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Lacunary Strongly Summable Sequences and q-Lacunary Almost Statistical Convergence

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Abstract. A lacunary sequence is an increasing sequence $\theta=(k_r)$ of positive integers such that $k_0=0$ and $k_r-k_{r-1}\to\infty$ as $r\to\infty$. A sequence $x=(x_k)$ is called $q-$lacunary almost statistical convergent to ξ provided that for each $\varepsilon>0$, $\lim_r(k_r-k_{r-1})^{-1}\{$ the number of $k{:}k_{r-1}<k\leqslant k_r{:}q(t_{km}(x)-\xi)\geqslant\varepsilon\}=0$. The purpose of this paper is to introduce the concept of $q-$ lacunary strongly almost convergence with respect to an Orlicz function and $q-$ lacunary almost statistical convergence, and examine some properties of these sequence spaces. We establish some connections between $q-$lacunary strongly almost convergence and $q-$lacunary almost statistical convergence. It is also shown that if a sequence is $q-$lacunary strongly almost convergent with respect to an Orlicz function then it is $q-$ lacunary almost statistical convergent.

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1. Introduction

Let w denote the set of all real sequences $x=(x_n)$. By ℓ_∞ and c, we denote the Banach spaces of bounded and convergent sequences $x=(x_n)$ normed by $\|x\|=\sup_n|x_n|$, respectively. A linear functional $\mathcal{L}$ on ℓ_∞ is said to be a Banach limit [1] if it has the properties:

i) $\mathcal{L}(x) \geqslant 0$ if $x \geqslant 0$ (i.e. $x_n \geqslant 0$ for all n),
ii) $\mathcal{L}(e) = 1$, where $e = (1, 1, \dots)$,
iii) $\mathcal{L}(Dx) = \mathcal{L}(x)$,
where D is the shift operator defined by $(Dx_n) = (x_{n+1})$.

Let $\mathfrak{B}$ be the set of all Banach limits on ℓ_∞. A sequence x is said to be almost convergent to a number ξ if $\mathcal{L}(x) = \xi$ for all $\mathcal{L} \in \mathfrak{B}$. Lorentz [12] has shown that x is almost convergent to ξ if and only if

$$t_{km} = t_{km}(x) = \frac{x_m + x_{m+1} + \ldots + x_{m+k}}{k+1} \to \xi \quad \text{as } k \to \infty, \text{ uniformly in } m.$$

Let f denote the set of all almost convergent sequences. We write $f - \lim x = \xi$ if x is almost convergent to ξ. Maddox [13] and (independently) Freedman et al. [7] have defined x to be strongly almost convergent to a number ξ if

$$\frac{1}{k+1} \sum_{i=0}^{k} |x_{i+m} - \xi| \to 0 \text{ as } k \to \infty, \text{ uniformly in } m.$$

Let $[f]$ denote the set of all strongly almost convergent sequences. If x is strongly almost convergent to ξ, we write $[f] - \lim x = \xi$. It is easy to see that $[f] \subset f \subset \ell_\infty$. Das and Sahoo [4] defined the sequence space

$$[w(p)] = \Big\{ x \in w : \frac{1}{n+1} \sum_{k=0}^{n} |t_{km}(x) - \xi|^{p_k} \to 0 \text{ as } n \to \infty, \text{ uniformly in } \quad m \Big\}$$

and investigated some of its properties.

The definition of statistical convergence was introduced by Fast [6] in a short note. Schoenberg [20] studied statistical convergence as a summability method and listed some of the elementary properties of statistical convergence. Recently, statistical convergence has been studied by various authors (cf. [3, 8, 9, 14, 17, 18]).

The statistical convergence depends on the density of the subsets of $\mathbb{N}$, the set of natural numbers. A subset E of $\mathbb{N}$ is said to have density $\delta(E)$ if

$$\delta(E) = \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \chi_E(k) \quad \text{exists},$$

where χ_E is the characteristic function of E.

A sequence (x_n) is said to be statistically convergent to ξ if for every $\varepsilon > 0$, $\delta\big(\{k \in \mathbb{N} : |x_k - \xi| \geqslant \varepsilon\}\big) = 0$. In this case we write stat-$\lim x_k = \xi$.

Let $\theta = (k_r)$ be the sequence of positive integers such that $k_0 = 0$, $0 < k_r < k_{r+1}$ and $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$. Then θ is called a lacunary sequence. The intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$ and the ratio k_r / k_{r-1} will be denoted by η_r.

Lacunary sequences have been studied in [2, 5, 7, 9, 19].

An Orlicz function is a function $M : [0, \infty) \to [0, \infty)$, which is continuous, non-decreasing and convex with $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$.

Lindenstrauss and Tzafriri [11] used the idea of Orlicz function to construct the sequence space

$$\ell_M = \Big\{x \in w : \sum_{k=1}^{\infty} M\Big(\frac{|x_k|}{\rho}\Big) < \infty \quad \text{for some} \quad \rho > 0\Big\}.$$

The space ℓ_M with the norm

$$\|x\| = \inf\Big\{\rho > 0 : \sum_{k=1}^{\infty} M\Big(\frac{|x_k|}{\rho}\Big) \leqslant 1\Big\}$$

becomes a Banach space, called an Orlicz sequence space. The space ℓ_M is closely related to the space ℓ_p which is an Orlicz sequence space with $M(x) = |x|^p$ for $1 \leqslant p < \infty$.

Recently Orlicz sequence spaces have been studied by Mursaleen et al. [15], Bhardwaj and Singh [2], Savaş and Rhoades [19] and many others.

A sequence space E is said to be solid (or normal) if $(\alpha_k x_k) \in E$ whenever $(x_k) \in E$ for all sequences (α_k) of scalars with $|\alpha_k| \leqslant 1$ for all $k \in \mathbb{N}$.

A sequence space E is said to be monotone if it contains the canonical preimages of its step spaces [10].

Remark. Two Orlicz functions M_1 and M_2 are said to be equivalent if there are positive constants α and β, and x_0 such that $M_1(\alpha x) \leqslant M_2(x) \leqslant M_1(\beta x)$ for all x with $0 \leqslant x \leqslant x_0$ [10].

It is well known that if M is a convex function and $M(0) = 0$, then $M(\lambda x) \leqslant \lambda M(x)$ for all λ with $0 < \lambda < 1$.

2. Main Results

Let M be an Orlicz function, $p = (p_k)$ be a sequence of positive real numbers and X be a seminormed space over the field $\mathbb{C}$ of complex numbers with the seminorm q. $w(X)$ denotes the space of all sequences $x = (x_k)$, where $x_k \in X$. We define the following sequence spaces:

$$(W, \theta, M, p, q) = \Big\{x \in w(X) : \lim_r \frac{1}{h_r} \sum_{k \in I_r} [M(q(\frac{t_{km}(x) - \xi}{\rho}))]^{p_k} = 0,$$
$$\text{uniformly in } m \text{ for some } \xi \text{ and for some } \rho > 0\Big\},$$

$$(W, \theta, M, p, q)_0 = \Big\{x \in w(X) : \lim_r \frac{1}{h_r} \sum_{k \in I_r} [M(q(\frac{t_{km}(x)}{\rho}))]^{p_k} = 0,$$
$$\text{uniformly in } m \text{ for some } \rho > 0\Big\},$$

$$(W, \theta, M, p, q)_\infty = \Big\{x \in w(X) : \sup_{r,m} \frac{1}{h_r} \sum_{k \in I_r} [M(q(\frac{t_{km}(x)}{\rho}))]^{p_k} < \infty,$$
$$\text{for some } \rho > 0\Big\}.$$

We get the following sequence spaces from the above sequence spaces on giving particular values to θ, M and p.
i) If $p_k = 1$ for all $k \in \mathbb{N}$, then we shall denote (W,θ,M,p,q), $(W,\theta,M,p,q)_0$ and $(W,\theta,M,p,q)_\infty$ by (W,θ,M,q), $(W,\theta,M,q)_0$ and $(W,\theta,M,q)_\infty$, respectively.

If $x \in (W,\theta,M,q)$ we say that x is $q-$ lacunary strongly almost convergent with respect to the Orlicz function M.
ii) Taking $p_k = 1$ for all $k \in \mathbb{N}$ and $M(x) = x$, we denote the above sequence spaces by (W,θ,q), $(W,\theta,q)_0$ and $(W,\theta,q)_\infty$, respectively.
iii) In the case $\theta = (2^r)$ we shall denote the above sequence spaces by (W,M,p,q), $(W,M,p,q)_0$ and $(W,M,p,q)_\infty$, respectively.

Theorem 2.1. *Let M be an Orlicz function. Then $(W,\theta,M,p,q)_0 \subset (W,\theta,M,p,q) \subset (W,\theta,M,p,q)_\infty$.*

Proof. Let $x \in (W,\theta,M,p,q)$. Then we have

$$\frac{1}{h_r}\sum_{k\in I_r}\Big[M\Big(\frac{t_{km}(x)}{\rho}\Big)\Big]^{p_k} \leqslant \frac{D}{h_r}\sum_{k\in I_r}\Big[M\Big(q\Big(\frac{t_{km}(x)-\xi}{\rho}\Big)\Big)\Big]^{p_k} + \frac{D}{h_r}\sum_{k\in I_r}\Big[M\Big(\frac{q(\xi)}{\rho}\Big)\Big]^{p_k}$$
$$\leqslant \frac{D}{h_r}\sum_{k\in I_r}\Big[M\Big(q\Big(\frac{t_{km}(x)-\xi}{\rho}\Big)\Big)\Big]^{p_k} + D\max\Big\{1, \sup\Big[M\Big(\frac{q(\xi)}{\rho}\Big)\Big]^{H}\Big\},$$

where $\sup_k p_k = G$, $H = \max(1,G)$ and $D = \max(1, 2^{G-1})$.
Thus we get $x \in (W,\theta,M,p,q)_\infty$. The inclusion $(W,\theta,M,p,q)_0 \subset (W,\theta,M,p,q)$ is obvious.

Theorem 2.2. *Let the sequence (p_k) be bounded, then $(W,\theta,M,p,q)_0, (W,\theta, M,p,q)$ and $(W,\theta,M,p,q)_\infty$ are linear spaces over the set of complex numbers.*

Proof. Omitted.

Theorem 2.3. *The spaces $(W,\theta,M,p,q)_0$, (W,θ,M,p,q) and $(W,\theta,M,p,q)_\infty$ are paranormed spaces (not totally paranormed), paranormed by*

$$g(x) = \inf\Big\{\rho^{p_r/H} : \sup_k M\Big(q\Big(\frac{t_{km}(x)}{\rho}\Big)\Big) \leqslant 1,\ \ \rho > 0, \quad \text{uniformly in } \ m\Big\},$$

Proof. Clearly $g(x) = g(-x)$, and $q\big(\frac{t_{km}(\bar\theta)}{\rho}\big) = q(\bar\theta) = 0$ where $\bar\theta$ is the zero sequence. Nothing that $M(0) = 0$, from the above one gets, $g(\bar\theta) = 0$. Next let (x_k), $(y_k) \in (W,\theta,M,p,q)_0$. Let $\rho_1 > 0$ and $\rho_2 > 0$ be such that

$$\sup_k M\Big(q\Big(\frac{t_{km}(x)}{\rho_1}\Big)\Big) \leqslant 1, \quad \text{uniformly in } \ m \tag{1}$$

and

$$\sup_k M\Big(q\Big(\frac{t_{km}(y)}{\rho_2}\Big)\Big) \leqslant 1, \quad \text{uniformly in } \ m. \tag{2}$$

Let $\rho = \rho_1 + \rho_2$. Then we have

$$\sup_k M\Big(q\Big(\frac{t_{km}(x+y)}{\rho}\Big)\Big) \leqslant \Big(\frac{\rho_1}{\rho_1+\rho_2}\Big)\sup_k M\Big(q\Big(\frac{t_{km}(x)}{\rho_1}\Big)\Big)$$
$$+\Big(\frac{\rho_2}{\rho_1+\rho_2}\Big)\sup_k M\Big(q\Big(\frac{t_{km}(y)}{\rho_2}\Big)\Big) \leqslant 1, \quad \text{uniformly in } m$$

by (1) and (2). Hence $g(x+y) \leqslant g(x)+g(y)$.

The continuity of scalar multiplication follows from the following equality:

$$g(\lambda x) = \inf\Big\{\rho^{p_r/H} : \sup_k M\Big(q\Big(\frac{t_{km}(\lambda x)}{\rho}\Big)\Big) \leqslant 1,\ \rho>0, \quad \text{uniformly in } m\Big\}$$
$$= \inf\Big\{(|\lambda|s)^{p_r/H} : \sup_k M\Big(q\Big(\frac{t_{km}(x)}{\rho}\Big)\Big) \leqslant 1,\ \rho>0, \quad \text{uniformly in } m\Big\},$$

where $s = \frac{\rho}{|\lambda|}$.

Theorem 2.4. *Let M_1 and M_2 be Orlicz functions. Then we have*

i) $(W,\theta,M_1,p,q)_0 \cap (W,\theta,M_2,p,q)_0 \subset (W,\theta,M_1+M_2,p,q)_0$,

ii) $(W,\theta,M_1,p,q) \cap (W,\theta,M_2,p,q) \subset (W,\theta,M_1+M_2,p,q)$,

iii) $(W,\theta,M_1,p,q)_\infty \cap (W,\theta,M_2,p,q)_\infty \subset (W,\theta,M_1+M_2,p,q)_\infty$.

Proof. It is straightforward and hence omitted.

Theorem 2.5. *Let $0 < p_k \leqslant t_k$ and $\big(\frac{t_k}{p_k}\big)$ be bounded. Then $(W,\theta,M,t,q) \subset (W,\theta,M,p,q)$.*

Proof. If we take $w_{k,m} = \big[M\big(q\big(\frac{t_{km}(x)}{\rho}\big)\big)\big]^{t_k}$ for all k,m and use the same technique of Theorem 2 of Nanda [16], the theorem is easily to be proved.

Theorem 2.6. *The sequence spaces $(W,\theta,M,p,q)_0$ and $(W,\theta,M,p,q)_\infty$ are neither solid nor monotone.*

Proof. We give the proof only for $(W,\theta,M,p,q)_0$. For this let $p_k = 1$, for $k \in \mathbb{N}$, $\theta = (2^r)$ $M(x) = x$ and $q(x) = |x|$. Consider two sequences $x_k = (-1)^k$ and $\alpha_k = (-1)^k$ for all $k \in \mathbb{N}$. Then $(x_k) \in (W,\theta,M,p,q)_0$ but $(\alpha_k x_k) \notin (W,\theta,M,p,q)_0$. Hence $(W,\theta,M,p,q)_0$ is not solid.

Consider the $J-$ stepspace $[(W,\theta,M,p,q)_0]_J$ of $(W,\theta,M,p,q)_0$. Given a sequence $x = (x_k) \in (W,\theta,M,p,q)_0$ let us define $y = (y_k) \in [(W,\theta,M,p,q)_0]_J$ as $y_k = x_k$ for odd k and $y_k = 0$, otherwise. Then $(y_k) \notin (W,\theta,M,p,q)_0$. Hence $(W,\theta,M,p,q)_0$ is not monotone.

The other cases can be proved on considering similar examples.

The following theorem can be proved using the same techniques of Theorem 2.5 and Theorem 2.6 of Savas and Rhoades [19], therefore we give without proof.

Theorem 2.7. *Let $\theta = (k_r)$ be a lacunary sequence with $1 < \liminf_r \eta_r \leqslant \limsup_r \eta_r < \infty$. Then for any Orlicz function M, we have $(W,M,p,q) = (W,\theta,M,p,q)$.*

Corollary 2.8. *$(W,\theta,M,p,q)_0$ and (W,θ,M,p,q) are nowhere dense subsets of $(W,\theta,M,p,q)_\infty$.*

Proof. Proof follows from Theorem 2.1.

Theorem 2.9. *Let M_1 and M_2 be two Orlicz functions. If M_1 and M_2 are equivalent then*

i) $(W,\theta,M_1,p,q)_0 = (W,\theta,M_2,p,q)_0$,

ii) $(W,\theta,M_1,p,q) = (W,\theta,M_2,p,q)$,

iii) $(W,\theta,M_1,p,q)_\infty = (W,\theta,M_2,p,q)_\infty$.

Proof. Proof follows from the equivalence of M_1 and M_2.

3. $q-$ Lacunary Almost Statistical Convergence

In this section we define $q-$ lacunary almost statistical convergence and give some relations between $q-$lacunary almost statistical convergence and $q-$ lacunary strongly almost convergence with respect to an Orlicz function.

Definition 3.1. *Let θ be a lacunary sequence, then the sequence $x=(x_k)$ is said to be $q-$lacunary almost statistically convergent to the number ξ provided that for every $\varepsilon>0$,*

$$\lim_r \frac{1}{h_r}|\{k\in I_r : q(t_{km}(x)-\xi)\geqslant\varepsilon\}| = 0, \quad \text{uniformly in } m.$$

In this case we write $[S_\theta]_q - \lim x = \xi$ or $x_k \to \xi([S_\theta]_q)$ and we define

$$[S_\theta]_q = \{x\in w(X) : [S_\theta]_q - \lim x = \xi, \quad \text{for some } \xi\}.$$

In the case $\theta=(2^r)$, we shall write $[S]_q$ instead of $[S_\theta]_q$.

Definition 3.2. *Let θ be a lacunary sequence and $0<p<\infty$. Then the sequence $x=(x_k)$ is said to be $q-$lacunary strongly almost convergent to the number ξ provided that*

$$\lim_r \frac{1}{h_r}\sum_{k\in I_r}(q(t_{km}(x)-\xi))^p = 0, \quad \text{uniformly in } m.$$

In this case we write $[w_\theta]_q - \lim x = \xi$ or $x_k\to\xi([w_\theta]_q)$ and we define

$$[w_\theta]_q = \{x\in w(X) : [w_\theta]_q - \lim x = \xi, \quad \text{for some } \xi\}.$$

Theorem 3.3. *Let θ be a lacunary sequence.*

i) *If a sequence (x_k) is $q-$lacunary strongly almost convergent to ξ, then it is $q-$lacunary almost statistically convergent to ξ.*

ii) *If a $q-$bounded sequence x (that is $x\in\ell_\infty(q)$) is $q-$lacunary almost statistically convergent to ξ, then it is $q-$lacunary strongly almost convergent to ξ.*

iii) $\ell_\infty(q)\cap[S_\theta]_q = \ell_\infty(q)\cap[w_\theta]_q$, *where,* $\ell_\infty(q) = \{x \in w(X) : \sup_k q(x) < \infty\}$.

Proof.
(i) Let $\varepsilon > 0$ and $x_k \to \xi([w_\theta]_q)$. Then we can write

$$\sum_{k\in I_r} (q(t_{km}(x) - \xi))^p \geqslant \sum_{\substack{k\in I_r \\ |t_{km}(x)-\xi|\geqslant\varepsilon}} (q(t_{km}(x) - \xi))^p \geqslant \varepsilon^p \big|\{k \in I_r : q(t_{km}(x) - \xi) \geqslant \varepsilon\}\big|.$$

Hence $x_k \to \xi([S_\theta]_q)$.
ii) Suppose that $x_k \to \xi([S_\theta]_q)$ and let $x \in \ell_\infty(q)$. Let $\varepsilon > 0$ be given and select N_ε such that

$$\frac{1}{h_r}\left|\left\{k \in I_r : q(t_{km}(x) - \xi) \geqslant \left(\frac{\varepsilon}{2}\right)^{\frac{1}{p}}\right\}\right| \leqslant \frac{\varepsilon}{2K^p}$$

for all m and $r > N_\varepsilon$ and set $L_{rm} = \{k \in I_r : q(t_{km}(x) - \xi) \geqslant \left(\frac{\varepsilon}{2}\right)^{\frac{1}{p}}\}$, where $K = \sup_{k,m}(q(t_{km}(x) - \xi))^p$. Now for all m and $r > N_\varepsilon$ we have

$$\begin{aligned} \frac{1}{h_r}\sum_{k\in I_r} q(t_{km}(x) - \xi)^p &= \frac{1}{h_r}\sum_{\substack{k\in I_r \\ k\in L_{rm}}} q(t_{km}(x) - \xi)^p + \frac{1}{h_r}\sum_{\substack{k\in I_r \\ k\notin L_{rm}}} q(t_{km}(x) - \xi)^p \\ &\leqslant \frac{1}{h_r}\Big(\frac{h_r\varepsilon}{2K^p}\Big)K^p + \frac{\varepsilon}{2h_r}h_r = \varepsilon. \end{aligned}$$

Thus $(x_k) \in [w_\theta]_q$. This completes the proof.
The proof of (iii) follows from (i) and (ii). ∎

Theorem 3.4. *For any lacunary sequence* θ, *if* $\liminf_{r\to\infty} \eta_r > 1$, *then* $[S]_q \subset [S_\theta]_q$.

Proof. If $\liminf_{r\to\infty} \eta_r > 1$, then there exists a $\delta > 0$ such that $1 + \delta \leqslant \eta_r$ for sufficiently large r. Since $h_r = k_r - k_{r-1}$, we have $\frac{k_r}{h_r} \leqslant \frac{1+\delta}{\delta}$. Let $x_k \to \xi([S_\theta]_q)$. Then for every $\varepsilon > 0$ and for all m we have

$$\begin{aligned} \frac{1}{k_r}\big|\{k \leqslant k_r : q(t_{km}(x) - \xi) \geqslant \varepsilon\}\big| &\geqslant \frac{1}{k_r}\big|\{k \in I_r : q(t_{km}(x) - \xi) \geqslant \varepsilon\}\big| \\ &\geqslant \frac{\delta}{1+\delta}\frac{1}{h_r}\big|\{k \in I_r : q(t_{km}(x) - \xi) \geqslant \varepsilon\}\big|. \end{aligned}$$

Hence $[S]_q \subset [S_\theta]_q$.

Theorem 3.5. *For any lacunary sequence* θ, *if* $\limsup_r q_r < \infty$, *then* $[S_\theta]_q \subset [S]_q$.

Proof. Suppose that $\limsup_r q_r < \infty$. Then there exists a $\beta > 0$ such that $\eta_r < \beta$ for all r. Let $x_k \to \xi([S_\theta]_q)$, and for each $m \geqslant 1$ set $E_{rm} = |\{k \in I_r : q(t_{km}(x) - \xi) \geqslant \varepsilon\}|$. Then there exists an $r_0 \in \mathbb{N}$ such that $\frac{E_{rm}}{h_r} < \varepsilon$ for all $r > r_0$

and for each $m \geqslant 1$. Let $K = \max\{E_{rm} : 1 \leqslant r \leqslant r_0\}$ and choose n such that $k_{r-1} < n \leqslant k_r$, then for each $m \geqslant 1$ we have

$$\frac{1}{n}\left|\left\{k \leqslant n : q(t_{km}(x) - \xi) \geqslant \varepsilon\right\}\right| \leqslant \frac{1}{k_{r-1}}\left|\left\{k \leqslant k_r : q(t_{km}(x) - \xi) \geqslant \varepsilon\right\}\right|$$
$$\leqslant \frac{1}{k_{r-1}}\left\{E_{1m} + E_{2m} + \cdots + E_{r_0 m} + E_{(r_0+1)m} + \cdots + E_{rm}\right\}$$
$$\leqslant \frac{K}{k_{r-1}}r_0 + \frac{1}{k_{r-1}}\left\{\frac{E_{(r_0+1)m}}{h_{r_0+1}}h_{r_0+1} + \cdots + \frac{E_{rm}}{h_r}h_r\right\}$$
$$\leqslant \frac{K}{k_{r-1}}r_0 + \frac{1}{k_{r-1}}\Big(\sup_{r>r_0}\frac{E_{rm}}{h_r}\Big)\{h_{r_0+1} + \cdots + h_r\}$$
$$\leqslant \frac{K}{k_{r-1}}r_0 + \varepsilon\frac{k_r - k_{r_0}}{k_{r-1}}$$
$$\leqslant \frac{K}{k_{r-1}}r_0 + \varepsilon q_r$$
$$\leqslant \frac{K}{k_{r-1}}r_0 + \varepsilon\beta.$$

This completes the proof. ■

Theorem 3.6. *Let M be an Orlicz function. Then $(W,\theta,M,p,q) \subset [S_\theta]_q$.*

Proof. Let $x \in (W,\theta,M,p,q)$. Then there exists a number $\rho > 0$ such that

$$\frac{1}{h_r}\sum_{k\in I_r}\Big[M\Big(q\Big(\frac{t_{km}(x)-\xi}{\rho}\Big)\Big)\Big]^{p_k} \to 0, \text{ as } r \to \infty.$$

Then given $\varepsilon > 0$ we have

$$\frac{1}{h_r}\sum_{k\in I_r}\Big[M\Big(q\Big(\frac{t_{km}(x)-\xi}{\rho}\Big)\Big)\Big]^{p_k} \geqslant \frac{1}{h_r}\sum_{\substack{k\in I_r\\ q(t_{km}(x)-\xi)\geqslant\varepsilon}}\Big[M\Big(q\Big(\frac{t_{km}(x)-\xi}{\rho}\Big)\Big)\Big]^{p_k}$$
$$\geqslant \frac{1}{h_r}\sum_{\substack{k\in I_r\\ q(t_{km}(x)-\xi)\geqslant\varepsilon}}[M(\varepsilon_1)]^{p_k}, \text{ where } \varepsilon/\rho = \varepsilon_1$$
$$\geqslant \frac{1}{h_r}\sum\min\left\{\big[M(\varepsilon_1)\big]^{\inf p_k}, [M(\varepsilon_1)]^G\right\}$$
$$\geqslant \frac{1}{h_r}\left|\left\{k \in I_r : q(t_{km}(x)-\xi) \geqslant \varepsilon\right\}\right|.\min\left\{\big[M(\varepsilon_1)\big]^{\inf p_k}, [M(\varepsilon_1)]^G\right\}.$$

Hence $x \in [S_\theta]_q$.

Theorem 3.7. $[S_\theta]_q \cap \ell_\infty(q) = (W,\theta,M,q) \cap \ell_\infty(q)$.

Proof. By Theorem 3.6, we need only show that

$$[S_\theta]_q \cap \ell_\infty(q) \subset (W,\theta,M,q) \cap \ell_\infty(q).$$

For each $m \geqslant 1$, let $y_{km} = (t_{km}(x) - \xi) \to 0(S_\theta)$. Since $(x_k) \in \ell_\infty(q)$, for each $m \geqslant 1$ there exists $K > 0$ such that

$$M\Big[q\Big(\frac{y_{km}}{\rho}\Big)\Big] \leqslant K$$

for all y_{km}. Then given $\varepsilon > 0$ and for each $m \in \mathbb{N}$, we have

$$\begin{aligned}\frac{1}{h_r}\sum_{k\in I_r} M\Big[q\Big(\frac{y_{km}}{\rho}\Big)\Big] &= \frac{1}{h_r}\sum_{\substack{k\in I_r\\ q(t_{km}(x)-L)\geqslant\varepsilon}} M\Big[q\Big(\frac{y_{km}}{\rho}\Big)] + \frac{1}{h_r}\sum_{\substack{k\in I_r\\ q(t_{km}(x)-L)<\varepsilon}} M[q(\frac{y_{km}}{\rho}\Big)\Big]\\ &\leqslant \frac{K}{h_r}|\big\{k \in I_r : q(y_{km}) \geqslant \varepsilon\rho\big\}| + M(\frac{\varepsilon}{\rho}).\end{aligned}$$

Hence $x \in (W, \theta, M, q) \cap \ell_\infty(q)$.

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