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Boundedness of Multilinear Littlewood-Paley Operators for the Extreme Cases*

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Abstract. The purpose of this paper is to study the boundedness properties of multilinear Littlewood-Paley operators for the extreme cases.

1. Introduction and Results

Fix $\delta > 0$. Let ψ be a fixed function which satisfies the following properties:
(1) $\int_{\mathbb{R}^n} \psi(x)dx = 0$,
(2) $|\psi(x)| \le C(1+|x|)^{-(n+1-\delta)}$,
(3) $|\psi(x+y) - \psi(x)| \le C|y|(1+|x|)^{-(n+2-\delta)}$ when $2|y| < |x|$.

We denote $\Gamma(x) = \{(y,t) \in \mathbb{R}_+^{n+1} : |x-y| < t\}$ and the characteristic function of $\Gamma(x)$ by $\chi_{\Gamma(x)}$. Let m be a positive integer and A be a function on $\mathbb{R}^n$. The multilinear Littlewood-Paley operator is defined by

$$S_\delta^A(f)(x) = \Big[\int\int_{\Gamma(x)} |F_t^A(f)(x,y)|^2 \frac{dydt}{t^{n+1}} \Big]^{1/2},$$

where

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$$F_t^A(f)(x,y) = \int_{\mathbb{R}^n} \frac{\mathbb{R}_{m+1}(A;x,z)}{|x-z|^m} f(z)\psi_t(y-z)dz,$$

$$R_{m+1}(A;x,y) = A(x) - \sum_{|\alpha|\le m} \frac{1}{\alpha!} D^\alpha A(y)(x-y)^\alpha$$

and $\psi_t(x) = t^{-n+\delta}\psi(x/t)$ for $t > 0$. Set $F_t(f)(y) = f * \psi_t(y)$. We also define

$$S_\delta(f)(x) = \Big(\int\int_{\Gamma(x)} |F_t(f)(y)|^2 \frac{dydt}{t^{n+1}} \Big)^{1/2},$$

which is the Littlewood-Paley operator (see [14]).

Let H be the Hilbert space $H = \Big\{ h : \|h\| = \Big(\int\int_{\mathbb{R}_+^{n+1}} |h(t)|^2 dydt/t^{n+1} \Big)^{1/2} < \infty \Big\}$. Then for each fixed $x \in \mathbb{R}^n$, $F_t^A(f)(x,y)$ may be viewed as a mapping from $(0,+\infty)$ to H, and it is clear that

$$S_\delta^A(f)(x) = \big\| \chi_{\Gamma(x)} F_t^A(f)(x,y) \big\|, \quad S_\delta(f)(x) = \big\| \chi_{\Gamma(x)} F_t(f)(y) \big\|.$$

We also consider the variant of S_δ^A, which is defined by

$$\tilde{S}_\delta^A(f)(x) = \Big(\int\int_{\Gamma(x)} |\tilde{F}_t^A(f)(x)|^2 \frac{dt}{t^{n+1}} \Big)^{1/2},$$

where

$$\tilde{F}_t^A(f)(x) = \int_{\mathbb{R}^n} \frac{Q_{m+1}(A;x,y)}{|x-y|^m} \psi_t(x-y) f(y)dy$$

and

$$Q_{m+1}(A;x,y) = R_m(A;x,y) - \sum_{|\alpha|=m} \frac{1}{\alpha!} D^\alpha A(x)(x-y)^\alpha.$$

Note that when $m = 0$, S_δ^A is just the commutator of Littlewood-Paley operator (see [1, 11, 12]). It is well known that multilinear operators, as the extension of Commutators, are of great interest in harmonic analysis and have been widely studied by many authors (see [3-6, 8]). In [2, 7], the $L^p (p > 1)$ boundedness of commutators generated by the Calderón-Zygmund operator or fractional integral operator and BMO functions are obtained, and in [11], the endpoint boundedness of commutators generated by the Calderón-Zygmund operator and BMO functions are obtained. The main purpose of this paper is to discuss the boundedness properties of the multilinear Littlewood-Paley operators for the extreme cases of p. Throughout this paper, the letter $C's$ will denote the positive constants which may have different values in each line; B will denote a ball of $\mathbb{R}^n$. For a ball B, set $f_B = |B|^{-1} \int_B f(x)dx$ and $f^\#(x) = \sup_{x \in B} |B|^{-1} \int_B |f(y) - f_B|dy$.

We shall prove the following theorems in Sec. 3.

Theorem 1. *Let $0 \le \delta < n$ and $D^\alpha A \in BMO(\mathbb{R}^n)$ for $|\alpha| = m$. Then S_δ^A is bounded from $L^{n/\delta}(\mathbb{R}^n)$ to $BMO(\mathbb{R}^n)$.*

Theorem 2. *Let $0 \le \delta < n$ and $D^\alpha A \in BMO(\mathbb{R}^n)$ for $|\alpha| = m$. Then $\tilde{S}_\delta^A$ is bounded from $H^1(\mathbb{R}^n)$ to $L^{n/(n-\delta)}(\mathbb{R}^n)$.*

Theorem 3. *Let $0 \le \delta < n$ and $D^\alpha A \in BMO(\mathbb{R}^n)$ for $|\alpha| = m$. Then S_δ^A is bounded from $H^1(\mathbb{R}^n)$ to weak $L^{n/(n-\delta)}(\mathbb{R}^n)$.*

Theorem 4. *Let $0 \le \delta < n$ and $D^\alpha A \in BMO(\mathbb{R}^n)$ for $|\alpha| = m$.*
(i) *If for any H^1-atom a supported on certain cube Q and $u \in 3Q \setminus 2Q$, there is*

$$\int_{(4Q)^c} \left\| \chi_{\Gamma(x)} \sum_{|\alpha|=m} \frac{1}{\alpha!} \frac{(x-u)^\alpha}{|x-u|^m} \psi_t(y-u) \int_Q D^\alpha A(z) a(z) dz \right\|^{n/(n-\delta)} dx \le C,$$

then S_δ^A is bounded from $H^1(\mathbb{R}^n)$ to $L^{n/(n-\delta)}(\mathbb{R}^n)$;
(ii) *If for any cube Q and $u \in 3Q \setminus 2Q$, there is*

$$\frac{1}{|Q|} \int_Q \Bigg\| \chi_{\Gamma(x)} \sum_{|\alpha|=m} \frac{1}{\alpha!} (D^\alpha A(x) - (D^\alpha A)_Q) \int_{(4Q)^c} \frac{(u-z)^\alpha}{|u-z|^m} \psi_t(u-z) f(z) dz \Bigg\| dx \le C ||f||_{L^{n/\delta}},$$

then $\tilde{S}_\delta^A$ is bounded from $L^{n/\delta}(\mathbb{R}^n)$ to $BMO(\mathbb{R}^n)$.

2. Proofs of Theorems

We begin with some preliminary lemmas.

Lemma 1. (see [6]) *Let A be a function on $\mathbb{R}^n$ and $D^\alpha A \in L^q(\mathbb{R}^n)$ for $|\alpha| = m$ and some $q > n$. Then*

$$|R_m(A; x, y)| \le C|x-y|^m \sum_{|\alpha|=m} \Big(\frac{1}{|\tilde{B}(x,y)|} \int_{\tilde{B}(x,y)} |D^\alpha A(z)|^q dz \Big)^{1/q},$$

where $\tilde{B}(x,y)$ is the ball centered at x and having radius $5\sqrt{n}|x-y|$.

Lemma 2. *Let $0 \le \delta < n$, $1 < p < n/\delta$ and $D^\alpha A \in BMO(\mathbb{R}^n)$ for $|\alpha| = m$, $1 < r \le \infty$, $1/q = 1/p + 1/r - \delta/n$. Then S_δ^A is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$, that is*

$$||S_\delta^A(f)||_{L^q} \le C \sum_{|\alpha|=m} ||D^\alpha A||_{BMO} ||f||_{L^p}.$$

Proof. By Minkowski inequality and by the condition of ψ, we have

$$\begin{aligned}
S_\delta^A(f)(x) &\le \int_{\mathbb{R}^n} \frac{|f(z)|\ |R_{m+1}(A;x,z)|}{|x-z|^m} \Big(\int_{\Gamma(x)} |\psi_t(y-z)|^2 \frac{dydt}{t^{1+n}} \Big)^{1/2} dz \\
&\le C \int_{\mathbb{R}^n} \frac{|f(z)||R_{m+1}(A;x,z)|}{|x-z|^m} \Big(\int_0^\infty \int_{|x-y|\le t} \frac{t^{-2n+2\delta}}{(1+|y-z|/t)^{2n+2-2\delta}} \frac{dydt}{t^{1+n}} \Big)^{1/2} dz \\
&\le C \int_{\mathbb{R}^n} \frac{|f(z)||R_{m+1}(A;x,z)|}{|x-z|^m} \Big(\int_0^\infty \int_{|x-y|\le t} \frac{2^{2n+2-2\delta} \cdot t^{1-n}}{(2t+|y-z|)^{2n+2-2\delta}} dydt \Big)^{1/2} dz.
\end{aligned}$$

Noting that $2t+|y-z| \ge 2t+|x-z|-|x-y| \ge t+|x-z|$ when $|x-y| \le t$ and

$$\int_0^\infty \frac{tdt}{(t+|x-z|)^{2n+2-2\delta}} = C|x-z|^{-2n+2\delta},$$

we obtain

$$\begin{aligned}
S_\delta^A(f)(x) &\le C \int_{\mathbb{R}^n} \frac{|f(z)||R_{m+1}(A;x,z)|}{|x-z|^m} \left(\int_0^\infty \frac{tdt}{(t+|x-z|)^{2n+2-2\delta}} \right)^{1/2} dz \\
&= C \int_{\mathbb{R}^n} \frac{|f(z)||R_{m+1}(A;x,z)|}{|x-z|^{m+n-\delta}} dz.
\end{aligned}$$

Thus, the lemma follows from [8].

Proof of Theorem 1. It suffices to prove that there exists a constant C depending on B such that

$$\frac{1}{|B|} \int_B |S_\delta^A(f)(x) - C_B| dx \le C_B ||f||_{L^{n/\delta}}$$

holds for any ball B. Fix a ball $B = B(x_0, l)$. Let $\tilde{B} = 5\sqrt{n}B$ and $\tilde{A}(x) = A(x) - \sum_{|\alpha|=m} \frac{1}{\alpha!} (D^\alpha A)_{\tilde{B}} x^\alpha$, then $R_m(A;x,y) = R_m(\tilde{A};x,y)$ and $D^\alpha \tilde{A} = D^\alpha A - (D^\alpha A)_{\tilde{B}}$ for $|\alpha| = m$. We write, for $f_1 = f\chi_{\tilde{B}}$ and $f_2 = f\chi_{\mathbb{R}^n \setminus \tilde{B}}$, $F_t^A(f)(x) = F_t^A(f_1)(x) + F_t^A(f_2)(x)$, then

$$\begin{aligned}
&\frac{1}{|B|}\int_B |S_\delta^A(f)(x)-S_\delta^A(f_2)(x_0)|dx\\
&=\frac{1}{|B|}\int_B \big|||\chi_{\Gamma(x)}F_t^A(f)(x,y)||-||\chi_{\Gamma(x)}F_t^A(f_2)(x_0,y)||\big|\,dx\\
&\le \frac{1}{|B|}\int_B S_\delta^A(f_1)(x)dx+\frac{1}{|B|}\int_B ||\chi_{\Gamma(x)}F_t^A(f_2)(x,y)-\chi_{\Gamma(x)}F_t^A(f_2)(x_0,y)||dx\\
&:= I+II.
\end{aligned}$$

Now, let us estimate I and II . First, taking $p>1$ and $q>1$ such that $1/q=1/p-\delta/n$, by the (L^p,L^q) boundedness of S_δ^A(Lemma 2), we gain

$$I\le \Big(\frac{1}{|B|}\int_B (S_\delta^A(f_1)(x))^q dx\Big)^{1/q}\le C|B|^{-1/q}||f_1||_{L^p}=C||f||_{L^{n/\delta}}.$$

To estimate II, we write

$$\begin{aligned}
&\chi_{\Gamma(x)}F_t^A(f_2)(x,y)-\chi_{\Gamma(x)}F_t^A(f_2)(x_0,y)\\
&=\int\Big[\frac{1}{|x-z|^m}-\frac{1}{|x_0-z|^m}\Big]\chi_{\Gamma(x)}\psi_t(y-z)R_m(A;x,z)f_2(z)dz\\
&\quad+\int\frac{\chi_{\Gamma(x)}\psi_t(y-z)f_2(z)}{|x_0-z|^m}[R_m(A;x,z)-R_m(A;x_0,z)]dz\\
&\quad+\int(\chi_{\Gamma(x)}-\chi_{\Gamma(x_0)})\frac{\psi_t(y-z)R_m(A;x_0,z)f_2(z)}{|x_0-z|^m}dz\\
&\quad-\sum_{|\alpha|=m}\frac{1}{\alpha!}\int\Big[\frac{\chi_{\Gamma(x)}(x-z)^\alpha}{|x-z|^m}-\frac{\chi_{\Gamma(x_0)}(x_0-z)^\alpha}{|x_0-z|^m}\Big]\psi_t(y-z)D^\alpha\tilde{A}(z)f_2(z)dz\\
&:= II_1^t(x)+II_2^t(x)+II_3^t(x)+II_4^t(x).
\end{aligned}$$

We choose $r>1$ such that $1/r+\delta/n=1$. Note that $|x-z|\sim|x_0-z|$ for $x\in\tilde{B}$ and $z\in\mathbb{R}^n\setminus\tilde{B}$, similar to the proof of Lemmas 2 and 1, we have

$$\begin{aligned}
&\frac{1}{|B|}\int_B ||II_1^t(x)||dx\\
&\le \frac{C}{|B|}\int_B\Big(\int_{\mathbb{R}^n\setminus\tilde{B}}\frac{|x-x_0||f(z)|}{|x-z|^{n+m+1-\delta}}|R_m(\tilde{A};x,z)|dz\Big)dx\\
&\le \frac{C}{|B|}\int_B\Big(\sum_{k=0}^\infty\int_{2^{k+1}\tilde{B}\setminus 2^k\tilde{B}}\frac{|x-x_0||f(z)|}{|x-z|^{n+m+1-\delta}}|R_m(\tilde{A};x,z)|dz\Big)dx
\end{aligned}$$

$$\leq C\sum_{k=0}^{\infty}\frac{l(2^kl)^m}{(2^kl)^{n+m+1-\delta}}k\sum_{|\alpha|=m}||D^\alpha A||_{BMO}\Big(\int_{2^k\tilde{B}}|f(z)|dz\Big)$$

$$\leq C\sum_{|\alpha|=m}||D^\alpha A||_{BMO}||f||_{L^{n/\delta}}\sum_{k=0}^{\infty}k2^{-k}$$

$$\leq C\sum_{|\alpha|=m}||D^\alpha A||_{BMO}||f||_{L^{n/\delta}}.$$

For $II_2^t(x)$, by the formula (see [6])

$$R_m(\tilde{A};x,z)-R_m(\tilde{A};x_0,z)$$
$$=R_m(\tilde{A};x,x_0)+\sum_{0<|\beta|<m}\frac{1}{\beta!}R_{m-|\beta|}(D^\beta\tilde{A};x_0,z)(x-x_0)^\beta$$

and by Lemma 1, we get

$$|R_m(\tilde{A};x,z)-R_m(\tilde{A};x_0,z)|$$
$$\leq C\sum_{|\alpha|=m}||D^\alpha A||_{BMO}(|x-x_0|^m+\sum_{0<|\beta|<m}|x_0-z|^{m-|\beta|}|x-x_0|^{|\beta|}),$$

thus, for $x\in B$,

$$||II_2^t(x)||\leq C\int_{\mathbb{R}^n}\frac{|f_2(z)|}{|x-z|^{m+n-\delta}}|R_m(\tilde{A};x,z)-R_m(\tilde{A};x_0,z)|dz$$

$$\leq C\sum_{|\alpha|=m}||D^\alpha A||_{BMO}\int_{\mathbb{R}^n}\frac{|x-x_0|^m+\sum_{0<|\beta|<m}|x_0-z|^{m-|\beta|}|x-x_0|^{|\beta|}}{|x_0-z|^{m+n-\delta}}|f_2(z)|dz$$

$$\leq C\sum_{|\alpha|=m}||D^\alpha A||_{BMO}\sum_{k=0}^{\infty}\frac{kl^m}{(2^kl)^{m+n-\delta}}\int_{2^k\tilde{B}}|f(z)|dz$$

$$\leq C\sum_{|\alpha|=m}||D^\alpha A||_{BMO}||f||_{L^{n/\delta}}\sum_{k=1}^{\infty}k2^{-km}$$

$$\leq C\sum_{|\alpha|=m}||D^\alpha A||_{BMO}||f||_{L^{n/\delta}}.$$

For $II_3^t(x)$, note that $|x+y-z|\sim|x_0+y-z|$ for $x\in\tilde{B}$ and $z\in\mathbb{R}^n\setminus\tilde{B}$, we obtain, similar to the estimate of II_1,

$$
\begin{aligned}
&||II_3^t(x)|| \\
&\le C\int_{\mathbb{R}^n}\Big(\int\int_{R_+^{n+1}}\Big[\frac{|\psi_t(y-z)||f_2(z)||R_m(\tilde{A};x_0,z)|}{|x_0-z|^m} \\
&\times|\chi_{\Gamma(x)}(y,t)-\chi_{\Gamma(x_0)}(y,t)|\Big]^2\frac{dydt}{t^{n+1}}\Big)^{1/2}dz \\
&\le C\int_{\mathbb{R}^n}\frac{|f_2(z)|R_m(\tilde{A};x_0,z)|}{|x_0-z|^m} \\
&\times\Big|\int\int_{\Gamma(x)}\frac{t^{1-n}dydt}{(t+|y-z|)^{2n+2-2\delta}}-\int\int_{\Gamma(x_0)}\frac{t^{1-n}dydt}{(t+|y-z|)^{2n+2-2\delta}}\Big|^{1/2}dz \\
&\le C\int_{\mathbb{R}^n}\frac{|f_2(z)|R_m(\tilde{A};x_0,z)|}{|x_0-z|^m} \\
&\times\Big(\int\int_{|y|\le t}\Big|\frac{1}{(t+|x+y-z|)^{2n+2-2\delta}}-\frac{1}{(t+|x_0+y-z|)^{2n+2-2\delta}}\Big|\frac{dydt}{t^{n-1}}\Big)^{1/2}dz \\
&\le C\int_{\mathbb{R}^n}\frac{|f_2(z)|R_m(\tilde{A};x_0,z)|}{|x_0-z|^m} \\
&\times\Big(\int\int_{|y|\le t}\frac{|x-x_0|t^{1-n}dydt}{(t+|x+y-z|)^{2n+3-2\delta}}\Big)^{1/2}dz \\
&\le C\int_{\mathbb{R}^n}\frac{|f_2(z)||x-x_0|^{1/2}|R_m(\tilde{A};x_0,z)|}{|x_0-z|^{m+n+1/2-\delta}}dz \\
&\le C\sum_{k=0}^{\infty}\frac{kl^{1/2}(2^kl)^m}{(2^kl)^{n+m+1/2-\delta}}||f||_{L^{n/\delta}}\sum_{|\alpha|=m}||D^\alpha A||_{BMO} \\
&\le C\sum_{|\alpha|=m}||D^\alpha A||_{BMO}||f||_{L^{n/\delta}}\sum_{k=0}^{\infty}k2^{-k/2}\le C\sum_{|\alpha|=m}||D^\alpha A||_{BMO}||f||_{L^{n/\delta}}.
\end{aligned}
$$

For $II_4^t(x)$, similar to the estimate of $II_3^t(x)$, we have

$$
\begin{aligned}
\|II_4^t(x)\| &\le C\int_{\mathbb{R}^n\setminus\tilde{B}}\Big[\frac{|x-x_0|}{|x-z|^{n+1-\delta}}+\frac{|x-x_0|^{1/2}}{|x-z|^{n+1/2-\delta}}\Big]\sum_{|\alpha|=m}|D^\alpha\tilde{A}(z)||f(z)|dz \\
&\le C\sum_{|\alpha|=m}||D^\alpha A||_{BMO}||f||_{L^{n/\delta}}\sum_{k=0}^{\infty}k(2^{-k}+2^{-k/2}) \\
&\le C\sum_{|\alpha|=m}||D^\alpha A||_{BMO}||f||_{L^{n/\delta}}.
\end{aligned}
$$

Combining these estimates, we complete the proof of Theorem 1. ■

Proof of Theorem 2. It suffices to show that there exists a constant $C>0$ such that for every H^1-atom a (that is: supp $a\subset B=B(x_0,r)$, $||a||_{L^\infty}\le |B|^{-1}$ and $\int_{\mathbb{R}^n} a(y)dy=0$ (see[9, 13])), we have

$$||\tilde{S}_\delta^A(a)||_{L^{n/(n-\delta)}}\le C.$$

We write

$$\int_{\mathbb{R}^n}[\tilde{S}_\delta^A(a)(x)]^{n/(n-\delta)}dx=\Big[\int_{|x-x_0|\le 2r}+\int_{|x-x_0|>2r}\Big][\tilde{S}_\delta^A(a)(x)]^{n/(n-\delta)}dx:=J+JJ.$$

For J, by the following equality

$$Q_{m+1}(A;x,y)=R_{m+1}(A;x,y)-\sum_{|\alpha|=m}\frac{1}{\alpha!}(x-y)^\alpha(D^\alpha A(x)-D^\alpha A(y)),$$

we have, similar to the proof of Lemma 2,

$$\tilde{S}_\delta^A(a)(x)\le S_\delta^A(a)(x)+C\sum_{|\alpha|=m}\int_{\mathbb{R}^n}\frac{|D^\alpha A(x)-D^\alpha A(y)|}{|x-y|^{n-\delta}}|a(y)|dy,$$

thus, $\tilde{S}_\delta^A$ is (L^p,L^q)-bounded by Lemma 2 and [1, 2], where $1/q=1/p-\delta/n$. We see that

$$J\le C||\tilde{S}_\delta^A(a)||_{L^q}^{n/((n-\delta)q)}|2B|^{1-n/((n-\delta)q)}\le C||a||_{L^p}^{n/(n-\delta)}|B|^{1-n/((n-\delta)q)}\le C.$$

To obtain the estimate of JJ, set $\tilde{A}(x)=A(x)-\sum_{|\alpha|=m}\frac{1}{\alpha!}(D^\alpha A)_{2B}x^\alpha$. Then $Q_m(A;x,y)=Q_m(\tilde{A};x,y)$. We write, by the vanishing moment of a and Q_{m+1} $(A;x,y)=R_m(A;x,y)-\sum_{|\alpha|=m}\frac{1}{\alpha!}(x-y)^\alpha D^\alpha A(x)$, for $x\in(2B)^c$,

$$\begin{aligned}
&\tilde{F}_t^A(a)(x,y)\\
=&\int_{\mathbb{R}^n}\frac{\psi_t(y-z)R_m(\tilde{A};x,z)}{|x-z|^m}a(z)dz\\
&-\sum_{|\alpha|=m}\frac{1}{\alpha!}\int_{\mathbb{R}^n}\frac{\psi_t(y-z)D^\alpha\tilde{A}(z)(x-z)^\alpha}{|x-z|^m}a(z)dz\\
=&\int_{\mathbb{R}^n}\left[\frac{\psi_t(y-z)R_m(\tilde{A};x,z)}{|x-z|^m}-\frac{\psi_t(y-x_0)R_m(\tilde{A};x,x_0)}{|x-x_0|^m}\right]a(z)dz\\
&-\sum_{|\alpha|=m}\frac{1}{\alpha!}\int_{\mathbb{R}^n}\left[\frac{\psi_t(y-z)(x-z)^\alpha}{|x-z|^m}-\frac{\psi_t(y-x_0)(x-x_0)^\alpha}{|x-x_0|^m}\right]D^\alpha\tilde{A}(x)a(z)dz,
\end{aligned}$$

thus, similar to the proof of II in Theorem 1, we obtain

$$\begin{aligned}&||\tilde{F}_t^A(a)(x,y)||\\ &\le C\Big(\sum_{|\alpha|=m}||D^\alpha A||_{BMO}|B|^{1/n}|x-x_0|^{-n-1+\delta}+|B|^{1/n}|x-x_0|^{-n-1+\delta}|D^\alpha\tilde{A}(x)|\Big),\end{aligned}$$

so that,

$$JJ\le C\Big(\sum_{|\alpha|=m}||D^\alpha A||_{BMO}\Big)^{n/(n-\delta)}\sum_{k=1}^{\infty}k2^{-kn/(n-\delta)}\le C,$$

which together with the estimate for J yields the desired result. This finishes the proof of Theorem 2. ■

Proof of Theorem 3. By the equality

$$R_{m+1}(A;x,y)=Q_{m+1}(A;x,y)+\sum_{|\alpha|=m}\frac{1}{\alpha!}(x-y)^\alpha(D^\alpha A(x)-D^\alpha A(y))$$

and similar to the proof of Lemma 2, we get

$$S_\delta^A(f)(x)\le\tilde{S}_\delta^A(f)(x)+C\sum_{|\alpha|=m}\int_{\mathbb{R}^n}\frac{|D^\alpha A(x)-D^\alpha A(y)|}{|x-y|^{n-\delta}}|f(y)|dy.$$

By Theorems 1 and 2 with [1, 2], we obtain

$$\begin{aligned}&|\{x\in\mathbb{R}^n:S_\delta^A(f)(x)>\lambda\}|\\ &\le|\{x\in\mathbb{R}^n:\tilde{S}_\delta^A(f)(x)>\lambda/2\}|\\ &\quad+\left|\left\{x\in\mathbb{R}^n:\sum_{|\alpha|=m}\int_{\mathbb{R}^n}\frac{|D^\alpha A(x)-D^\alpha A(y)|}{|x-y|^{n-\delta}}|f(y)|dy>C\lambda\right\}\right|\\ &\le C(||f||_{H^1}/\lambda)^{n/(n-\delta)}.\end{aligned}$$

This completes the proof of Theorem 3. ■

Proof of Theorem 4 (i). It suffices to show that there exists a constant $C>0$ such that for every $H^1(w)$-atom a with supp$a\subset Q=Q(x_0,d)$, there is

$$||S_\delta^A(a)||_{L^{n/(n-\delta)}}\le C.$$

Let $\tilde{A}(x)=A(x)-\sum_{|\alpha|=m}\frac{1}{\alpha!}(D^\alpha A)_Q x^\alpha$, then $R_m(A;x,y)=R_m(\tilde{A};x,y)$ and $D^\alpha\tilde{A}=D^\alpha A-(D^\alpha A)_Q$ for all α with $|\alpha|=m$. We write, by the vanishing moment of a and for $u\in 3Q\setminus 2Q$,

$$F_t^A(a)(x,y) = \chi_{4Q}(x)F_t^A(a)(x,y)$$
$$+ \chi_{(4Q)^c}(x)\int_{\mathbb{R}^n}\Big[\frac{R_m(\tilde{A};x,z)\psi_t(y-z)}{|x-y|^m} - \frac{R_m(\tilde{A};x,u)\psi_t(y-u)}{|x-u|^m}\Big]a(z)dz$$
$$- \chi_{(4Q)^c}(x)\sum_{|\alpha|=m}\frac{1}{\alpha!}\int_{\mathbb{R}^n}\Big[\frac{\psi_t(y-z)(x-z)^\alpha}{|x-z|^m} - \frac{\psi_t(y-u)(x-u)^\alpha}{|x-u|^m}\Big]D^\alpha\tilde{A}(z)a(z)dz$$
$$- \chi_{(4Q)^c}(x)\sum_{|\alpha|=m}\frac{1}{\alpha!}\int_{\mathbb{R}^n}\frac{(x-u)^\alpha}{|x-u|^m}\psi_t(y-u)D^\alpha\tilde{A}(z)a(z)dz,$$

then

$$S_\delta^A(a)(x) = \Big\|\chi_{\Gamma(x)}(y,t)F_t^A(a)(x,y)\Big\|$$
$$\le i_{4Q}(x)\big\|\chi_{\Gamma(x)}(y,t)F_t^A(a)(x,y)\big\| + \chi_{(4Q)^c}(x)$$
$$\times \Big\|\chi_{\Gamma(x)}(y,t)\int_{\mathbb{R}^n}\Big[\frac{R_m(\tilde{A};x,z)\psi_t(y-z)}{|x-z|^m} - \frac{R_m(\tilde{A};x,u)\psi_t(y-u)}{|x-u|^m}\Big]a(z)dz\Big\|$$
$$+ \chi_{(4Q)^c}(x)\Big\|\chi_{\Gamma(x)}(y,t)\sum_{|\alpha|=m}\frac{1}{\alpha!}\int_{\mathbb{R}^n}\Big[\frac{\psi_t(y-z)(x-z)^\alpha}{|x-z|^m}$$
$$- \frac{\psi_t(y-u)(x-u)^\alpha}{|x-u|^m}\Big]D^\alpha\tilde{A}(z)a(z)dz\Big\|$$
$$+ \chi_{(4Q)^c}(x)\Big\|\chi_{\Gamma(x)}(y,t)\sum_{|\alpha|=m}\frac{1}{\alpha!}\int_{\mathbb{R}^n}\frac{(x-u)^\alpha}{|x-u|^m}\psi_t(y-u)D^\alpha\tilde{A}(z)a(z)dz\Big\|$$
$$= L_1(x) + L_2(x,u) + L_3(x,u) + L_4(x,u).$$

By the (L^p, L^q)-boundedness of S_δ^A for $n/(n-\delta) < q$ and $1/q = 1/p - \delta/n$ (see Lemma 2), we get

$$\|L_1(\cdot)\|_{L^{n/(n-\delta)}} \le \|S_\delta^A(a)\|_{L^q}|4Q|^{(n-\delta)/n-1/q} \le C\|a\|_{L^p}|Q|^{1-1/p} \le C.$$

Similar to the proof of Theorem 1, we obtain

$$\|L_2\|_{L^{n/(n-\delta)}} \le C \quad \text{and} \quad \|L_3(\cdot,u)\|_{L^{n/(n-\delta)}} \le C.$$

Thus, using the condition of $L_4(x,u)$, we obtain

$$\|S_\delta^A(a)\|_{L^{n/(n-\delta)}} \le C.$$

(ii). We write, for $f = f\chi_{4Q} + f\chi_{(4Q)^c} = f_1 + f_2$ and $u \in 3Q \setminus 2Q$,

$$\tilde{F}_t^A(f)(x,y) = \tilde{F}_t^A(f_1)(x,y) + \int_{\mathbb{R}^n}\frac{R_m(\tilde{A};x,z)}{|x-z|^m}\psi_t(y-z)f_2(z)dz$$
$$-\sum_{|\alpha|=m}\frac{1}{\alpha!}(D^\alpha A(x)-(D^\alpha A)_Q)\int_{\mathbb{R}^n}\Big[\frac{\psi_t(y-z)(x-z)^\alpha}{|x-z|^m} - \frac{\psi_t(u-z)(u-z)^\alpha}{|u-z|^m}\Big]f_2(z)dz$$
$$-\sum_{|\alpha|=m}\frac{1}{\alpha!}(D^\alpha A(x)-(D^\alpha A)_Q)\int_{\mathbb{R}^n}\frac{(u-z)^\alpha}{|u-z|^m}\psi_t(u-z)f_2(z)dz,$$

then

$$\begin{aligned}
&\Big|\tilde S_\delta^A(f)(x)-S_\delta\Big(\frac{R_m(\tilde A;x_0,\cdot)}{|x_0-\cdot|^m}f_2\Big)(x_0)\Big|\\
&=\Big|\big\|\chi_{\Gamma(x)}\tilde F_t^A(f)(x,y)\big\|-\Big\|\chi_{\Gamma(x_0)}F_t\Big(\frac{R_m(\tilde A;x_0,\cdot)}{|x_0-\cdot|^m}f_2\Big)(y)\Big\|\Big|\\
&\le\Big\|\chi_{\Gamma(x)}(y,t)\tilde F_t^A(f)(x,y)-\chi_{\Gamma(x_0)}(y,t)F_t\Big(\frac{R_m(\tilde A;x_0,\cdot)}{|x_0-\cdot|^m}f_2\Big)(y)\Big\|\\
&\le\big\|\chi_{\Gamma(x)}(y,t)\tilde F_t^A(f_1)(x,y)\big\|\\
&\quad+\Big\|\Big[\chi_{\Gamma(x)}(y,t)\int_{\mathbb{R}^n}\frac{R_m(\tilde A;x,z)}{|x-z|^m}\psi_t(y-z)\\
&\quad-\chi_{\Gamma(x_0)}(y,t)\int_{\mathbb{R}^n}\frac{R_m(\tilde A;x_0,z)}{|x_0-z|^m}\psi_t(y-z)\Big]f_2(z)dz\Big\|\\
&\quad+\Big\|\chi_{\Gamma(x)}(y,t)\sum_{|\alpha|=m}\frac{1}{\alpha!}(D^\alpha A(x)-(D^\alpha A)_Q)\\
&\quad\times\int_{\mathbb{R}^n}\Big[\frac{\psi_t(y-z)(x-z)^\alpha}{|x-z|^m}-\frac{\psi_t(u-z)(u-z)^\alpha}{|u-z|^m}\Big]f_2(z)dz\Big\|\\
&\quad+\Big\|\chi_{\Gamma(x)}(y,t)\sum_{|\alpha|=m}\frac{1}{\alpha!}(D^\alpha A(x)-(D^\alpha A)_Q)\int_{\mathbb{R}^n}\frac{(u-z)^\alpha}{|u-z|^m}\psi_t(u-z)f_2(z)dz\Big\|\\
&=M_1(x)+M_2(x)+M_3(x,u)+M_4(x,u).
\end{aligned}$$

By the (L^p,L^q)-boundedness of $\tilde S_\delta^A$ for $1<p<n/\delta$ and $1/q=1/p-\delta/n$, we get

$$\frac{1}{|Q|}\int_Q M_1(x)dx\le|Q|^{-1/q}||\tilde S_\delta^A(f_1)||_{L^q}\le C|Q|^{-1/q}||f_1||_{L^p}\le C||f||_{L^{n/\delta}}.$$

Similar to the proof of Theorem 1, we obtain

$$\frac{1}{|Q|}\int_Q M_2(x)dx\le C||f||_{L^{n/\delta}}\quad\text{and}\quad\frac{1}{|Q|}\int_Q M_3(x,u)dx\le C||f||_{L^{n/\delta}}.$$

Thus, by using the condition of $M_4(x,u)$, we obtain

$$\frac{1}{|Q|}\int_Q\Big|\tilde S_\delta^A(f)(x)-S_\delta\Big(\frac{R_m(\tilde A;x_0,\cdot)}{|x_0-\cdot|^m}f_2\Big)(x_0)\Big|dx\le C||f||_{L^{n/\delta}}.$$

This completes the proof of Theorem 4.

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References

1. J. Alvarez, R. J. Babgy, D. S. Kurtz, and C. Perez, Weighted estimates for commutators of linear operators, *Studia Math.* **104** (1993) 195–209.
2. S. Chanillo, A note on commutators, *Indiana Univ. Math. J.* **31** (1982) 7–16.
3. W. Chen and G. Hu, Weak type (H^1, L^1) estimate for multilinear singular integral operator, *Adv. in Math.* **30** (2001) 63–69 (Chinese).
4. J. Cohen, A sharp estimate for a multilinear singular integral on $\mathbb{R}^n$, *Indiana Univ. Math. J.* **30** (1981) 693–702.
5. J. Cohen and J. Gosselin, On multilinear singular integral operators on $\mathbb{R}^n$, *Studia Math.* **72** (1982) 199–223.
6. J. Cohen and J. Gosselin, A BMO estimate for multilinear singular integral operators, *Illinois J. Math.* **30** (1986) 445–465.
7. R. Coifman, R. Rochberg, and G. Weiss, Factorization theorems for Hardy spaces in several variables, *Ann. Math.* **103** (1976) 611–635.
8. Y. Ding and S. Z. Lu,Weighted boundedness for a class rough multilinear operators, *Acta Math. Sinica* **17** (2001) 517–526.
9. J. Garcia-Cuerva and J. L. Rubio de Francia, *Weighted Norm Inequalities and Related Topics*, North-Holland Math. 16, Amsterdam, 1985.
10. E. Harboure, C. Segovia, and J. L.Torrea, Boundedness of commutators of fractional and singular integrals for the extreme values of p, *Illinois J. Math.* **41** (1997) 676–700.
11. L. Z. Liu , Weighted weak type estimates for commutators of Littlewood-Paley operator, *Japanese J. Math.* **29** (2003) 1–13.
12. L. Z. Liu, Weighted weak type (H^1, L^1) estimates for commutators of Littlewood-Paley operator, *Indian J. Math.* **45** 71–78.
13. E. M. Stein, *Harmonic Analysis: Real Variable Methods, Orthogonality and Oscillatory Integrals*, Princeton Univ. Press, Princeton NJ, 1993.
14. A. Torchinsky, *The Real Variable Methods in Harmonic Analysis*, Pure and Applied Math. 123, Academic Press, New York, 1986.