in (0,T] \times B(0;r)$. Since the function $z \mapsto (H_1^{\#}(z) \vee u_0(x - tz)) \wedge K, z \in \bar{B}(0;N)$ is finite (bounded) and lower semicontinuous on a compact set, it holds for any $(t,x) \in (0,T] \times B(0;r)$,

$$\begin{aligned} v(t,x) &= \inf_{|z| \leq N} \Big\{ H_1^{\#}(z) \vee u_0(x - tz) \Big\} \wedge K \\ &= \inf_{|z| \leq N} \Big\{ \big[H_1^{\#}(z) \vee u_0(x - tz) \big] \wedge K \Big\} \\ &= \min_{|z| \leq N} \Big\{ \big[H_1^{\#}(z) \vee u_0(x - tz) \big] \wedge K \Big\} \\ &= \min_{|z| \leq N} \{ H_1^{\#}(z) \vee u_0(x - tz) \}. \end{aligned}$$

Thus, for every $(t,x) \in (0,T] \times B(0;r)$, the set

$$k(t,x) := \big\{ y_0 \in \mathbb{R}^n : H_1^{\#}(y_0) \vee u_0(x - ty_0) = \inf_{z \in \mathbb{R}^n} \{ H_1^{\#}(z) \vee u_0(x - tz) \} \big\}$$

is not empty. Since r is arbitrary, we can extend the definition of $k(t,x)$ to the whole domain $(0,T] \times \mathbb{R}^n$. The above arguments enable us to say

$$\|k(t,x)\| := \sup\{ |y_0| : y_0 \in k(t,x) \} \leq N, \quad (t,x) \in (0,T] \times B(0;r). \tag{53}$$

For any $(t,x), (t',x') \in (0,T] \times B(0;r)$, choosing $\xi \in k(t,x), |\xi| \leq N$ (by virtue of (52)), we get

$$\begin{aligned} v(t',x') - v(t,x) &= \inf_{z\in\mathbb{R}^n} \left\{H_1^{\#}(z) \vee u_0(x'-t'z)\right\} \\ &- H_1^{\#}(\xi) \vee u_0(x-t\xi) \\ &\le H_1^{\#}(\xi) \vee u_0(x'-t'\xi) - H_1^{\#}(\xi) \vee u_0(x-t\xi) \\ &\le |u_0(x'-t'\xi) - u_0(x-t\xi)|. \end{aligned} \tag{54}$$

Exchanging (t,x) and (t',x'), we can select $\xi' \in k(t',x'), |\xi'| \le N$ so that

$$v(t,x) - v(t',x') \le |u_0(x'-t'\xi') - u_0(x-t\xi')|. \tag{55}$$

The estimates (54) and (55) yield

$$\lim_{(t',x')\to(t,x)} v(t',x') = v(t,x), \quad \forall (t,x) \in (0,T] \times B(0,r).$$

Since r is arbitrary, it follows that $u \in C((0,T] \times \mathbb{R}^n)$.

Next, let us verify that the function v is continuous until the boundary $\{0\} \times \mathbb{R}^n$, i.e.,

$$\lim_{(t,x)\to(0,x_0)} v(t,x) = u_0(x_0), \quad \forall x_0 \in \mathbb{R}^n. \tag{56}$$

Indeed, by what was shown above one has for some fixed $p^* \in \mathbb{R}^n$ at which $H_1^{\#}(p^*) = -\infty$, that

$$v(t,x) \le H_1^{\#}(p^*) \vee u_0(x-tp^*) = u_0(x-tp^*), \quad \forall (t,x) \in (0,T] \times \mathbb{R}^n.$$

Consequently,

$$\limsup_{(t,x)\to(0,x_0)} v(t,x) \le \lim_{(t,x)\to(0,x_0)} u_0(x-tp^*) = u_0(x_0). \tag{57}$$

On the other hand, in view of (52) where $r > 0$ is arbitrarily given, it holds true that

$$v(t,x) = H_1^{\#}(\xi) \vee u_0(x-t\xi) \ge u_0(x-t\xi),$$

for every $(t,x) \in (0,T] \times B(0;r)$ with some fixed $\xi \in k(t,x), |\xi| \le N$. Sending $(t,x) \to (0,x_0)$, we have

$$\liminf_{(t,x)\to(0,x_0)} v(t,x) \ge \lim_{(t,x)\to(0,x_0)} u_0(x-t\xi) = u_0(x_0). \tag{58}$$

The combination of (57) and (58) yields (56). The proof of Lemma 4.4 is complete. ■

Proof of Theorem 4.1.
i) First, we will show that u_- is continuous in $(0,T) \times \mathbb{R}^n$. Indeed, u_- can be rewritten as

$$u_-(t,x) = \sup_z \{v(t,x-tz) \wedge H_{2\#}(z)\},$$

where $v(t,x)$ is defined by (50). By virtue of Lemma 4.3, there is $q^* \in \mathbb{R}^n$, $H_{2\#}(q^*) = +\infty$. Hence, if $|x| \le M$ for some constant $M > 0$ then

$$\begin{aligned} u_-(t,x) &\geq v(t, x - tq^*) \wedge H_{2\#}(q^*) = v(t, x - tq^*) \\ &\geq \min_{s\in[0,T],|y|\leq M+T|q^*|} v(s,y) := K > -\infty. \end{aligned}$$

Also, there is $N > 0$, $H_{2\#}(z) < K, \forall |z| > N$. Therefore,

$$u_-(t,x) = \sup_{|z|\leq N} \{v(t, x - tz) \wedge H_{2\#}(z)\}, \quad \forall t \in [0,T], |x| \leq M.$$

Since both v and $H_{2\#}$ are upper semicontinuous in the variable $z \in \mathbb{R}^n$, so is their minimum. Hence, the last expression becomes

$$u_-(t,x) = \max_{|z|\leq N} \{v(t, x - tz) \wedge H_{2\#}(z)\}, \quad \forall t \in [0,T], |x| \leq M. \tag{59}$$

By virtue of (59), let $|x| \leq M, |x'| \leq M$, and let for some fixed $z_0 \in \mathbb{R}^n, |z_0| \leq N$,

$$u_-(t,x) = v(t, x - tz_0) \wedge H_{2\#}(z_0).$$

Then

$$\begin{aligned} u_-(t,x) - u_-(t',x') &= v(t, x - tz_0) \wedge H_{2\#}(z_0) \\ &\quad - \max_{|z|\leq N} \{v(t', x' - t'z) \wedge H_{2\#}(z)\} \\ &\leq v(t, x - tz_0) \wedge H_{2\#}(z_0) \\ &\quad - v(t', x' - t'z_0) \wedge H_{2\#}(z_0) \\ &\leq |v(t', x' - t'z_0) - v(t, x - tz_0)|. \end{aligned} \tag{60}$$

Interchanging (t', x') and (t, x) we get for some $z_1, |z_1| \leq N$,

$$u_-(t',x') - u_-(t,x) \leq |v(t', x' - t'z_1) - v(t, x - tz_1)|. \tag{61}$$

The estimates (60), (61) and the continuity of v imply that u_- is continuous on $(0,T) \times \{x : |x| \leq M\}$. Since M is arbitrarily chosen, the continuity in $(0,T) \times \mathbb{R}^n$ of u_- follows.

Next we claim that for every $(t,x) \in (0,T) \times \mathbb{R}^n$, $0 < s < t$,

$$u_-(t,x) \leq \inf_z \{H_1^\#\big(\frac{x-z}{t-s} - z_0\big) \vee u_-(s,z)\},$$

where $z_0 \in \mathbb{R}^n$ such that

$$u_-(t,x) = v(t, x - tz_0) \wedge H_{2\#}(z_0). \tag{62}$$

Actually, in view of (59), it holds

$$\begin{aligned} u_-(t,x) &= v(t, x - tz_0) \wedge H_{2\#}(z_0) \\ &\leq \big[H_1^\#\big(\frac{x-y}{t} - z_0\big) \vee u_0(y)\big] \wedge H_{2\#}(z_0), \quad \forall y \in \mathbb{R}^n. \end{aligned}$$

Since $H_1^{\#}$ is quasiconvex, we have for each fixed $z \in \mathbb{R}^n$,

$$H_1^{\#}\Big(\frac{x-y}{t} - z_0\Big) \le H_1^{\#}\Big(\frac{x-z}{t-s} - z_0\Big) \vee H_1^{\#}\Big(\frac{z-y}{s} - z_0\Big), \quad \forall y \in \mathbb{R}^n.$$

Thus,

$$u_-(t,x) \le \left[H_1^{\#}\Big(\frac{x-z}{t-s} - z_0\Big) \vee H_1^{\#}\Big(\frac{z-y}{s} - z_0\Big) \vee u_0(y)\right] \wedge H_{2\#}(z_0), \quad \forall y \in \mathbb{R}^n.$$

By changing variable $p := (z-y)/s - z_0, \forall y \in \mathbb{R}^n$, we obtain from the last estimate that

$$u_-(t,x) \le \left[H_1^{\#}\Big(\frac{x-z}{t-s} - z_0\Big) \vee \big(H_1^{\#}(p) \vee u_0(z - s(p+z_0))\big)\right] \wedge H_{2\#}(z_0), \quad \forall p \in \mathbb{R}^n.$$

Taking infimum both sides in $p \in \mathbb{R}^n$, we obtain

$$\begin{aligned} u_-(t,x) &\le \left[H_1^{\#}\Big(\frac{x-z}{t-s} - z_0\Big) \vee v(s, z - sz_0)\right] \wedge H_{2\#}(z_0) \\ &\le H_1^{\#}\Big(\frac{x-z}{t-s} - z_0\Big) \vee \big[v(s, z-sz_0) \wedge H_{2\#}(z_0)\big] \\ &\le H_1^{\#}\Big(\frac{x-z}{t-s} - z_0\Big) \vee u_-(s,z). \end{aligned}$$

Since z is arbitrary, the last inequality verified (62).

Now, the fact that u_- is a viscosity subsolution of the equation (42) will be proved as follows. We have known that it is not restrictive to suppose that the maximum and the minimum in the definition of viscosity sub- and supersolutions are zero and global. Assume the contrary that u_- is not a viscosity subsolution. Then there exist a constant $\varepsilon_0 > 0$, a point $(t_0, x_0) \in (0,T) \times \mathbb{R}^n$, a function $\varphi \in C^1$, such that $u_- - \varphi$ has zero as its maximum value at (t_0, x_0) and

$$\varphi_t(t_0,x_0) + H(u_-(t_0,x_0), D_x\varphi(t_0,x_0)) > \varepsilon_0.$$

Set $\gamma_0 := u_-(t_0,x_0)$. Since H is continuous, there exists a number $\delta > 0$ such that

$$\varphi_t(t_0,x_0) + H(\gamma_0 - \delta, D_x\varphi(t_0,x_0)) > \varepsilon_0.$$

Using $H_1^{\#*} = H_1$, $H_{2\#*} = H_2$ from Lemma 4.3, we have

$$\varphi_t(t_0,x_0) + \sup_{\{p : H_1^{\#}(p) \le \gamma_0 - \delta\}} (p, D_x\varphi(t_0,x_0)) + \inf_{\{q : H_{2\#}(q) \ge \gamma_0 - \delta\}} (q, D_x\varphi(t_0,x_0)) > \varepsilon_0.$$

Thus there exists $p_0 \in \mathbb{R}^n, H_1^{\#}(p_0) \le \gamma_0 - \delta$ such that

$$\varphi_t(t_0, x_0) + (p_0 + q, D_x\varphi(t_0, x_0)) > \varepsilon_0, \quad \forall q \in \mathbb{R}^n, H_{2\#}(q) \geq \gamma_0 - \delta. \tag{63}$$

On the other hand, let z_0 be selected and fixed at which the maximum in (59) corresponding to (t_0, x_0) is attained, i.e.,

$$\gamma_0 = u_-(t_0, x_0) = v(t_0, x_0 - t_0 z_0) \wedge H_{2\#}(z_0) \leq H_{2\#}(z_0).$$

By virtue of (62) for every $0 < s < t_0, \mu := t_0 - s > 0$,

$$\gamma_0 = u_-(t_0, x_0) \leq \inf_z \{H_1^\#\big(\frac{x_0 - z}{t_0 - s} - z_0\big) \vee u_-(s, z)\}.$$

Changing variable $p := (x_0 - z)/(t_0 - s) - z_0, \forall z \in \mathbb{R}^n$, and then replacing $s = t_0 - \mu$ in the right-hand side of the last inequality, we obtain

$$\begin{aligned} u_-(t_0, x_0) &\leq \inf_p \{H_1^\#(p) \vee u_-(t_0 - \mu, x_0 - \mu(p + z_0)\} \\ &\leq H_1^\#(p_0) \vee u_-(t_0 - \mu, x_0 - \mu(p_0 + z_0)). \end{aligned} \tag{64}$$

Besides, since $u_-(t_0, x_0) - \delta \geq H_1^\#(p_0)$ and u_- is continuous in $(0, T) \times \mathbb{R}^n$, there exists $\mu_0 > 0$ such that

$$H_1^\#(p_0) < u_-(t_0 - \mu, x_0 - \mu(p_0 + z_0)), \quad 0 < \forall \mu < \mu_0.$$

This coupled with (64) gives

$$\varphi(t_0, x_0) = \gamma_0 \leq u_-(t_0 - \mu, x_0 - \mu(p_0 + z_0)) \leq \varphi(t_0 - \mu, x_0 - \mu(p_0 + z_0)),$$

$0 < \forall \mu < \mu_0$. Consequently,

$$\frac{\varphi(t_0 - \mu, x_0 - \mu(p_0 + z_0)) - \varphi(t_0, x_0)}{-\mu} \leq 0, \quad 0 < \forall \mu < \mu_0.$$

Letting $\mu \to 0$ in the last inequality, we see that

$$\varphi_t(t_0, x_0) + (p_0 + z_0, D_x\varphi(t_0, x_0)) \leq 0,$$

which contradicts (63) where z_0 plays the role of a $q \in \mathbb{R}^n$. This contradiction proved that u_- is a viscosity subsolution of the equation (42). We remain to prove (46). By Lemma 4.3, let $q^* \in \mathbb{R}^n$ be taken so that $H_{2\#}(q^*) = +\infty$. Then

$$u_-(t, x) \geq H_{2\#}(q^*) \wedge v(t, x - tq^*) = v(t, x - tq^*). \tag{65}$$

Besides, it follows from (59) that for every $|x| < M$, there exists $z_0 \in \mathbb{R}^n, |z_0| \leq N$ at which

$$u_-(t, x) = v(t, x - tz_0) \wedge H_{2\#}(z_0) \leq v(t, x - tz_0). \tag{66}$$

From (65) and (66), letting $t \to 0$ and using the continuity of v on $[0, T] \times \mathbb{R}^n$ with $v(0, x) = u_0(x)$, we obtain

$$u_-(0, x) = u_0(x), \quad |x| \leq M.$$

Since M is arbitrary, (46) follows. The part i) of Theorem 4.1 is thus completely proved.

ii) By a similar argument, we also get ii). Instead of (62), the following estimate is invoked

$$u_+(t,x) \geq \sup_z \left\{ H_{2\#}\Big(\frac{x-y}{t-s} - y_0\Big) \wedge u_+(s,y) \right\},$$

where $y_0 \in \mathbb{R}^n$ is arbitrary so that

$$u_+(t,x) = w(t, x - ty_0) \vee H_1^{\#}(y_0).$$

■

Proof of Corollary 4.2. If $u_0 \in BUC(\mathbb{R}^n)$, then we can choose the constant N in (60) and (61) independent of $(t,x),(t',x') \in [0,T]\times\mathbb{R}^n$ so that these estimates still hold true. This implies that $u_-, u_+ \in BUC([0,T]\times\mathbb{R}^n)$. Hence, the conclusion follows from Theorem IV.1 of Barles [9]. ■

Example 3. Consider the following Cauchy problem

$$u_t + |D_x u|\mathrm{sh}u = 0 \quad \text{in} \quad (0,T)\times\mathbb{R}^n, \tag{67}$$

$$u(0,x) = u_0(x), \quad \text{in} \quad \mathbb{R}^n, \tag{68}$$

where sh$x$ is the *hyperbolic sine* function

$$\mathrm{sh}x = \frac{\mathrm{e}^x - \mathrm{e}^{-x}}{2}, \quad x \in \mathbb{R}.$$

The Hamiltonian $H(\gamma,p) = |p|\mathrm{sh}\gamma$ can be written as

$$H = H_1 + H_2, \quad H_1(\gamma,p) := \frac{\mathrm{e}^{\gamma}|p|}{2}, \quad H_2(\gamma,p) := -\frac{\mathrm{e}^{-\gamma}|p|}{2}, \quad (\gamma,p)\in\mathbb{R}\times\mathbb{R}^n,$$

meetting the assumption (A). A direct calculation yields

$$H_1^{\#}(q) = \log 2|q|, \quad H_{2\#}(q) = -\log 2|q|, \quad q \in \mathbb{R}^n.$$

Hence, it is derived from the formulas (44) and (45) that

$$u_-(t,x) = \sup_z \inf_y \{[\log 2|y| \vee u_0(x - t(y+z))] \wedge (-\log 2|z|)\},$$

$$u_+(t,x) = \inf_y \sup_z \{\log 2|y| \vee [u_0(x - t(y+z)) \wedge (-\log 2|z|)]\}, \quad (t,x)\in(0,T)\times\mathbb{R}^n.$$

are subsolution and supersolution of (67)-(68)

Example 4. Let $f(\gamma), \gamma \in \mathbb{R}$, be an any continuous nondecreasing function. Our results can be applied to a Hamiltonian of the form

$$H(\gamma,p) := f(\gamma)|p|, \quad (\gamma,p)\in\mathbb{R}\times\mathbb{R}^n.$$

Actually, we need only to determine

$$H_1(\gamma, p) := \max\{f(\gamma), 0\}|p|,$$
$$H_2(\gamma, p) := \min\{f(\gamma), 0\}|p|, \quad (\gamma, p) \in \mathbb{R} \times \mathbb{R}^n.$$

Clearly, these functions satisfy the hypothesis (A).

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