Vietnam Journal of Mathematics **33**:3 (2005) 261–270

Vietnam Journal
of
MATHEMATICS

On the Asymptotic Distribution of the Bootstrap Estimate with Random Resample Size*

Nguyen Van Toan

Department of Mathematics, College of Science,
Hue University, 77 Nguyen Hue, Hue, Vietnam

Received Demcember 19, 2003

Abstract In this paper, we study the bootstrap with random resample size which is not independent of the original sample. We find sufficient conditions on the random resample size for the central limit theorem to hold for the bootstrap sample mean.

1. Introduction

Efron [5] discusses a "bootstrap" method for setting confidence intervals and estimating significance levels. This method consists of approximating the distribution of a function of the observations and the underlying distribution, such as a pivot, by what Efron calls the bootstrap distribution of this quantity. This distribution is obtained by replacing the unknown distribution by the empirical distribution of the data in the definition of the statistical function, and then resampling the data to obtain a Monte Carlo distribution for the resulting random variable. Efron gives a series of examples in which this principle works, and establishes the validity of the approach for a general class of statistics when the sample space is finite.

The first necessary condition for the bootstrap of the mean for independent identically distributed (i.i.d.) sequences and resampling size equal to the sample size was given in [8] showing that the bootstrap works a.s. if and only if the common distribution of the sequence has finite second moment, while it works

*This research is supported in part by the National Fundamental Research Program in Natural Science Vietnam, No. 130701.

in probability if and only if that distribution belongs to the domain of attraction of the normal law. Hall [10] completes the analysis in this setup showing that when there exists a bootstrap limit law (in probability) then either the parent distribution belongs to the domain of attraction of the normal law or it has slowly varying tails and one of the two tails completely dominates the other.

The interest of considering resampling sizes different to the sample size was noted among others by Bickel and Freedman [3], Swanepoel [19] and Athreya [1].

In sufficiently regular cases, the bootstrap approximation to an unknown distribution function has been established as an improvement over the simpler normal approximation (see [2, 6 - 7]). In the case where the bootstrap sample size N is in itself a random variable, Mammen [11] has considered bootstrap with a Poisson random sample size which is independent of the sample. Stemming from Efron's observation that the information content of a bootstrap sample is based on approximately $(1-\mathrm{e}^{-1})100\% \approx 63\%$ of the original sample, Rao, Pathak and Koltchinskii [17] have introduced a sequential resampling method in which sampling is carried out one-by-one (with replacement) until $(m+1)$ distinct original observation appear, where m denotes the largest integer not exceeding $(1-\mathrm{e}^{-1})n$. It has been shown that the empirical characteristics of this sequential bootstrap are within a distance $O(n^{-3/4})$ from the usual bootstrap. The authors provide a heuristic argument in favor of their sampling scheme and establish the consistency of the sequential bootstrap. Our work on this problem is limited to [12 - 16] and [20 - 21]. In these references we consider bootstrap with a random resample size which is independent of the original sample and find sufficient conditions for random resample size that random sample size bootstrap distribution can be used to approximate the sampling distribution. The purpose of this paper is to study bootstrap with a random resample size which is not independent of the original sample.

2. Results

Let $S_n = (X_1,\ X_2, \dots ,\ X_n)$ be a random sample from a distribution F and $\theta(F)$ a parameter of interest. Let F_n denote the empirical distribution function based on S_n and suppose that $\theta(F_n)$ is an estimator of $\theta(F)$. The Efron bootstrap method approximates the sampling distribution of a standardized version of $\sqrt{n}(\theta(F_n) - \theta(F))$ by the resampling distribution of a corresponding statistic $\sqrt{n}(\theta(F_n^*) - \theta(F_n))$ based on a bootstrap sample S_n^*. Here the original F has been replaced by the empirical distribution based on the original sample S_n and F_n of the former statistic has been replaced by the empirical distribution based on a bootstrap sample F_n^*. In Efron's bootstrap resampling scheme, $S_n^* = (X_{n1}^*,\ X_{n2}^*, \dots ,\ X_{nn}^*)$ is a random sample of size n drawn from S_n by simple random sampling with replacement. In Rao, Pathak and Koltchinskii [17] sequential scheme, observations are drawn from S_n sequentially by simple random sampling with replacement until there are $m+1 = [n(1-\mathrm{e}^{-1})] + 2$ distinct original observations in the bootstrap sample; the last observation is discarded to ensure technical simplicity. Thus an observed bootstrap sample under the Rao-Pathak-Koltchinskii scheme admits the form

$$S^*_{N_n} = (X^*_{n1},\ X^*_{n2}, \dots ,\ X^*_{nN_n})$$

where $X^*_{n1},\ X^*_{n2}, \dots ,\ X^*_{nN_n}$ have $m \approx n(1-\mathrm{e}^{-1})$ distinct observations from S_n. The random sample size N_n admits the following decomposition in terms of the independent random variables:

$$N_n = N_{n1} + N_{n2} + \dots + N_{nm}$$

where $m = [n(1-\mathrm{e}^{-1})] + 1;\ N_1 = 1$ and for each $k,\ 2 \le k \le m$,

$$P^*(N_{nk} = i) = \Big(1 - \frac{k-1}{n}\Big)\Big(\frac{k-1}{n}\Big)^{i-1},$$

where P^* denotes conditional probability $P(\dots|X_1, \dots , X_n)$.

Rao, Pathak and Koltchinskii [17] have established the consistency of this sampling scheme. In this paper we investigate the random bootstrap sample size N_n such that the following condition is satisfied:

(1) Along almost all sample sequences $X_1,\ X_2,\ \dots$, given $S_n = (X_1, X_2, \dots , X_n)$, as n tends to infinity, the sequence $\left(\frac{N_n}{k_n}\right)_{1\le n<\infty}$ converges in conditional probability to a positive random variable ν, where $(k_n)_{1\le n<\infty}$ is an increasing sequence of positive integer number tending to infinity when n tends to infinity: that is, for $\varepsilon > 0$,

$$P^*\Big\{\Big|\frac{N_n}{k_n} - \nu\Big| > \varepsilon\Big\} \to 0 \quad \text{a.s.}$$

We state now our main result.

Theorem 2.1. *Let $X_1, X_2, \dots$ be a sequence of i.i.d random variables on a probability space $(\Omega,\ \mathcal{A},\ P)$ with mean μ and finite positive variance σ^2. Let F_n be the empirical distribution of $S_n = (X_1, \dots , X_n)$. Given $S_n = (X_1, \dots , X_n)$, let $X^*_{n1}, \dots , X^*_{nm}, \dots$ be conditionally independent random variables with common distribution F_n and $(N_n)_{n\ge1}$ be a sequence of positive integer valued random variables such that condition* (1) *holds. Denote*

$$\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i,\ \ \bar{X}^*_{N_n} = \frac{1}{N_n}\sum_{i=1}^{N_n} X^*_{ni},\ \ s^{*2}_{N_n} = \frac{1}{N_n}\sum_{i=1}^{N_n}(X^*_{ni} - \bar{X}^*_{N_n})^2.$$

Along almost all sample sequences, as n tends to infinity:

$$\sup_{-\infty<x<+\infty}\big|P\big\{\sqrt{n}(\bar{X}_n - \mu) < x\big\} - P^*\big\{\sqrt{N_n}(\bar{X}^*_{N_n} - \bar{X}_n) < x\big\}\big| \to 0.$$

3. Proofs

For the proof of Theorem 2.1 we will need the following results.

Lemma 3.1. (Guiasu, [9]) *Let*

$$(W_n)_{1\leq n<\infty},\ (x_{mn})_{\substack{1\leq n<\infty\\1\leq m<\infty}},\ (y_{mn})_{\substack{1\leq n<\infty\\1\leq m<\infty}}$$

be sequences of random variables such that for every m and n we have

$$W_n = x_{mn} + y_{mn}.$$

Let us suppose that the following conditions are satisfied:
(A) *The distribution functions of the sequence $(x_{mn})_{1\leq n<\infty}$ converge to the distribution function F for each fixed m;*
(B) $\forall\varepsilon>0:\ \lim_{m\to\infty}\limsup_n P(|y_{mn}|>\varepsilon)=0$
then distribution functions of sequence $(W_n)_{1\leq n<\infty}$ converge also to F.

Lemma 3.2. [4, Lemma 3] *Let $(\eta_n)_{1\leq n<\infty}$ be a sequence of independent random variables, further let $(k_n)_{1\leq n<\infty}$ and $(m_n)_{1\leq n<\infty}$, $k_n\leq m_n$, be two (not constant) sequences of natural numbers. If for each n, A_n is an event depending only on the random variables $\eta_{k_n},\dots,\ \eta_{m_n}$ then for every event A, having positive probability:*

$$\limsup_n P(A_n|A) = \limsup_n P(A_n).$$

The proof of Theorem 2.1 is somewhat long, so we shall separate out the major steps and present them in the form of lemmas.

Denote

$$s_n^2=\frac{1}{n}\sum_{i=1}^{n}(X_i-\bar{X}_n)^2,\ \bar{X}^*_{nm}=\frac{1}{m}\sum_{i=1}^{m}X^*_{ni},$$
$$s_m^{*2}=\frac{1}{m}\sum_{i=1}^{m}(X^*_{ni}-\bar{X}^*_{nm})^2 \text{ and } Y^*_{nm}=\frac{\sqrt{m}}{s_n}\left(\bar{X}^*_{nm}-\bar{X}_n\right).$$

Lemma 3.3. *For every event A, having positive probability, we have*

$$\lim_{\substack{m\to\infty\\n\to\infty}} P^*_A(Y^*_{nm}\leq x)=\Phi(x)\quad a.s.,$$

where $P^*_A(\dots)$ is conditional probability $P^*(\dots|A)$ and $\Phi(x)$ is the standard normal distribution function.

Proof. For every event A, $P^*(A)>0$, we have

$$\lim_{\substack{m\to\infty\\n\to\infty}} P^*_A(Y^*_{nm}\leq x)=\Phi(x)\Leftrightarrow \lim_{\substack{m\to\infty\\n\to\infty}} E^*(e^{itY^*_{nm}}|A)=e^{-\frac{t^2}{2}},\quad \forall t,$$

where $E^*(\dots)$ is the conditional expectation $E(\dots|X_{n1},\dots,X_{nn})$.

Therefore, the lemma follows if we show that for all t

$$\lim_{\substack{m\to\infty\\n\to\infty}} E^*(e^{itY^*_{nm}}|A)=e^{-\frac{t^2}{2}}\quad \text{a.s.}$$

For every natural number n denote by $\mathcal{F}_n$ the tail σ-field of the sequence $(X^*_{nm})_{1\le m<\infty}$ and let $\mathcal{F}$ be the σ-field generated by $\bigcup_{n=1}^{\infty} \mathcal{F}_n$.

Since $\mathcal{F}_n$ is trivial on the probability space $(\Omega,\ \mathcal{A},\ P^*)$ for every n ($n = 1,\ 2,\ \dots$), $\mathcal{F}$ is also trivial on the probability space $(\Omega,\ \mathcal{A},\ P^*)$.

Consider, for fixed t, the sequence $\xi^*_{nm} = e^{itY^*_{nm}}$ of bounded random variables on the probability space $(\Omega,\ \mathcal{A},\ P^*)$ which is necessarily uniformly integrable.

It is well known that a sequence of random variables is relatively sequentially $L_1(\Omega,\ \mathcal{A},\ P^*)$-weakly compact if and only if it is uniformly integrable.

Hence, there exists a subsequence random variables of ξ_{nm} that converges weakly in $L_1(\Omega,\ \mathcal{A},\ P^*)$ to some random variable $\alpha(t)$. It is easy to check that $\alpha(t)$ is $\mathcal{F}$-measurable. But $\mathcal{F}$ is trivial, and so $\alpha(t)$ must be a constant (P^*-a.s.).

By Theorem 2.1 of Bickel and Freedman [3], the conditional distribution function of Y^*_{mn} converges almost surely to the standard normal distribution function as n and m tend to ∞. Hence $\alpha(t)$ has to be $e^{-\frac{t^2}{2}}$ and

$$\lim_{\substack{m\to\infty\\ n\to\infty}} E^*(e^{itY^*_{nm}}|A) = e^{-\frac{t^2}{2}} \quad \text{a.s.}$$

Thus all subsequences of ξ_{nm} which converge weakly in $L_1(\Omega,\ \mathcal{A},\ P^*)$, converge to $e^{-\frac{t^2}{2}}$ a.s. and so the original sequence must converge weakly in $L_1(\Omega,\ \mathcal{A},\ P^*)$ to $e^{-\frac{t^2}{2}}$ a.s. also. This holds for all real t, the lemma is proved. ∎

Lemma 3.4. *For every $\varepsilon > 0$ and $\eta > 0$ there exists a positive real number $s_0 = s_0(\varepsilon,\eta)$ and a natural number $m_0 = m_0(\varepsilon,\eta)$ such that for every $m > m_0$, we have*

$$P^*\big(\max_{i:|i-m|<s_0 m} |Y^*_{ni} - Y^*_{nm}| > \varepsilon\big) < \eta$$

for every natural number n.

Proof. It is easy to check that

$$P^*\big(\max_{i:|i-m|<s_0 m} |Y^*_{ni} - Y^*_{nm}| > \varepsilon\big) \le P^*\big(\max_{i:|i-m|<s_0 m} |Y^*_{ni} - Y^*_{n[(1-s_0)m]}| > \frac{\varepsilon}{2}\big) + P^*\big(|Y^*_{nm} - Y^*_{n[(1-s_0)m]}| > \frac{\varepsilon}{2}\big),$$

where $[x]$ is the largest integer $\le x$.

Applying the well-known inequalities of Tchebychev and Kolmogorov one obtains the following inequalities:

$$P^*\Big(\max_{i:|i-m|<s_0 m} |Y^*_{ni} - Y^*_{n[(1-s_0)m]}| > \frac{\varepsilon}{2}\Big) \le \frac{16}{\varepsilon^2}\Big(\frac{u}{v} + \frac{v}{u} - 2\sqrt{\frac{u}{v}}\Big)$$

$$P^*\Big(|Y^*_{nm} - Y^*_{n[(1-s_0)m]}| > \frac{\varepsilon}{2}\Big) \le \frac{32}{\varepsilon^2}\Big(1 - \sqrt{\frac{u}{m}}\Big),$$

where $u = [(1-s_0)m],\ v = [(1+s_0)m]$.

From the above inequalities we obtain the result desired.

Lemma 3.5. *For every $\varepsilon > 0$ and $\eta > 0$ there exists a positive real number $s_0 = s_0(\varepsilon, \eta)$ and a natural number $m_0 = m_0(\varepsilon, \eta)$ such that for every $m > m_0$ we have*

$$P_A^*\big(\max_{i:|i-m|<s_0 m} |Y_{ni}^* - Y_{nm}^*| > \varepsilon\big) < \eta$$

for every natural number n and every $A \in \mathcal{A}$, $(P^*(A) > 0)$.

Proof. By Lemma 3.4, for every $\varepsilon > 0$ and $\eta > 0$ there exists a positive real number $s_0 = s_0(\varepsilon, \eta)$ such that

$$\limsup_m P^*\big(\max_{i:|i-m|<s_0 m} |Y_{ni}^* - Y_{nm}^*| > \varepsilon\big) < \eta$$

for every natural number n.

We notice also that for every $\varepsilon > 0$ and $\eta > 0$ the event

$$\big(\max_{i:|i-m|<s_0 m} |Y_{ni}^* - Y_{nm}^*| > \varepsilon\big) \in \mathcal{K}_{[(1-s_0)m]+1},$$

where $\mathcal{K}_{[(1-s_0)m]+1}$ is the σ-algebra generated by the sequence of random variables $(Y_{nk})_{[(1-s_0)m]+1 \le k < \infty}$.

Therefore

$$\begin{aligned} &\limsup_m P_A^*\big(\max_{i:|i-m|<s_0 m} |Y_{ni}^* - Y_{nm}^*| > \varepsilon\big) \\ &= \limsup_m P^*\big(\max_{i:|i-m|<s_0 m} |Y_{ni}^* - Y_{nm}^*| > \varepsilon\big) < \eta \end{aligned}$$

for every natural number n and every $A \in \mathcal{A}$, $(P^*(A) > 0)$, by Lemma 3.2.

Thus, for every $\varepsilon > 0$ and $\eta > 0$ there exists a positive real number $s_0 = s_0(\varepsilon, \eta)$ and a natural number $m_0 = m_0(\varepsilon, \eta)$ such that for every $m > m_0$, we have

$$P_A^*\big(\max_{i:|i-m|<s_0 m} |Y_{ni}^* - Y_{nm}^*| > \varepsilon\big) < \eta$$

for every natural number n and every $A \in \mathcal{A}$, $(P^*(A) > 0)$, which completes the proof. ∎

Proof of Theorem 2.1.
If $EX^2 < \infty$ then $s_n^2 \to \sigma^2$ a.s. Therefore, the theorem follows if we show that the conditional distribution of $Y_{nN_n}^*$ converges weakly to $N(0,\ 1)$ a.s.

Let $(\nu_m)_{1 \le m < \infty}$ be the usual sequence of elementary random variables which approximates the random variable ν on the probability space $(\Omega,\ \mathcal{A},\ P^*)$. For every natural number m and h define

$$A_{hm} = \{(h-1)2^{-m} < \nu \le h2^{-m}\} = \{\nu_m = h2^{-m}\}.$$

Obviously

$$A_{hm} \bigcap A_{km} = \emptyset,\ h \neq k,$$
$$\bigcup_{h=1}^{\infty} A_{hm} = \Omega,\ m = 1,\ 2,\ \ldots$$

Since for every m $(m = 1,\ 2,\ \ldots)$

$$\sum_{h=1}^{\infty} P^*(A_{hm}) = 1$$

then, for every $\eta > 0$ and every m there exists a natural number $l^* = l^*(m,\ \eta)$ such that

$$\sum_{h=l^*+1}^{\infty} P^*(A_{hm}) < \eta,$$

or equivalently:

$$\sum_{h=1}^{l^*} P^*(A_{hm}) \leq 1 - \eta.$$

We shall denote the set of events $\{A_{1m},\ A_{2m},\ \ldots,\ A_{l^*m}\}$ by $\varepsilon(l^*(m,\ \eta))$ and the sequence $(\varepsilon(l^*(m,\ \eta)))_{1 \leq m < \infty}$ by $\varepsilon_\nu(\eta)$.

According to the notation of Lemma 3.1, we put

$$x^*_{mn} = Y^*_{n[k_n \nu_m]},\ y^*_{mn} = Y^*_{nN_n} - Y^*_{n[k_n \nu_m]},\ W^*_n = Y^*_{nN_n}.$$

Obviously,

$$W^*_n = x^*_{mn} + y^*_{mn}$$

for any $n,\ m$ $(n,\ m = 1,\ 2,\ \ldots)$.

Let us show that all conditions of Lemma 3.1 are satisfied. Indeed, $([k_n h 2^{-m}])_{1 \leq n < \infty}$ is a sequence of natural number, for every m and h $(m,\ h = 1,\ 2,\ \ldots)$. Lemma 3.3 implies that for every $\eta > 0,\ A_{hm} \in \varepsilon_\nu(\eta)$ and every real number x there exits a natural number $n_0 = n_0(\eta,\ x,\ h,\ m)$ such that for every $n > n_0$ we have

$$\Big| P^*_{A_{hm}}\big(Y^*_{n[k_n h 2^{-m}]} \leq x\big) - \Phi(x) \Big| < \eta \text{ a.s.}$$

We put now

$$n^* = n^*(\eta,\ x,\ m) = \max_{1 \leq k \leq l^*} n_0(\eta,\ x,\ h,\ m) \quad (l^* = l^*(m,\ \eta))$$

and for simplicity of notation, we let

$$\Delta^1_{mn} = \Big| \sum_{h=1}^{\infty} P^*\big(\big(Y^*_{n[k_n \nu_m]} \leq x\big) \bigcap A_{hm}\big) - \Phi(x) \Big|,$$
$$\Delta^{11}_{mn} = \Big| \sum_{h=1}^{l^*} P^*\big(\big(Y^*_{n[k_n \nu_m]} \leq x\big) \bigcap A_{hm}\big) - \Phi(x) \sum_{h=1}^{l^*} P^*\big(A_{hm}\big) \Big|,$$

$$\Delta_{mn}^{12} = \sum_{h=l^*+1}^{\infty} P^*\big((Y^*_{n[k_n\nu_m]} \le x\big) \bigcap A_{hm}\big),$$

$$\Delta_{mn}^{13} = \Phi(x) \sum_{h=l^*+1}^{\infty} P^*\big(A_{hm}\big),$$

then for every m $(m = 1,\ 2,\ \dots)$ if $n > n^*$ we have

$$\begin{aligned}\big|P^*(x^*_{mn} \le x) - \Phi(x)\big| &= \big|P^*\big(Y^*_{n[k_n\nu_m]} \le x\big) - \Phi(x)\big| = \Delta^1_{mn} \le \Delta^{11}_{mn} + \Delta^{12}_{mn} + \Delta^{13}_{mn}\\ &\le \sum_{h=1}^{l^*} \big|P^*_{A_{hm}}\big(Y^*_{n[k_n h 2^{-m}]} \le x\big) - \Phi(x)\big| P^*(A_{hm}) + 2\sum_{h=l^*+1}^{\infty} P^*(A_{hm})\\ &< \eta \sum_{h=1}^{l^*} P^*(A_{hm}) + 2\eta < 3\eta \text{ a.s.}\end{aligned}$$

i.e.

$$\lim_{n\to\infty} P^*(x_{mn} \le x) = \Phi(x) \text{ a.s.}$$

for any m $(m = 1,\ 2,\ \dots)$.

Therefore condition (A) of Lemma 3.1 is satisfied a.s.

Now, for all $\varepsilon > 0$, consider the following events:

$$\begin{aligned} B_{mn} &= \big\{\big|Y^*_{nN_n} - Y^*_{n[k_n\nu_m]}\big| > \varepsilon\big\},\\ C_{mn} &= \Big\{\Big|\frac{N_n}{k_n} - \nu\Big| < 2^{-m}\Big\},\\ D_{mn} &= \Big\{\Big|\frac{N_n}{k_n} - \nu\Big| \ge 2^{-m}\Big\},\\ E_{mn} &= \bigcup_{h=1}^{\infty} \Big(\Big\{\max_{i:\left|\frac{i}{N_n} - \nu\right| < 2^{-m}} \big|Y^*_{ni} - Y^*_{n[k_n h 2^{-m}]}\big| > \varepsilon\Big\} \bigcap A_{hm}\Big),\\ F_{mn} &= \bigcup_{h=1}^{\infty} \big(\{\max_{i:(h-2)2^{-m}k_n < i < (h+1)2^{-m}k_n} \big|Y^*_{ni} - Y^*_{n[k_n h 2^{-m}]}\big| > \varepsilon\} \bigcap A_{hm}\big).\end{aligned}$$

From condition (1) we have

$$\begin{aligned}\lim_{m\to\infty} \limsup_n P^*(|y^*_{mn}| > \varepsilon) &= \lim_{m\to\infty} \limsup_n P^*\big(B_{mn}\big) \le \lim_{m\to\infty} \limsup_n P^*\big(B_{mn} \cap C_{mn}\big)\\ &+ \lim_{m\to\infty} \limsup_n P^*\big(D_{mn}\big) = \lim_{m\to\infty} \limsup_n P^*\Big(\bigcup_{h=1}^{\infty} \big(B_{mn} \cap C_{mn} \cap A_{hm}\big)\Big)\\ &\le \lim_{m\to\infty} \limsup_n P^*\big(E_{mn}\big) \le \lim_{m\to\infty} \limsup_n P^*\big(F_{mn}\big) \text{ a.s.,}\end{aligned} \tag{1}$$

where in the last inequality we have taken into account that the inequality

$$\Big|\frac{i}{k_n} - \nu\Big| < 2^{-m}$$

implies

$$(h-2)2^{-m}k_n < i < (h+1)2^{-m}k_n, \tag{2}$$

because on the set A_{hm} we have $(h-1)2^{-m} < \nu < h2^{-m}$.

From Lemma 3.5 it follows that for every $\varepsilon > 0$ and $\eta > 0$ there exists a positive real number $s_0 = s_0(\varepsilon, \eta)$ such that

$$\limsup_j P^*_{A_{hm}}\Big(\max_{i:|i-j|<s_0 j} |Y^*_{ni} - Y^*_{nj}| > \varepsilon\Big) < \eta \tag{3}$$

for every natural number n and every $A_{hm} \in \varepsilon_\nu(\eta)$.

Let us choose the natural number $m_0 = m_0(\varepsilon,\ \eta)$ such that $m_0 s_0 > 2$ and such that for $m > m_0$

$$P^*(\nu < m2^{-m}) < \eta \quad \text{a.s.} \tag{4}$$

Some simple calculations show that for every $m > m_0$ and $h \geq m$ if n is sufficiently large, the inequality (2) implies

$$|i - [k_n h 2^{-m}]| < s_0[k_n h 2^{-m}]. \tag{5}$$

Now, using (3) and (4) it follows that for $m > m_0$ we have

$$\begin{aligned}\lim_{m\to\infty}\limsup_n P^*\big(F_{mn}\big) &\leq \Delta^* + P^*(\nu < m2^{-m}) + \sum_{h=l^*+1}^{\infty} P^*(A_{hm}) \\ &< \eta \sum_{h=m}^{l^*} P^*(A_{hm}) + \eta + \eta < 3\eta \text{ a.s.,}\end{aligned} \tag{6}$$

where

$$\Delta^* = \sum_{h=m}^{l^*} \limsup_n P^*_{A_{hm}}\Big(\max_{i:|i-[k_n h2^{-m}]|<s_0[k_n h2^{-m}]} \big|Y^*_{ni} - Y^*_{n[k_n h2^{-m}]}\big| > \varepsilon\Big) P^*(A_{hm}).$$

Thus from (1) and (6) it results

$$\lim_{m\to\infty}\limsup_n P^*(|y^*_{mn}| > \varepsilon) = 0 \text{ a.s.,} \quad \forall \varepsilon > 0.$$

Therefore the condition (B) of Lemma 3.1 is satisfied too and we have

$$\lim_{n\to\infty} P^*\big(Y^*_{nN_n} \leq x\big) = \lim_{n\to\infty} P^*\big(W^*_n \leq x\big) = \lim_{n\to\infty} P^*\big(x^*_{mn} \leq x\big) = \Phi(x) \text{ a.s.,}$$

which proves the theorem.

References

1. K. B. Athreya, Bootstrap of the Mean in the infinite variance Case, *Proceedings of the 1st World Congress of the Bernoulli Society,* Y. Prohorov and V. V. Sazonov (Eds.) VNU Science Press, The Netherlands, **2** (1987) 95–98.

2. R. Beran, Bootstrap method in statistics, *Jahsesber. Deutsch. Math. -Verein* **86** (1984) 14–30.
3. P. J. Bickel and D. A. Freedman, Some asymptotic theory for the bootstrap, *Ann. Statist.* **9** (1981) 1196–1217.
4. J. Blum, D. Hanson, and J. Rosenblatt, On the central limit theorem for the sum of a random number of independent random variables, *J. Z. Wahrscheinlichkeitstheorie verw. Gebiete* **1** (1963) 389–393.
5. B. Efron, Bootstrap methods: Another look at the Jackknife, *Ann. Statist.* **7** (1979) 1–26.
6. B. Efron, Nonparametric standard errors and confidence intervals (with discussion), *Canad. J. Statist.* **9** (1981) 139–172.
7. B. Efron and R. Tibshirani, Bootstrap methods for standard errors, confidence intervals, and other measures of statistical accuracy (with discussion), *Statist. Sci.* **1** (1986) 54–77.
8. E. Giné and J. Zinn, Necessary conditions for the bootstrap of the mean, *Ann. Statist.* **17** (1989) 684–691.
9. S. Guiasu, On the asymptotic distribution of the sequences of random variables with random indices, *J. Ann. Math. Statist.* **42** (1971) 2018–2028.
10. P. Hall, Asymptoyic Properties of the Bootstrap of Heavy Tailed Distribution, *Ann. Statist.* **18** (1990) 1342–1360.
11. E. Mammen, Bootstrap, wild bootstrap, and asymptotic normality, *Prob. Theory Relat. Fields* **93** (1992) 439–455
12. Nguyen Van Toan, Wild bootstrap and asymptotic normality, *Bulletin, College of Science, Hue University,* **10** (1996) 48–52.
13. Nguyen Van Toan, On the bootstrap estimate with random sample size, *Scientific Bulletin of Universities* (1998) 31–34.
14. Nguyen Van Toan, On the asymptotic accuracy of the bootstrap with random sample size, *Vietnam J. Math.* **26** (1998) 351–356.
15. Nguyen Van Toan, On the asymptotic accuracy of the bootstrap with random sample size, *Pakistan J. Statist.* **14** (1998) 193–203.
16. Nguyen Van Toan, Rate of convergence in bootstrap approximations with random sample size, *Acta Math. Vietnam.* **25** (2000) 161–179.
17. C. R. Rao, P. K. Pathak, and V. I. Koltchinskii, Bootstrap by sequential resampling, *J. Statist. Plann. Inference* **64** (1997) 257–281.
18. A. Renyi, On the central limit theorem for the sum of a random number of independent random variables, *Acta Math. Acad. Sci. Hungar.* **11** (1960) 97–102.
19. J. W. H. Swanepoel, A note in proving that the (Modified) Bootstrap works, *Commun. Statist. Theory Meth.* **15** (1986) 3193–3203.
20. Tran Manh Tuan and Nguyen Van Toan, On the asymptotic theory for the bootstrap with random sample size, *Proceedings of the National Centre for Science and Technology of Vietnam* **10** (1998) 3–8.
21. Tran Manh Tuan and Nguyen Van Toan, An asymptotic normality theorem of the bootstrap sample with random sample size, *VNU J. Science Nat. Sci.* **14** (1998) 1–7.